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DETERMINATION OF A MINOR COEFFICIENT IN A TIME FRACTIONAL DIFFUSION EQUATION

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For a time fractional diffusion equation on bounded cylindrical domain the inverse problem is studied. It consists of the determination of a pair of functions: a classical solution of the second boundary value problem for such an equation and unknown, depending on all variables, minor coefficient in the equation under some integral type over-determination condition. Conditions of the existence and uniqueness of a solution are found.

1. Introduction. Equations with fractional derivatives are appearing in the study of anomalous diffusion and other important processes. Conditions of the classical solvability of the first boundary value problem to the equation

$$D_t^\beta u(x,t) - a^2\Delta u(x,t) = F_0(x,t), \quad a^2 = \text{const} > 0$$

with regularized (Caputo–Djrbashian) fractional derivative ([1, 2])

$$D_t^\beta u(x,t) = \frac{1}{\Gamma(1-\beta)}\int_0^t \frac{u_\tau(x,\tau)}{(t-\tau)^\beta}d\tau = \frac{1}{\Gamma(1-\beta)}\Big[\frac{\partial}{\partial t}\int_0^t \frac{u(x,\tau)}{(t-\tau)^\beta}d\tau - \frac{u(x,0)}{t^\beta}\Big],\ \beta\in(0,1)$$

were obtained in [3, 4]. There were proved the existence and uniqueness theorems and the representations (in terms of the Green vector-function) of classical solutions of fractional Cauchy problems to equations of the kind in [5]–[7].

Inverse problems to such equations appear in many regions of industry. Some inverse boundary value problems to diffusion-wave equation with different unknown functions or parameters were investigated, for example, in [8]–[15]. Comparison on the well posedness of the inverse problems to the equation of fractional diffusion and corresponding ordinary diffusion equation was considered in [16].

In this note we find conditions of the existence and uniqueness of a solution of the inverse problem for a time fractional diffusion equation on bounded cylindrical domain consisting of the determination of a pair of functions: a classical solution of the second boundary value problem for such an equation and unknown, depending on all variables, minor coefficient in the equation under some integral type over-determination condition.

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We study the inverse boundary value problem

$$D_t^\beta u - a(x,t)\Delta u - b(x,t)u = F_0(x,t), \quad (x,t) \in \Omega_0 \times (0,T], \tag{1}$$

$$\frac{\partial u(x,t)}{\partial \nu_x} = 0, \quad (x,t) \in \Omega_1 \times (0,T], \tag{2}$$

$$u(x,0) = F_1(x), \quad x \in \bar{\Omega}_0, \tag{3}$$

$$\int_0^t \int_{\Omega_1} K(x,t,z,s)u(z,s)dzds = F_2(x,t), \quad (x,t) \in \bar{\Omega}_0 \times (0,T] \tag{4}$$

where $\beta \in (0,1)$, Ω_0 is a boundary domain in $\mathbb{R}^n$, $n \geq 2$ with a smooth boundary Ω_1, $\nu(x) = (\nu_1(x), \ldots, \nu_n(x))$ is the unit vector of the interior normal to the surface Ω_1 at the point $x \in \Omega_1$, where a, F_0, F_1, F_2, K are given functions. We use the Green function method in the study of this problem.

Note that some inverse boundary value problems on determination of a pair (u,b) with $b = b(t)$ for ordinary diffusion equation ($\beta = 1$) were studied in [17, 18] and other papers, where the existence and uniqueness theorems were proved. In [9] the unique solvability of the inverse problem on determination of a solution u of an abstract fractional Cauchy problem in a Hilbert space X and a minor coefficient $b(t)$ was studied under the over-determination condition $(u, \varphi_0) = h$, where (u, φ_0) is the inner product of elements $u, \varphi_0 \in X$, φ_0 is a given element of X, $h = h(t)$ is a given function.

2. Green vector-function and auxiliary results. Assume that $Q_i = \Omega_i \times (0,T]$, $i = 0,1$, Ω_1 is a surface of the class $C^{1+\gamma}$, $\gamma \in (0,1)$, $C(\bar{Q}_0)$ is the space of continuous functions on $\bar{Q}_0$, $C^\gamma(\bar{\Omega}_0)$ is the space of Hölder continuous functions on $\bar{\Omega}_0$, $C^\gamma(\bar{Q}_i)$ ($C^\gamma(Q_i)$) is the space of Hölder continuous functions in space variables $x \in \bar{\Omega}_i$ for all $t \in [0,T]$ ($t \in (0,T]$) and jointly continuous in $(x,t) \in \bar{Q}_i$ ($(x,t) \in \bar{\Omega}_i \times (0,T]$), $i = 0,1$, $C_{2,\beta}(\bar{Q}_0) = \{v \in C(\bar{Q}_0) | \Delta v, D_t^\beta v \in C(Q_0)\}$, $C^1_{2,\beta}(\bar{Q}_0) = \{v \in C_{2,\beta}(\bar{Q}_0) | \frac{\partial v}{\partial \nu} \in C(\bar{Q}_1)\}$, $\mathcal{D}(\mathbb{R})$ is the space of indefinitely differentiable functions compactly supported in $\mathbb{R}$, $\mathcal{D}'(\mathbb{R})$ is the space of linear continuous functionals (distributions) over $\mathcal{D}(\mathbb{R})$, $\mathcal{D}'_+(\mathbb{R}) = \{f \in \mathcal{D}'(\mathbb{R}) : f = 0 \text{ for } t < 0\}$.

We denote by $f*g$ the convolution of distributions f and g, use the function $f_\lambda \in \mathcal{D}'_+(R)$:

$$f_\lambda(t) = \begin{cases} \frac{\theta(t)t^{\lambda-1}}{\Gamma(\lambda)}, & \lambda > 0, \\ f_\lambda(t) = f'_{1+\lambda}(t), & \lambda \leq 0 \end{cases}$$

where $\Gamma(z)$ is the Gamma-function, $\theta(t)$ is the Heaviside function, understand the derivative in $\mathcal{D}'(\mathbb{R})$-sense. Note that $f_\lambda * f_\mu = f_{\lambda+\mu}$.

We suppose that $a \in C^\gamma(\bar{Q}_0)$, $\min\limits_{(x,t)\in\bar{Q}_0} a(x,t) = a_0 > 0$.

Definition 1. A pair of functions $(u,b) \in \mathcal{M}_\beta(Q_0) := C^1_{2,\beta}(\bar{Q}_0) \times C^\gamma(\bar{Q}_0)$ satisfying the equation (1) on Q_0 and conditions (2)–(4) is called a *solution of problem (1)–(4)*.

Let $(L^{\mathrm{reg}}v)(x,t) \equiv D_t^\beta v(x,t) - a(x,t)\Delta v(x,t)$, $(x,t) \in Q_0$.

Definition 2. A pair $(G_0(x,t,y,\tau), G_1(x,t,y))$, such that for rather regular g_0, g_1 there exists a classical (from $C^1_{2,\beta}(\bar{Q}_0)$) solution

$$u(x,t) = \int_0^t d\tau \int_{\Omega_0} G_0(x,t,y,\tau)g_0(y,\tau)dy + \int_{\Omega_0} G_1(x,t,y)g_1(y)dy, \quad (x,t) \in \bar{Q}_0 \tag{5}$$

of the problem

$$(L^{reg}u)(x,t) = g_0(x,t), \quad (x,t) \in Q_0, \tag{6}$$

$$\frac{\partial u(x,t)}{\partial \nu_x} = 0, \quad (x,t) \in \bar{Q}_1, \quad u(x,0) = g_1(x), \quad x \in \bar{\Omega}_0, \tag{7}$$

is called a Green vector-function of the problem (6), (7).

By Definition 2

$$(L^{\text{reg}}G_0)(x,t,y,\tau) = \delta(x-y,t-\tau), \quad (x,t),(y,\tau) \in Q_0$$

where δ is the Dirac delta-function,

$$(L^{\text{reg}}G_1)(x,t,y) = 0, \quad (x,t) \in Q_0, y \in \Omega_0, \quad G_1(x,0,y) = \delta(x-y), \quad x,y \in \Omega_0.$$

The Green vector-function $(G_0^0(x,t,y,\tau), G_1^0(x,t,y))$ of the first boundary value problem with homogeneous boundary condition is defined similarly, and it follows from the maximum principle ([3]) that $G_0^0(x,t,y,\tau) > 0$ for $(x,t),(y,\tau) \in Q_0$, $G_1^0(x,t,y) > 0$, $(x,t) \in Q_0$, $y \in \Omega_0$.

Lemma 1. *The following relations hold*

$$G_1(x,t,y) = \int_0^t f_{1-\beta}(\tau) G_0(x,t,y,\tau) d\tau, \ (x,t) \in \bar{Q}_0, \ y \in \Omega_0,$$
$$G_0^0(x,t,y,\tau) \le G_0(x,t,y,\tau), \quad (x,t),(y,\tau) \in \bar{Q}_0,$$

and therefore,

$$G_0(x,t,y,\tau) > 0 \ \textit{for} \ (x,t),(y,\tau) \in Q_0, \quad G_1(x,t,y) > 0, \quad (x,t) \in Q_0, \quad y \in \Omega_0.$$

Proof. The needed relations we prove by the scheme of corresponding results in [19]. The needed inequality between G_0^0 and G_0 follows from the maximum principle ([3]–[5]). Indeed,

$$G_0^0(x,t,y,\tau) = G(x,t,y,\tau) + z_0(x,t,y,\tau), \quad G_0(x,t,y,\tau) = G(x,t,y,\tau) + z(x,t,y,\tau),$$

where $G(x,t,y,\tau)$ is the fundamental function of the operator $L = f_{-\beta}(t) * -a(x,t)\Delta$, $z_0(\cdot,\cdot,y,\tau)$, $z(\cdot,\cdot,y,\tau)$ are solutions (from $C_{2,\beta}(\bar{Q}_0)$) of the equation $D_t^\beta u - a(x,t)\Delta u = 0$, $(x,t) \in Q_0$ for all $(y,\tau) \in Q_0$, such that $z_0(x,t,y,\tau) = -G(x,t,y,\tau)$, $\frac{\partial}{\partial \nu_x} z(x,t,y,\tau) = -\frac{\partial}{\partial \nu_x} G(x,t,y,\tau)$ for $(x,t) \in \bar{Q}_1$, $(y,\tau) \in Q_0$. Therefore, $G_0^0(x,t,y,\tau) - G_0(x,t,y,\tau) = z_0(x,t,y,\tau) - z(x,t,y,\tau)$ belongs to $C_{2,\beta}(\bar{Q}_0)$ as a function of (x,t) for all $(y,\tau) \in Q_0$,

$$\frac{\partial z_0(x,t,y,\tau)}{\partial \nu_x} - \frac{\partial z(x,t,y,\tau)}{\partial \nu_x} = \frac{\partial G_0^0(x,t,y,\tau)}{\partial \nu_x} > 0, \ (x,t) \in \bar{Q}_1, \ (y,\tau) \in Q_0.$$

As in [20, 2.5] we obtain

$$G_0^0(x,t,y,\tau) - G_0(x,t,y,\tau) = z_0(x,t,y,\tau) - z(x,t,y,\tau) \le 0 \ \text{ for all } \ (x,t),(y,\tau) \in \bar{Q}_0.$$

Since $z_0(x,t,y,\tau) - z(x,t,y,\tau) = -G_0(x,t,y,\tau) \ne 0$ for $(x,t) \in \bar{Q}_1$, $(y,\tau) \in Q_0$, one has $G_0(x,t,y,\tau) > 0$ for all $(x,t),(y,\tau) \in Q_0$. □

The existence of a Green vector-function of second boundary value problem (6), (7) can be proved by the Levi method (see [6, 20]), also as in [15] for the case $a = a(t)$.

We use the notation

$$(\mathcal{G}_0\varphi)(x,t) = \int_0^t d\tau \int_{\Omega_0} G_0(x,t,y,\tau)\varphi(y,\tau)dy, \quad (\mathcal{G}_1\varphi)(x,t) = \int_{\Omega_0} G_1(x,t,y)\varphi(y)dy.$$

As in the cases of the time fractional Cauchy problem ([6, 7]), the first boundary value problem ([14]) and the second boundary value problem for ordinary diffusion equation ([20]) we obtain the following result.

Theorem 1. *If $g_0 \in C^\gamma(\bar{Q}_0)$, $g_1 \in C^\gamma(\bar{\Omega}_0)$, $\operatorname{supp} g_1 \subset \Omega_0$ then there exists a unique solution $u \in C^1_{2,\beta}(\bar{Q}_0)$ of problem (6), (7) which is defined by*

$$u(x,t) = \big(\mathcal{G}_0 g_0\big)(x,t) + \big(\mathcal{G}_1 g_1\big)(x,t), \quad (x,t) \in \bar{Q}_0. \tag{8}$$

3. The existence and uniqueness theorems. Let the following assumptions hold:

(A1) $F_0 \in C^\gamma(\bar{Q}_0)$, $F_1 \in C^\gamma(\bar{\Omega}_0)$, $\operatorname{supp} F_1 \subset \Omega_0$, $\|F_1\|_{C(\bar{\Omega}_0)} > 0$,

(A2) $F_2 \in C^\gamma(\bar{Q}_0)$, $F_2(x,0) = 0$, $\min\limits_{(x,t)\in\bar{Q}_0} t^{-p}|F_2(x,t)| \neq 0$ for some $p \in (0, 1-\beta)$,

$K(x,t,z,s)$ $((x,t) \in \bar{Q}_0, (z,s) \in \bar{Q}_1)$ is a nonnegative function, Hölder continuous with respect to $x \in \bar{\Omega}_0$ and such that

$$K(x,t,z,s) \le A_0(t-s)^\alpha, \;\; x \in \bar{\Omega}_0, \;\; z \in \Omega_1, \;\; 0 \le s < t \le T, \tag{9}$$

where $A_0 = \text{const} > 0$, $p + \frac{\beta}{2} - 1 < \alpha \le -\frac{\beta}{2}$,

(A3) $F_1(x) \ge 0$, $x \in \bar{\Omega}_0$, $\quad F_2(x,t) \ge 0$, $(x,t) \in \bar{Q}_0$,

or

$F_1(x) \le 0$, $x \in \bar{\Omega}_0$, $\quad F_2(x,t) \le 0$, $(x,t) \in \bar{Q}_0$.

Note that there exists the convolution $f_{-\alpha-\frac{\beta}{2}}(t) * F_2(x,t)$ from $C^\gamma(\bar{Q}_0)$ under assumptions (A2).

It follows from Theorem 1 that under assumptions (A1) the solution $u \in C^1_{2,\beta}(\bar{Q}_0)$ of second boundary value problem (1)–(3) satisfies the integral equation

$$u(x,t) = \big(\mathcal{G}_0\big(bu + F_0\big)\big)(x,t) + \big(\mathcal{G}_1 F_1\big)(x,t), \quad (x,t) \in \bar{Q}_0, \tag{10}$$

and conversely, any solution $u \in C(\bar{Q}_0)$ of integral equation (10) belongs to $C^1_{2,\beta}(\bar{Q}_0)$ and is a solution of problem (1)–(3).

Substituting the right-hand side of (10) in condition (4) we obtain

$$\int_0^t ds \int_{\Omega_1} K(x,t,z,s)\Big[\int_0^s d\tau \int_{\Omega_0} G_0(z,s,y,\tau)(b(y,\tau)u(y,\tau) + F_0(y,\tau))dy +$$
$$+ \int_{\Omega_0} G_1(z,s,y)F_1(y)dy\Big]dz = F_2(x,t), \quad (x,t) \in \bar{Q}_0,$$

that is

$$\int_0^t d\tau \int_{\Omega_0} K_0(x,t,y,\tau)v(y,\tau)dy = h(x,t), \quad (x,t) \in \bar{Q}_0, \tag{11}$$

where

$$v(y,\tau) = b(y,\tau)u(y,\tau) + F_0(y,\tau), \quad (y,\tau) \in \bar{Q}_0,$$

$$h(x,t) = F_2(x,t) - \int_{\Omega_0} K_1(x,t,y)F_1(y)dy, \quad (x,t) \in \bar{Q}_0,$$
$$K_0(x,t,y,\tau) = \int_\tau^t ds \int_{\Omega_1} K(x,t,z,s)G_0(z,s,y,\tau)dz,$$
$$K_1(x,t,y) = \int_0^t ds \int_{\Omega_1} K(x,t,z,s)G_1(z,s,y)dz.$$

It follows from the properties of the H-functions of Fox ([21]) and the results [5, 6, 7], that for the case $n \geq 3$ the following estimates hold:

$$G_0(x,t,y,\tau) \leq \frac{C}{a_0(t-\tau)|x-y|^{n-2}},$$
$$G_1(x,t,y) \leq \frac{C}{a_0 t^\beta |x-y|^{n-2}} \quad \text{when} \quad |x-y|^2 < 4a_0(t-\tau)^\beta,$$
$$G_0(x,t,y,\tau) \leq \frac{C(t-\tau)^{\beta-1}}{|x-y|^n} \cdot \Big(\frac{|x-y|^2}{4a_0(t-\tau)^\beta}\Big)^{1+\frac{n}{2(2-\beta)}} e^{-c\left(\frac{|x-y|^2}{4a_0(t-\tau)^\beta}\right)^{\frac{1}{2-\beta}}},$$
$$G_1(x,t,y) \leq \frac{C}{|x-y|^n} \cdot \Big(\frac{|x-y|^2}{4a_0 t^\beta}\Big)^{\frac{n}{2(2-\beta)}} e^{-c\left(\frac{|x-y|^2}{4a_0 t^\beta}\right)^{\frac{1}{2-\beta}}} \quad \text{if} \quad |x-y|^2 > 4a_0(t-\tau)^\beta,$$

where C, c are positive constants. Using them and (9) for the case $n \geq 3$ we obtain

$$\int_\tau^t ds \int_{\Omega_1} K(x,t,z,s)G_0(z,s,y,\tau)dz \leq$$
$$\leq \int_\tau^t \Big[\int_{\{z\in\Omega_1 : |y-z|<2\sqrt{a_0}(s-\tau)^{\beta/2}\}} K(x,t,z,s)G_0(z,s,y,\tau)dz +$$
$$+ \int_{\{z\in\Omega_1 : |y-z|>2\sqrt{a_0}(s-\tau)^{\beta/2}\}} K(x,t,z,s)G_0(z,s,y,\tau)dz\Big]ds \leq$$
$$\leq A_0 C \int_\tau^t (t-s)^\alpha \Big[\int_{\{z\in\Omega_1 : |y-z|<2\sqrt{a_0}(s-\tau)^{\beta/2}\}} \frac{dz}{a_0(s-\tau)|y-z|^{n-2}} dz +$$
$$+ \int_{\{z\in\Omega_1 : |y-z|>2\sqrt{a_0}(s-\tau)^{\beta/2}\}} \frac{(s-\tau)^{\beta-1}}{|z-y|^n} \Big(\frac{|z-y|^2}{4a_0(s-\tau)^\beta}\Big)^{1+\frac{n}{2(2-\beta)}} e^{-c\left(\frac{|z-y|^2}{4a_0(s-\tau)^\beta}\right)^{\frac{1}{2-\beta}}} dz\Big] ds \leq$$
$$\leq C_1 \int_\tau^t (t-s)^\alpha \Big[\frac{1}{\sqrt{a_0}(s-\tau)} \int_0^{2\sqrt{a_0}(s-\tau)^{\beta/2}} dr +$$
$$+ (s-\tau)^{-1-\frac{n\beta}{2(2-\beta)}} \int_{2\sqrt{a_0}(s-\tau)^{\beta/2}}^{\operatorname{diam}\Omega_1} r^{\frac{n}{2-\beta}} e^{-c\left(\frac{r^2}{4a_0(s-\tau)^\beta}\right)^{\frac{1}{2-\beta}}} dr\Big] ds \leq$$
$$\leq C_2 \int_\tau^t (t-s)^\alpha (s-\tau)^{\frac{\beta}{2}-1} \Big[1 + \int_1^\infty \eta^{\frac{n-\beta}{2}} e^{-c\eta} d\eta\Big] ds \leq \hat{k}(t-\tau)^{\alpha+\frac{\beta}{2}},$$

and similarly,

$$\int_0^t ds \int_{\Omega_1} K(x,t,z,s)G_1(z,s,y)dz \leq$$
$$\leq A_0 C \int_0^t (t-s)^\alpha \Big[\int_{\{z\in\Omega_1 : |y-z|<2\sqrt{a_0}s^{\beta/2}\}} \frac{dz}{a_0 s^\beta |y-z|^{n-2}} dz +$$

$$+\int_{\{z\in\Omega_1:|y-z|>2\sqrt{a_0}s^{\beta/2}\}}\frac{1}{|z-y|^n}\Big(\frac{|z-y|^2}{4a_0s^\beta}\Big)^{\frac{n}{2(2-\beta)}}e^{-c\left(\frac{|z-y|^2}{4a_0s^\beta}\right)^{\frac{1}{2-\beta}}}dz\Big]ds\le$$

$$\le C_3\int_0^t(t-s)^\alpha\Big[\frac{1}{\sqrt{a_0}s^\beta}\int_0^{2\sqrt{a_0}s^{\beta/2}}dr\quad+s^{-\frac{n\beta}{2(2-\beta)}}\int_{2\sqrt{a_0}s^{\beta/2}}^{\operatorname{diam}\Omega_1}r^{\frac{n}{2-\beta}-2}e^{-c\left(\frac{r^2}{4a_0(s-\tau)^\beta}\right)^{\frac{1}{2-\beta}}}dr\Big]ds\le$$

$$\le C_4\int_0^t(t-s)^\alpha s^{-\beta/2}\Big[1+\int_1^\infty\eta^{\frac{n-\beta}{2}-2}e^{-c\eta}d\eta\Big]ds\le\hat{k}t^{\alpha-\frac{\beta}{2}+1},\quad(x,t)\in\bar{Q}_0$$

where $\hat{k}, C_i,\ i\in\{1,2,3,4\}$ are positive constants. The same kind of estimates we obtain in the case $n=2$. So,

$$K_0(x,t,y,\tau)\le\hat{k}(t-\tau)^{\alpha+\frac{\beta}{2}}=\hat{k}\Gamma\Big(1+\alpha+\frac{\beta}{2}\Big)f_{1+\alpha+\frac{\beta}{2}}(t-\tau),\ x,y\in\bar{\Omega}_0,\ 0\le\tau<t\le T,$$

$$K_1(x,t,y)\le\hat{k}t^{\alpha-\frac{\beta}{2}+1}=\hat{k}\Gamma\Big(2+\alpha-\frac{\beta}{2}\Big)f_{2+\alpha-\frac{\beta}{2}}(t),(x,t)\in\bar{Q}_0,t\in\bar{\Omega}_0.\tag{12}$$

The function

$$\mathcal{R}_0(x,t,y,\tau)=f_{-\alpha-\frac{\beta}{2}}(t-\tau)*K_0(x,t-\tau,y,\tau)\ne0,\ (x,t),(y,\tau)\in\bar{Q}_0,$$

is continuous on $\bar{Q}_0\times\bar{Q}_0$ ($\mathcal{R}_0=K_0$ if $\alpha=-\frac{\beta}{2}$), the function

$$\mathcal{R}_1(x,t,y)=f_{-\alpha-\frac{\beta}{2}}(t)*K_1(x,t,y)$$

is continuous on $\bar{Q}_0\times\bar{\Omega}_0$ and has the estimate

$$\mathcal{R}_1(x,t,y)\le\hat{k}\Gamma\Big(\alpha-\frac{\beta}{2}+2\Big)f_{-\alpha-\frac{\beta}{2}}(t)*f_{\alpha-\frac{\beta}{2}+2}(t)=$$

$$=\hat{k}\Gamma\Big(\alpha-\frac{\beta}{2}+2\Big)f_{2-\beta}(t)=\hat{k}_1t^{1-\beta},\ x\in\bar{\Omega}_0,y\in\Omega_1,\ t\in(0,T]$$

where $\hat{k}_1=\hat{k}\Gamma(\alpha-\frac{\beta}{2}+2)/\Gamma(2-\beta)$.

Thus, from (11) we obtain the linear integral Volterra equation of the first kind

$$\int_0^t d\tau\int_{\Omega_0}\mathcal{R}_0(x,t,y,\tau)v(y,\tau)dy=h_0(x,t),\quad(x,t)\in\bar{Q}_0\tag{13}$$

with the continuous positive kernel $\mathcal{R}_0(x,t,y,\tau)$ and

$$h_0(x,t)=f_{-\alpha-\frac{\beta}{2}}(t)*h(x,t),\quad(x,t)\in\bar{Q}_0.\tag{14}$$

Note that $h_0\in C^\gamma(\bar{Q}_0)$ and $h_0(x,0)=0$.

Conversely, if $v\in C^\gamma(\bar{Q}_0)$ is a solution of equation (13), which is equivalent to equation (11), then the function given by (10), that is

$$u(x,t)=(\mathcal{G}_0v)(x,t)+(\mathcal{G}_1F_1)(x,t),\quad(x,t)\in\bar{Q}_0,\tag{15}$$

is a solution (from $C^1_{2,\beta}(\bar{Q}_0)$) of problem (1)–(4).

Moreover, multiplying (15) by b, we obtain

$$bu=b\mathcal{G}_0v+b\mathcal{G}_1F_1\quad\Longleftrightarrow\quad v=b\big(\mathcal{G}_0v+\mathcal{G}_1F_1\big)+F_0$$

and hence

$$b(x,t) = \frac{v(x,t) - F_0(x,t)}{u(x,t)}, \quad (x,t) \in \bar{Q}_0 \tag{16}$$

if the ratio is not zero.

If $v(x,t) \geq 0$, $(x,t) \in \bar{Q}_0$ and $F_1(x) \geq 0$, $x \in \bar{\Omega}_0$ then, according to the above remarks, $u = \mathcal{G}_0 v + \mathcal{G}_1 F_1 \geq 0$ on $\bar{Q}_0$, and for $v(x,t) \leq 0$, $(x,t) \in \bar{Q}_0$, $F_1(x) \leq 0$, $x \in \bar{\Omega}_0$ we obtain $u \leq 0$ on $\bar{Q}_0$. In the case $\|F_1\|_{C(\bar{Q}_0)} > 0$ we obtain $|u(x,t)| \geq u_0 > 0$, $(x,t) \in \bar{Q}_0$.

Let $\mathcal{V}(\bar{Q}_0)$ be the class of functions from $C^\gamma(\bar{Q}_0)$ preserving sine on $\bar{Q}_0$.

Theorem 2. *Assume that (A1), (A2), (A3) hold, there exists a solution $v \in C^\gamma(\bar{Q}_0)$ of integral equation (13). Then there exist some $T^* \in (0,T]$ ($Q_0^* = \Omega \times (0,T^*]$, respectively) and a solution $(u,b) \in \mathcal{M}_\beta(Q_0^*)$ of problem (1)–(4). It is defined by formulas (15), (16) for $(x,t) \in \bar{Q}_0^*$.*

Proof. By the above remarks the kernel $\mathcal{R}_0(x,t,y,\tau)$ of integral equation (13) is continuous on $\bar{Q}_0 \times \bar{Q}_0$, satisfies the Hölder condition with respect to $x \in \bar{\Omega}_0$ and is positive. So, if $h_0 \in \mathcal{V}(\bar{Q}_0)$ then a solution v of equation (13) belongs to $\mathcal{V}(\bar{Q}_0)$, namely, for nonnegative (nonpositive) $h_0(x,t)$, $(x,t) \in \bar{Q}_0$ we obtain $v(x,t) \geq 0$ ($v(x,t) \leq 0$), $(x,t) \in \bar{Q}_0$ (see Lemma 1 and assumption (A2)).

Consider the right-hand side of (13) $h_0(x,t)$. Under assumption (A2), using (12), we obtain

$$\Big| \int_{\Omega_0} K_1(x,t,y) F_1(y) dy \Big| \leq k_1 t^{\alpha - \frac{\beta}{2} + 1} \|F_1\|_{C(\bar{\Omega}_0)}, \quad k_1 = \text{const} > 0.$$

Note that $F_2(x,t) \cdot \int_{\Omega_0} K_1(x,t,y) F_1(y) dy \geq 0$, $(x,t) \in \bar{Q}_0$, in both cases of assumption (A3). Then under assumptions (A2), (A3) there exists $T^* \in (0,T]$ such that $h_0(x,t) \geq 0$, $(x,t) \in \bar{Q}_0^*$ in the first case of assumption (A3), $h_0(x,t) \leq 0$, $(x,t) \in \bar{Q}_0^*$ in the second case of this assumption. Indeed,

$$t^{-p} \Big| \int_{\Omega_0} K_1(x,t,y) F_1(y) dy \Big| \leq k_1 t^{\alpha - \frac{\beta}{2} + 1 - p} \|F_1\|_{C(\bar{\Omega}_0)} \leq \min_{(x,t) \in \bar{Q}_0} |F_2(x,t)| t^{-p} \quad \forall t \in [0, T^*],$$

$$T^* = \left[\frac{\min_{(x,t) \in \bar{Q}_0} |F_2(x,t)| t^{-p}}{k_1 \|F_1\|_{C(\bar{\Omega}_0)}} \right]^{\frac{1}{\alpha - \frac{\beta}{2} + 1 - p}}.$$

Thus, the solution v of equation (13) belongs to $\mathcal{V}(\bar{Q}_0)$. □

Remark. If in Theorem 2 we assume in addition that $F_0 \leq 0$ on $\bar{Q}_0$ in the case of the first assumption of (A3) (or $F_0 \geq 0$ on $\bar{Q}_0$ in the case of the second assumption of (A3)) then we find the solution $(u,b) \in \mathcal{M}_\beta(Q_0^*)$ of problem (1)–(4) such that $u > 0$ on $\bar{Q}_0^*$ (respectively, $u < 0$ on $\bar{Q}_0^*$) and $b(x,t) \geq 0$, $(x,t) \in \bar{Q}_0^*$ in both cases.

Theorem 3. *Under conditions of the uniqueness of a solution of the uniqueness of a solution of equation (13), a solution $(u,b) \in \mathcal{M}_\beta(Q_0)$ of problem (1)–(4) such that $u \neq 0$ on $\bar{Q}_0$ is unique.*

Proof. Take two solutions $(u_1,b_1), (u_2,b_2) \in \mathcal{M}_\beta(Q_0)$ of problem (1)–(4), $u = u_1 - u_2$, $b = b_1 - b_2$. Then

$$D_t^\beta u = a \Delta u + b_2 u + b u_1, \tag{17}$$

$$\frac{\partial u(x,t)}{\partial \nu_x} = 0, \quad (x,t) \in \bar{Q}_1, \quad u(x,0) = 0, \quad x \in \bar{\Omega}_0, \tag{18}$$

and, by Theorem 1, the solution $u(x,t)$ of problem (17), (18) satisfies the integral equation

$$u(x,t) = \int_0^t d\tau \int_{\Omega_0} G_0(x,t,y,\tau)\Big(b(y,\tau)u_1(y,\tau) + b_2(y,\tau)u(y,\tau)\Big)dy, \quad (x,t) \in \bar{Q}_0. \tag{19}$$

Use the condition

$$\int_0^t \int_{\Omega_1} K(x,t,z,s)u(z,s)dzds = 0, \quad (x,t) \in Q_0$$

which we obtain from (4). As it has been shown before we obtain Volterra integral equation of the first kind

$$\int_0^t d\tau \int_{\Omega_0} \mathcal{R}_0(x,t,y,\tau)\Big(b(y,\tau)u_1(y,\tau) + b_2(y,\tau)u(y,\tau)\Big)dy = 0, \quad (x,t) \in \bar{Q}_0.$$

By the theorem's assumption,

$$b(y,\tau)u_1(y,\tau) + b_2(y,\tau)u(y,\tau) = 0, \quad (y,\tau) \in \bar{Q}_0. \tag{20}$$

Then from (19) we obtain $u = 0$ on $\bar{Q}_0$, and from (20) $b(x,t)u_1(x,t) = 0$, $(y,\tau) \in \bar{Q}_0$. Since $u_1 \neq 0$ on $\bar{Q}_0$, under the assumption of this theorem, one has $b(x,t) = 0$, $(x,t) \in \bar{Q}_0$. □

4. The one-dimensional case. Consider the problem

$$D_t^\beta u - A(x,t,D)u - b(x,t)u = F_0(x,t), \quad (x,t) \in Q := (0,l) \times (0,T], \tag{21}$$

$$\frac{\partial u(0,t)}{\partial x} = \frac{\partial u(l,t)}{\partial x} = 0, \quad t \in [0,T], \tag{22}$$

$$u(x,0) = F_1(x), \quad x \in [0,l], \tag{23}$$

$$\int_0^t K(x,t,s)u(0,s)ds = F_2(x,t), \quad (x,t) \in \bar{Q}, \tag{24}$$

where $Au = A(x,t,D)u = a(x,t)u_{xx} + a_1(x,t)u_x + a_2(x,t)u$ is an elliptic differential second order expression with Hölder continuous coefficients and $\min_{(x,t)\in\bar{Q}} a(x,t) = a_0 > 0$, $\beta \in (0,1)$, F_0, F_1, F_2, K are given functions.

A pair of functions $(u,b) \in \mathcal{M}(Q) := C^1_{2,\beta}(\bar{Q}) \times C^\gamma(\bar{Q})$ is called a solution of problem (21)–(24) if it satisfies equation (21) on Q and conditions (22)–(24).

Let assumptions (A1), (A2), (A3) with $[0,l]$ instead of $\bar{\Omega}_0$ hold.

It follows from Theorem 1 that under assumption (A1) the solution u of problem (21)–(23) satisfies the integral equation

$$u(x,t) = \int_0^t d\tau \int_0^l G_0(x,t,y,\tau)\big(b(y,\tau)u(y,\tau) + F_0(y,\tau)\big)dy + \\ + \int_0^l G_1(x,t,y)F_1(y)dy, \quad (x,t) \in \bar{Q}, \tag{25}$$

and conversely, any solution $u \in C^\gamma(\bar{Q})$ of integral equation (25) belongs to $C^1_{2,\beta}(\bar{Q})$ and is a solution of problem (21)–(23).

Substituting the right-hand side of (25) in condition (24) we obtain

$$\int_0^t K(x,t,s)\Big[\int_0^s d\tau \int_0^l G_0(0,s,y,\tau)\Big(b(y,\tau)u(y,\tau)+F_0(y,\tau)\Big)dy+ \\ +\int_0^l G_1(0,s,y)F_1(y)dy\Big]ds = F_2(x,t), \quad (x,t)\in\bar{Q},$$

that is

$$\int_0^t d\tau \int_0^l K_0(x,t,y,\tau)v(y,\tau)dy = h(x,t), \quad (x,t)\in\bar{Q}, \tag{26}$$

where

$$v(y,\tau)=b(y,\tau)u(y,\tau)+F_0(y,\tau), \quad (y,\tau)\in\bar{Q},$$

$$h(x,t)=F_2(x,t)-\int_0^l K_1(x,t,y)F_1(y)dy, \quad (x,t)\in\bar{Q},$$

$$K_0(x,t,y,\tau)=\int_\tau^t K(x,t,s)G_0(0,s,y,\tau)ds, \quad K_1(x,t,y)=\int_0^t K(x,t,s)G_1(0,s,y)ds.$$

Results of [6, 7] imply the following estimates:

$$G_0(x,t,y,\tau)\le C(t-\tau)^{\frac{\alpha}{2}-1}, \quad x,y\in[0,l],\ 0\le\tau<t\le T,$$
$$G_1(x,t,y)\le Ct^{-\frac{\alpha}{2}}, \quad x,y\in[0,l],\ t\in[0,T].$$

The Green vector-function possesses Hölder properties also.

Using the above estimates and (12) we find

$$\int_\tau^t K(x,t,s)G_0(0,s,y,\tau)ds \le A_0C\int_\tau^t (t-s)^\alpha(s-\tau)^{\frac{\beta}{2}-1}ds \le k_2(t-\tau)^{\alpha+\frac{\beta}{2}},$$

and similarly,

$$\int_0^t K(x,t,s)G_1(0,s,y)ds \le A_0C\int_0^t (t-s)^\alpha s^{-\frac{\beta}{2}}ds \le k_2 t^{\alpha-\frac{\beta}{2}+1}, \quad (x,t)\in\bar{Q}$$

where $k_2=\text{const}>0$. Repeating the proof of Theorems 2 and 3 we obtain the following result.

Theorem 4. *Assume that (A1), (A2), (A3) with* $[0,l]$ *instead of* $\bar{\Omega}_0$ *hold, there exists a solution* $v\in C^\gamma(\bar{Q})$ *of the integral equation*

$$\int_0^t d\tau\int_0^l \mathcal{R}_0(x,t,y,\tau)v(y,\tau)dy = h_0(x,t), \quad (x,t)\in\bar{Q}$$

with $\mathcal{R}_0(x,t,y,\tau)=f_{-\alpha-\frac{\beta}{2}}(t-\tau)*K_0(x,t,y,\tau)\ne 0,\ (x,t),(y,\tau)\in\bar{Q},\ h_0(x,t)=f_{-\alpha-\frac{\beta}{2}}(t)*h(x,t),\quad (x,t)\in\bar{Q}$. *Then there exist some* $T^*\in(0,T]$ ($Q^*=(0,l)\times(0,T^*]$, *respectively) and a solution* $(u,b)\in\mathcal{M}(Q^*)$ *of problem (21)–(24). It is defined by the formulas*

$$u(x,t)=\int_0^t d\tau\int_0^l G_0(x,t,y,\tau)v(y,\tau)dy+\int_0^l G_1(x,t,y)F_1(y)dy$$

and (16) for $(x,t)\in\bar{Q}^*$.

5. Remarks. In the same way we can investigate problem (1)–(3) with the over-determination condition $\int_0^t\int_{\Omega_0} K(x,t,z,s)u(z,s)dzds = F_2(x,t),\ (x,t)\in\bar{Q}_0$ instead of condition (4).

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