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EXISTENCE AND UNIQUENESS OF THE SOLUTIONS FOR NONLINEAR DEGENERATE ELLIPTIC EQUATIONS IN WEIGHTED SOBOLEV SPACES

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In this paper we are interested in the existence and uniqueness of solutions for the Dirichlet problem associated to the degenerate nolinear elliptic equation

$$-\sum_{j=1}^{n} D_j\big[\mathcal{A}_j(x,\nabla u)\,\omega_2(x)\big] + b(x,u)\,\omega_1(x) + g(x)\,u(x) = f_0(x) - \sum_{j=1}^{n} D_j f_j(x) \ \ \text{on} \ \ \Omega$$

in the setting of the weighted Sobolev spaces $\mathrm{W}_0^{1,p}(\Omega,\omega_1,\omega_2)$.

1. Introduction. In this paper we prove the existence and uniqueness of (weak) solutions in the weighted Sobolev spaces $\mathrm{W}_0^{1,p}(\Omega,\omega_1,\omega_2)$ (see Definition 2) for the Dirichlet problem

$$(P) \begin{cases} Lu(x) = f_0(x) - \sum\limits_{j=1}^{n} D_j f_j(x) & \text{in} \ \ \Omega, \\ u(x) = 0 & \text{on} \ \ \partial\Omega, \end{cases}$$

where L is the partial differential operator

$$Lu(x) = -\sum_{j=1}^{n} D_j\big[\mathcal{A}_j(x,\nabla u(x))\,\omega_2(x)\big] + b(x,u(x))\,\omega_1(x) + g(x)\,u(x),$$

$D_j = \partial/\partial x_j$, Ω is a bounded connected open set in $\mathbb{R}^n$, ω_i $(i=1,2)$ are weights functions and the functions $\mathcal{A} = (\mathcal{A}_1,...,\mathcal{A}_n)\colon \Omega\times\mathbb{R}^n \to \mathbb{R}^n$, $b\colon \Omega\times\mathbb{R} \to \mathbb{R}$, $g\colon \Omega \to \mathbb{R}$ satisfy the following conditions:

(H1) $x\mapsto\mathcal{A}_j(x,\xi)$ is measurable on Ω for all $\xi \in \mathbb{R}^n$, $\xi\mapsto\mathcal{A}_j(x,\xi)$ is continuous on $\mathbb{R}^n$ for almost all $x \in \Omega$, and $\mathcal{A}_j(x,0) = 0$ for a.e. $x \in \Omega$;

(H2) there exist constants $p > 2$ and $\lambda > 0$ such that $[\mathcal{A}(x,\xi)-\mathcal{A}(x,\xi')]\cdot(\xi-\xi') \geq \lambda\,|\xi - \xi'|^p$, whenever $\xi,\xi' \in \mathbb{R}^n$, where $\mathcal{A}(x,\xi) = (\mathcal{A}_1(x,\xi),...,\mathcal{A}_n(x,\xi))$, a dot denote here the Euclidian scalar product in $\mathbb{R}^n$;

(H3) $|\mathcal{A}(x,\xi)| \leq K_2(x) + h_2(x)|\xi|^{p/p'}$ such that h_2 and K_2 are nonnegative functions, with $h_2 \in L^\infty(\Omega)$ and $K_2 \in L^{p'}(\Omega, \omega_2)$ (with $1/p + 1/p' = 1$);

(H4) $x \mapsto b(x, \eta)$ is measurable on Ω for all $\eta \in \mathbb{R}$, $\eta \mapsto b(x, \eta)$ is continuous on $\mathbb{R}$ for almost all $x \in \Omega$, and $b(x, 0) = 0$ for a.e. $x \in \Omega$;

(H5) there exists a constant $\Lambda > 0$ such that $[b(x, \eta) - b(x, \eta')](\eta - \eta') \geq \Lambda |\eta - \eta'|^p$, whenever $\eta, \eta' \in \mathbb{R}$;

(H6) $|b(x, \eta)| \leq K_1(x) + h_1(x)|\eta|^{p/p'}$, where K_1 and h_1 are nonnegative functions such that $K_1 \in L^{p'}(\Omega, \omega_1)$ and $h_1 \in L^\infty(\Omega)$;

(H7) $g/\omega_1 \in L^q(\Omega, \omega_1)$, where $1/q = 1/p' - 1/p$, and $g(x) \geq 0$ a.e. $x \in \Omega$;

(H8) ω_1 and ω_2 are weights from the Muckenhoupt class A_p, with $2 < p < \infty$;

(H9) $f_0/\omega_1 \in L^{p'}(\Omega, \omega_1)$ and $f_j/\omega_2 \in L^{p'}(\Omega, \omega_2)$ $(j = 1, ..., n)$.

Under a *weight*, we mean a locally integrable function ω on $\mathbb{R}^n$ such that $0 < \omega(x) < \infty$ for a.e. $x \in \mathbb{R}^n$ (see [13]). Every weight ω gives rise to a measure on the measurable subsets on $\mathbb{R}^n$ through integration. This measure will be denoted by μ. Thus, $\mu(E) = \int_E \omega(x)\,dx$ for all measurable sets $E \subset \mathbb{R}^n$.

In general, the Sobolev spaces $W^{k,p}(\Omega)$ without weights occur as spaces of solutions for elliptic and parabolic partial differential equations. For the degenerate partial differential equations, i.e., the equations with various types of singularities in coefficients, it is natural to look for solutions in the weighted Sobolev spaces (see [3], [5] and [6]).

A class of weights, which is particulary well understood, is the class of A_p-weights (or the Muckenhoupt class) these classes weve introduced by B. Muckenhoupt (see [11]). They have found many usefull applications in harmonic analysis (see [12]). Another reason for studying A_p-weights is the fact that powers of distance to submanifolds of $\mathbb{R}^n$ often belong to A_p (see [9]). There are, in fact, many interesting examples of weights (see [8] for p-admissible weights).

In the non-degenerate case (i.e., with $\omega_1(x) = \omega_2(x) = 1$) the problem (P) has been studied in [1] and [2].

Remark 1. (a) From (H2) and the fact that $\mathcal{A}_j(x, 0) = 0$ for a.e. $x \in \Omega$, we have

$$\mathcal{A}(x,\xi) \cdot \xi = \Big(\mathcal{A}(x,\xi) - \mathcal{A}(x,0) \Big) \cdot (\xi - 0) \geq \lambda |\xi|^p.$$

(b) Since $b(x, 0) = 0$ for a.e. $x \in \Omega$, we deduce from (H5) that

$$b(x, \eta)\,\eta = (b(x, \eta) - b(x, 0))\,(\eta - 0) \geq \Lambda |\eta|^p.$$

Definition 1. We say that an element $u \in W_0^{1,p}(\Omega, \omega_1, \omega_2)$ is a *(weak) solution of problem (P)* if

$$\sum_{j=1}^n \int_\Omega \mathcal{A}_j(x, \nabla u(x))\, D_j\varphi(x)\, \omega_2(x)dx + \int_\Omega b(x, u(x))\varphi(x)\omega_1(x)dx +$$

$$+ \int_\Omega g(x)u(x)\varphi(x)dx = \int_\Omega f_0(x)\varphi(x)dx + \sum_{j=1}^n \int_\Omega f_j(x)D_j\varphi(x)dx,$$

for all $\varphi \in W_0^{1,p}(\Omega, \omega_1, \omega_2)$.

The following theorem will be proved in Section 3.

Theorem 1. *Suppose that conditions (H1)–(H9) hold. Then the problem (P) has the unique solution $u \in W_0^{1,p}(\Omega, \omega_1, \omega_2)$. Moreover,*

$$\|u\|_{W_0^{1,p}(\Omega,\omega_1,\omega_2)} \leq \frac{1}{\gamma^{p'/p}} \left(\left\| \frac{f_0}{\omega_1} \right\|_{L^{p'}(\Omega,\omega_1)} + \sum_{j=1}^{n} \left\| \frac{f_j}{\omega_2} \right\|_{L^{p'}(\Omega,\omega_2)} \right)^{p'/p},$$

where $\gamma = \min\{\lambda, \Lambda\}$.

Corollary 1. *Under the assumptions of Theorem 1, if $u_1 \in W_0^{1,p}(\Omega, \omega_1, \omega_2)$ is a solution of the problem*

$$(P_1) \begin{cases} Lu_1(x) = f_0(x) - \sum\limits_{j=1}^{n} D_j f_j(x) & \text{in } \Omega, \\ u_1(x) = 0 & \text{on } \partial\Omega \end{cases}$$

$u_2 \in W_0^{1,p}(\Omega, \omega_1, \omega_2)$ is a solutions of

$$(P_2) \begin{cases} Lu_2(x) = \tilde{f}_0(x) - \sum\limits_{j=1}^{n} D_j \tilde{f}_j(x) & \text{in } \Omega, \\ u_2(x) = 0 & \text{on } \partial\Omega, \end{cases}$$

and $\gamma = \min\{\lambda, \Lambda\}$, then

$$\|u_1 - u_2\|_{W_0^{1,p}(\Omega,\omega_1,\omega_2)} \leq \frac{1}{\gamma^{p/p'}} \left(\left\| \frac{f_0 - \tilde{f}_0}{\omega_1} \right\|_{L^{p'}(\Omega,\omega_1)} + \sum_{j=1}^{n} \left\| \frac{f_j - \tilde{f}_j}{\omega_2} \right\|_{L^{p'}(\Omega,\omega_2)} \right)^{p/p'}.$$

Corollary 2. *Let the assumptions of Theorem 1 be fulfilled, and let $\{f_{0m}\}$ and $\{f_{jm}\}$ $(j = 1, ..., n)$ be sequences of functions satisfying $\frac{f_{0m}}{\omega_1} \to \frac{f_0}{\omega_1}$ in $L^{p'}(\Omega, \omega_1)$ and $\frac{f_{jm}}{\omega_2} \to \frac{f_j}{\omega_2}$ in $L^{p'}(\Omega, \omega_2)$. If $u_m \in W_0^{1,p}(\Omega, \omega_1, \omega_2)$ is a solution of the problem*

$$(P_m) \begin{cases} Lu_m(x) = f_{0m}(x) - \sum\limits_{j=1}^{n} D_j f_{jm}(x) & \text{in } \Omega, \\ u_m(x) = 0 & \text{on } \partial\Omega, \end{cases}$$

then $u_m \to u$ in $W_0^{1,p}(\Omega, \omega_1, \omega_2)$, where u is a solution of (P).

2. Definitions and basic results. Let ω be a locally integrable nonnegative function in $\mathbb{R}^n$ and assume that $0 < \omega < \infty$ almost everywhere. We say that ω belongs to the Muckenhoupt class A_p, $1 < p < \infty$, or that ω is an A_p-weight, if there is a constant $C = C_{p,\omega}$ such that

$$\left(\frac{1}{|B|} \int_B \omega(x) dx \right) \left(\frac{1}{|B|} \int_B \omega^{1/(1-p)}(x) dx \right)^{p-1} \leq C$$

for all balls $B \subset \mathbb{R}^n$, where $|.|$ denotes the n-dimensional Lebesgue measure in $\mathbb{R}^n$. If $1 < q \leq p$, then $A_q \subset A_p$ (see [7], [8] or [12] for more information about A_p-weights).

As an example of A_p-weight, the function $\omega(x) = |x|^\alpha$, $x \in \mathbb{R}^n$, is in A_p if and only if $-n < \alpha < n(p-1)$ (see Corollary 4.4, Chapter IX in [12]).

The measure μ and the Lebesgue measure $|\cdot|$ are mutually absolutely continuous, i.e., they have the same zero sets ($\mu(E) = 0$ if and only if $|E| = 0$); so there is no need to specify the measure when using the ubiquitous expression almost everywhere and almost every, both abbreviated a.e..

Definition 2. Let ω be a weight, and let $\Omega \subset \mathbb{R}^n$ be open. For $1 < p < \infty$ we define $L^p(\Omega, \omega)$ as the set of measurable functions f on Ω such that

$$\|f\|_{L^p(\Omega,\omega)} = \left(\int_\Omega |f(x)|^p \omega(x) dx \right)^{1/p} < \infty.$$

If $\omega \in A_p$, $1 < p < \infty$, then $\omega^{-1/(p-1)}$ is locally integrable and $L^p(\Omega, \omega) \subset L^1_{\text{loc}}(\Omega)$ for every open set Ω (see Remark 1.2.4 in [13]). It thus makes sense to talk about weak derivatives of functions in $L^p(\Omega, \omega)$.

Definition 3. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, and let ω_1 and ω_2 be A_p-weights ($1 < p < \infty$). We define the *weighted Sobolev space* $W^{1,p}(\Omega, \omega_1, \omega_2)$ as the set of functions $u \in L^p(\Omega, \omega_1)$ with the weak derivatives $D_j u \in L^p(\Omega, \omega_2)$ ($j = 1, ..., n$). The norm of u in $W^{1,p}(\Omega, \omega_1, \omega_2)$ is defined by

$$\|u\|_{W^{1,p}(\Omega,\omega_1,\omega_2)} = \left(\int_\Omega |u(x)|^p \omega_1(x) dx + \int_\Omega |\nabla u(x)|^p \omega_2(x) dx \right)^{1/p}. \tag{1}$$

The space $W_0^{1,p}(\Omega, \omega_1, \omega_2)$ is the clousure of $C_0^\infty(\Omega)$ with respect norm (1). Equipped with this norm, $W_0^{1,p}(\Omega, \omega_1, \omega_2)$ is a reflexixe Banach space (see [10] for more information about the spaces $W^{1,p}(\Omega, \omega_1, \omega_2)$). The dual space of $W_0^{1,p}(\Omega, \Omega_1, \omega_2)$ is the space

$$[W_0^{1,p}(\Omega, \omega_2, \omega_2)]^* =$$

$$= \left\{ T = f_0 - \text{div}(F), F = (f_1, ..., f_n) \colon \frac{f_0}{\omega_1} \in L^{p'}(\Omega, \omega_1), \frac{f_j}{\omega_2} \in L^{p'}(\Omega, \omega_2), j = 1, ..., n \right\}.$$

In this paper we use the following theorem.

Theorem 2. *Let $\omega \in A_p$, $1 < p < \infty$, and let Ω be a bounded open set in $\mathbb{R}^n$. If $u_m \to u$ in $L^p(\Omega, \omega)$, then there exist a subsequence $\{u_{m_k}\}$ and a function $\Phi \in L^p(\Omega, \omega)$ such that*

(i) $u_{m_k}(x) \to u(x)$, $m_k \to \infty$, a.e. on Ω;

(ii) $|u_{m_k}(x)| \leq \Phi(x)$ a.e. on Ω.

The proof of this theorem follows the lines of Theorem 2.8.1 in [4].

Proof of Theorem 1. The basic idea is to reduce the problem (P) to an operator equation $Au = T$ and apply the theorem below. □

Theorem 3. *Let $A \colon X \to X^*$ be a monotone, coercive and hemicontinuous operator on the real, separable, reflexive Banach space X. Then the following assertions hold:*
(a) for each $T \in X^*$ the equation $Au = T$ has a solution $u \in X$;
(b) if the operator A is strictly monotone, then equation $Au = T$ is uniquely solvable in X (Theorem 26.A in [15]).

To prove Theorem 1, we define $B, B_1, B_2 \colon W_0^{1,p}(\Omega, \omega_1, \omega_2) \times W_0^{1,p}(\Omega, \omega_1, \omega_2) \to \mathbb{R}$ and $T \colon W_0^{1,p}(\Omega, \omega_1, \omega_2) \to \mathbb{R}$ by

$$B(u, \varphi) = B_1(u, \varphi) + B_2(u, \varphi),$$

$$B_2(u, \varphi) = \sum_{j=1}^{n} \int_{\Omega} \mathcal{A}_j(x, \nabla u) D_j \varphi \omega_2 dx = \int_{\Omega} \mathcal{A}(x, \nabla u).\nabla \varphi \omega_2 dx,$$

$$B_1(u, \varphi) = \int_{\Omega} b(x, u(x)) \varphi(x) \omega_1(x) dx + \int_{\Omega} g(x) \varphi(x) u(x) dx,$$

$$T(\varphi) = \int_{\Omega} f_0(x) \varphi(x) dx + \sum_{j=1}^{n} \int_{\Omega} f_j(x) D_j \varphi(x) dx.$$

Then $u \in W_0^{1,p}(\Omega, \omega_1, \omega_2)$ is a (weak) solution of problem (P) if and only if

$$B(u, \varphi) = T(\varphi) \text{ for all } \varphi \in W_0^{1,p}(\Omega, \omega_1, \omega_2).$$

Step 1. For $j = 1, ..., n$ we define the operator $F_j : W_0^{1,p}(\Omega, \omega_1, \omega_2) \to L^{p'}(\Omega, \omega_2)$ as $(F_j u)(x) = \mathcal{A}_j(x, \nabla u(x))$. We now show that the operator F_j is bounded and continuous.
(i) From (H3) we obtain

$$\|F_j u\|_{L^{p'}(\Omega, \omega_2)}^{p'} = \int_{\Omega} |F_j u(x)|^{p'} \omega_2 dx = \int_{\Omega} |\mathcal{A}_j(x, \nabla u)|^{p'} \omega_2 dx \leq \int_{\Omega} \left(K_2 + h_2 |\nabla u|^{p/p'} \right)^{p'} \omega_2 dx \leq$$

$$\leq C_p \int_{\Omega} (K_2^{p'} + h_2^{p'} |\nabla u|^p) \omega_2 dx = C_p \left[\int_{\Omega} K_2^{p'} \omega_2 dx + \int_{\Omega} h_2^{p'} |\nabla u|^p \omega_2 dx \right] \leq$$

$$\leq C_p \left[\|K_2\|_{L^{p'}(\Omega, \omega_2)}^{p'} + \|h_2\|_{L^\infty(\Omega)}^{p'} \|u\|_{W_0^{1,p}(\Omega, \omega_1, \omega_2)}^p \right], \tag{2}$$

where the constant C_p depends on p only. Therefore,

$$\|F_j u\|_{L^{p'}(\Omega, \omega_2)} \leq C_p \left(\|K_2\|_{L^{p'}(\Omega, \omega_2)} + \|h_2\|_{L^\infty(\Omega)} \|u\|_{W_0^{1,p}(\Omega, \omega_1, \omega_2)}^{p/p'} \right).$$

(ii) Let $u_m \to u$ in $W_0^{1,p}(\Omega, \omega_1, \omega_2)$ as $m \to \infty$. We need to show that $F_j u_m \to F_j u$ in $L^{p'}(\Omega, \omega_2)$. We will apply the Lebesgue Dominated Theorem. If $u_m \to u$ in $W_0^{1,p}(\Omega, \omega_1, \omega_2)$, then $u_m \to u$ in $L^p(\Omega, \omega_1)$ and $|\nabla u_m| \to |\nabla u|$ in $L^p(\Omega, \omega_2)$. By Theorem 2, there exist a subsequence $\{u_{m_k}\}$, functions $\Phi_1 \in L^p(\Omega, \omega_1)$ and $\Phi_2 \in L^p(\Omega, \omega_2)$ such that

$$u_{m_k}(x) \to u(x) \text{ a.e. in } \Omega, \quad |u_{m_k}(x)| \leq \Phi_1(x) \text{ a.e. in } \Omega,$$
$$D_j u_{m_k}(x) \to D_j u(x) \text{ a.e. in } \Omega \ (j = 1, ..., n), \quad |\nabla u_{m_k}(x)| \leq \Phi_2(x) \text{ a.e. in } \Omega.$$

Next, applying (H3) yeilds

$$|F_j u_{m_k} - F_j u|^{p'} \omega_2 = |\mathcal{A}_j(x, \nabla u_{m_k}) - \mathcal{A}_j(x, \nabla u)|^{p'} \omega_2 \leq$$

$$\leq C_p \left(|\mathcal{A}_j(x, \nabla u_{m_k})|^{p'} + |\mathcal{A}_j(x, \nabla u)|^{p'} \right) \omega_2 \leq$$

$$\leq C_p \left[\left(K_2 + h_2 |\nabla u_{m_k}|^{p/p'} \right)^{p'} + \left(K_2 + h_2 |\nabla u|^{p/p'} \right)^{p'} \right] \omega_2 \leq$$

$$\leq C_p^2 \left[K_2^{p'} + \|h_2\|_{L^\infty(\Omega)}^{p'} |\nabla u_{m_k}|^p + K_2^{p'} + \|h_2\|_{L^\infty(\Omega)}^{p'} |\nabla u|^p \right] \omega_2 \leq$$

$$\leq 2C_p^2 \left[K_2^{p'} + \|h_2\|_{L^\infty(\Omega)}^{p'} |\Phi_2|^p \right] \omega_2 \in L^1(\Omega).$$

By condition (H1), we have

$$F_j u_{m_k}(x) = \mathcal{A}_j(x, \nabla u_{m_k}(x)) \to \mathcal{A}_j(x, \nabla u(x)) = F_j u(x),$$

as $m_k \to +\infty$. Therefore, by the Lebesgue Dominated Convergence Theorem, we obtain $\|F_j u_{m_k} - F_j u\|_{L^{p'}(\Omega, \omega_2)} \to 0$, that is, $F_j u_{m_k} \to F_j u$ in $L^{p'}(\Omega, \omega_2)$. We conclude from the Convergence Principle in Banach spaces (see Proposition 10.13 in [14]) that

$$F_j u_m \to F_j u \quad \text{in} \quad L^{p'}(\Omega, \omega_2). \tag{3}$$

Step 2. We define the operator $G \colon W_0^{1,p}(\Omega, \omega_1, \omega_2) \to L^{p'}(\Omega, \omega_1)$ by $(Gu)(x) = b(x, u(x))$.

This operator is continuous and bounded. In fact,

(i) By $(H6)$, we obtain

$$\|Gu\|_{L^{p'}(\Omega,\omega_1)}^{p'} = \int_\Omega |Gu|^{p'} \omega_1 dx = \int_\Omega |b(x,u)|^{p'} \omega_1 dx \le \int_\Omega \left(K_1 + h_1 |u|^{p/p'} \right)^{p'} \omega_1 dx \le$$

$$\le C_p \int_\Omega (K_1^{p'} + h_1^{p'} |u|^p) \omega_1 dx = C_p \left[\int_\Omega K_1^{p'} \omega_1 dx + \int_\Omega h_1^{p'} |u|^p \omega_1 dx \right] \le$$

$$\le C_p \left(\|K_1\|_{L^{p'}(\Omega,\omega_1)}^{p'} + \|h_2\|_{L^\infty(\Omega)}^{p'} \|u\|_{W_0^{1,p}(\Omega,\omega_1,\omega_2)}^{p} \right).$$

(ii) Applying (H6) together with a similar argument as in Step 1(ii) yields

$$|Gu_{m_k} - Gu|^{p'} \omega_1 = |b(x, u_{m_k}) - b(x, u)|^{p'} \omega_1 \le C_p \left(|b(x, u_{m_k})|^{p'} + |b(x,u)|^{p'} \right) \omega_1 \le$$

$$\le C_p \left[\left(K_1 + h_1 |u_{m_k}|^{p/p'} \right)^{p'} + \left(K_1 + h_1 |u|^{p/p'} \right)^{p'} \right] \omega_1 \le$$

$$\le C_p^2 \left[K_1^{p'} + \|h_1\|_{L^\infty(\Omega)}^{p'} |u_{m_k}|^p + K_1^{p'} + \|h_1\|_{L^\infty(\Omega)}^{p'} |u|^p \right] \omega_1 \le$$

$$\le 2C_p^2 \left[K_1^{p'} + \|h_1\|_{L^\infty(\Omega)}^{p'} |\Phi_1|^p \right] \omega_1 \in L^1(\Omega).$$

Repeated application of the Convergence Principle in Banach Spaces given as: if $u_m \to u$ in $W_0^{1,p}(\Omega, \omega_1, \omega_2)$ we obtain

$$Gu_m \to Gu \quad \text{in} \quad L^{p'}(\Omega, \omega_1). \tag{4}$$

Step 3. By (H9) we have

$$|T(\varphi)| \le \int_\Omega |f_0||\varphi| dx + \sum_{j=1}^n \int_\Omega |f_j||D_j\varphi| dx = \int_\Omega \frac{|f_0|}{\omega_1} |\varphi| \omega_1 dx + \sum_{j=1}^n \int_\Omega \frac{|f_j|}{\omega_2} |D_j\varphi| \omega_2 dx \le$$

$$\le \|f_0/\omega_1\|_{L^{p'}(\Omega,\omega_1)} \|\varphi\|_{L^p(\Omega,\omega_1)} + \sum_{j=1}^n \|f_j/\omega_2\|_{L^{p'}(\Omega,\omega_2)} \|D_j\varphi\|_{L^p(\Omega,\omega_2)} \le$$

$$\le \left(\|f_0/\omega_1\|_{L^{p'}(\Omega,\omega_1)} + \sum_{j=1}^n \|f_j/\omega_2\|_{L^{p'}(\Omega,\omega_2)} \right) \|\varphi\|_{W_0^{1,p}(\Omega,\omega_1,\omega_2)}.$$

Moreover, using (H3), (H6), (H7) and the generalized Hölder inequality, we also have

$$|B(u,\varphi)| \le |B_1(u,\varphi)| + |B_2(u,\varphi)| \le$$

$$\le \sum_{j=1}^{n} \int_{\Omega} |\mathcal{A}_j(x,\nabla u)||D_j\varphi|\omega_2 dx + \int_{\Omega} |b(x,u)||\varphi|\omega_1 dx + \int_{\Omega} |g||u||\varphi| dx. \qquad (5)$$

The right hand side of (5) admits the estimation:

$$\int_{\Omega} |\mathcal{A}(x,\nabla u)||\nabla\varphi|\omega_2 dx \le \int_{\Omega} \left(K_2 + h_2|\nabla u|^{p/p'} \right)|\nabla\varphi|\omega_2 dx \le$$

$$\le \|K_2\|_{L^{p'}(\Omega,\omega_2)}\|\nabla\varphi\|_{L^p(\Omega,\omega_2)} + \|h_2\|_{L^\infty(\Omega)}\|\nabla u\|_{L^p(\Omega,\omega_2)}^{p/p'}\|\nabla\varphi\|_{L^p(\Omega,\omega_2)} \le$$

$$\le \left(\|K_2\|_{L^{p'}(\Omega,\omega_2)} + \|h_2\|_{L^\infty(\Omega)}\|u\|_{W_0^{1,p}(\Omega,\omega_1,\omega_2)}^{p/p'} \right)\|\varphi\|_{W_0^{1,p}(\Omega,\omega_1,\omega_2)},$$

$$\int_{\Omega} |b(x,u)||\varphi|\omega_1 dx \le \int_{\Omega} \left(K_1 + h_1|u|^{p/p'} \right)|\varphi|\,y\omega_1\,ydx \le$$

$$\le \int_{\Omega} K_1\,y|\varphi|\,y\omega_1\,ydx + \|h_1\|_{L^\infty(\Omega)} \int_{\Omega} |u|^{p/p\,y'}|\varphi|\,y\omega_1\,ydx \le$$

$$\le \left(\|K_1\|_{L^{p'}(\Omega,\omega_1)} + \|h_1\|_{L^\infty(\Omega)}\|u\|_{W_0^{1,p}(\Omega,\omega_1,\omega_2)}^{p/p'} \right)\|\varphi\|_{W_0^{1,p}(\Omega,\omega_1,\omega_2)}$$

and, by (H2), since $1/q + 1/p + 1/p' = 1$ (by (H7)),

$$\int_{\Omega} |g||u||\varphi| dx = \int_{\Omega} \frac{|g|}{\omega_1}|u||\varphi|\omega_1 dx \le \|g/\omega_1\|_{L^q(\Omega,\omega_1)}\|u\|_{L^p(\Omega,\omega_1)}\|\varphi\|_{L^p(\Omega,\omega_1)} \le$$

$$\le \|g/\omega_1\|_{L^q(\Omega,\omega_1)}\|u\|_{W_0^{1,p}(\Omega,\omega_1,\omega_2)}\|\varphi\|_{W_0^{1,p}(\Omega,\omega_1,\omega_2)}.$$

Finally, we obtain

$$|B(u,\varphi)| \le \left[\|K_1\|_{L^{p'}(\Omega,\omega_1)} + \|h_1\|_{L^\infty(\Omega)}\|u\|_{W_0^{1,p}(\Omega,\omega_1,\omega_2)}^{p/p'} + \|h_2\|_{L^\infty(\Omega)}\|u\|_{W_0^{1,p}(\Omega,\omega_1,\omega_2)}^{p/p'} + \right.$$

$$\left. + \|K_2\|_{L^{p'}(\Omega,\omega_2)} + \|g/\omega_1\|_{L^q(\Omega,\omega_1)}\|u\|_{W_0^{1,p}(\Omega,\omega_1,\omega_2)} \right]\|\varphi\|_{W_0^{1,p}(\Omega,\omega_1,\omega_2)}$$

for all $u, \varphi \in W_0^{1,p}(\Omega,\omega_1,\omega_2)$.

Since $B(u,.)$ is linear, for each $u \in W_0^{1,p}(\Omega,\omega_1,\omega_2)$, there exists a linear continuous functional on $W_0^{1,p}(\Omega,\omega_1,\omega_2)$ denoted by Au such that $\langle Au,\varphi \rangle = B(u,\varphi)$, for all $\varphi \in W_0^{1,p}(\Omega,\omega_1,\omega_2)$. Here $\langle f,x \rangle$ denotes the value of the linear functional f at the point x. Moreover

$$\|Au\|_* \le \|K_1\|_{L^{p'}(\Omega,\omega_1)} + \|K_2\|_{L^{p'}(\Omega,\omega_2)} + \|h_1\|_{L^\infty(\Omega)}\|u\|_{W_0^{1,p}(\Omega,\omega_1,\omega_2)}^{p/p'} +$$

$$+ \|h_2\|_{L^\infty(\Omega)}\|u\|_{W_0^{1,p}(\Omega,\omega_1,\omega_2)}^{p/p'} + \|g/\omega_1\|_{L^q(\Omega,\omega_1)}\|u\|_{W_0^{1,p}(\Omega,\omega_1,\omega_2)}.$$

Hence, we obtain the operator

$$A \colon W_0^{1,p}(\Omega,\omega_1,\omega_2) \to [W_0^{1,p}(\Omega,\omega_1,\omega_2)]^*, \quad u \mapsto Au.$$

Consequently, problem (P) is equivalent to the operator equation

$$u \in W_0^{1,p}(\Omega, \omega_1, \omega_2) : \quad Au = T.$$

Step 4. Using condition (H2), (H5) and (H7), we have

$$\langle Au_1 - Au_2, u_1 - u_2 \rangle = B(u_1, u_1 - u_2) - B(u_2, u_1 - u_2) =$$

$$= \int_\Omega \mathcal{A}(x, \nabla u_1) \cdot \nabla(u_1 - u_2)\omega_2 dx + \int_\Omega b(x, u_1)(u_1 - u_2)\omega_1 dx + \int_\Omega (u_1 - u_2)gu_1 dx -$$

$$- \int_\Omega \mathcal{A}(x, \nabla u_2) \cdot \nabla(u_1 - u_2)\omega_2 dx - \int_\Omega b(x, u_2)(u_1 - u_2)\omega_1 dx - \int_\Omega g(u_1 - u_2)u_2 dx$$

$$= \int_\Omega \left(\mathcal{A}(x, \nabla u_1) - \mathcal{A}(x, \nabla u_2) \right) \cdot \nabla(u_1 - u_2)\omega_2 dx +$$

$$+ \int_\Omega (b(x, u_1) - b(x, u_2))(u_1 - u_2)\omega_1 dx + \int_\Omega g(u_1 - u_2)^2 dx \geq$$

$$\geq \lambda \int_\Omega |\nabla(u_1 - u_2)|^p \omega_2 dx + \Lambda \int_\Omega |u_1 - u_2|^p \omega_1 dx \geq \gamma \|u_1 - u_2\|_{W_0^{1,p}(\Omega,\omega_1,\omega_2)}^p,$$

where $\gamma = \min\{\lambda, \Lambda\}$.

Therefore, the operator A is strongly monotone, and this implies that A is strictly monotone. Moreover, from Remark 1 and (H7) it follows that

$$\langle Au, u \rangle = B(u, u) = B_1(u, u) + B_2(u, u) = \int_\Omega \mathcal{A}(x, \nabla u) \cdot \nabla u \omega_2 dx + \int_\Omega b(x, u)u\omega_1 dx +$$

$$+ \int_\Omega gu^2 dx \geq \int_\Omega \lambda |\nabla u|^p \omega_2 dx + \int_\Omega \Lambda |u|^p \omega_1 dx \geq \gamma \|u\|_{W_0^{1,p}(\Omega,\omega_1,\omega_2)}^p,$$

where $\gamma = \min\{\lambda, \Lambda\} > 0$. Since $p > 2$, we have

$$\frac{\langle Au, u \rangle}{\|u\|_{W_0^{1,p}(\Omega,\omega_1,\omega_2)}} \to +\infty, \text{ as } \|u\|_{W_0^{1,p}(\Omega,\omega_1,\omega_2)} \to +\infty,$$

that is, A is coercive.

Step 5. We need to show that the operator A is continuous. Let $u_m \to u$ in $W_0^{1,p}(\Omega, \omega_1, \omega_2)$ as $m \to \infty$. Then

$$|B_2(u_m, \varphi) - B_2(u, \varphi)| \leq \sum_{j=1}^n \int_\Omega |\mathcal{A}_j(x, \nabla u_m) - \mathcal{A}_j(x, \nabla u)||D_j\varphi|\omega_2 dx =$$

$$= \sum_{j=1}^n \int_\Omega |F_j u_m - F_j u||D_j\varphi|\omega_2 dx \leq \sum_{j=1}^n \|F_j u_m - F_j u\|_{L^{p'}(\Omega,\omega_2)} \|D_j\varphi\|_{L^p(\Omega,\omega_2)} \leq$$

$$\leq \sum_{j=1}^n \|F_j u_m - F_j u\|_{L^{p'}(\Omega,\omega_2)} \|\varphi\|_{W_0^{1,p}(\Omega,\omega_1,\omega_2)},$$

$$|B_1(u_m, \varphi) - B_1(u, \varphi)| = \left| \int_\Omega (b(x, u_m) - b(x, u))\varphi\omega_1 dx + \int_\Omega g\varphi(u_m - u)dx \right| \leq$$

$$\leq \int_\Omega |Gu_m - Gu||\varphi|\omega_1 dx + \int_\Omega |g||\varphi||u_m - u|dx \leq \|Gu_m - Gu\|_{L^{p'}(\Omega,\omega_1)} \|\varphi\|_{L^p(\Omega,\omega_1)} +$$

$$+\|g/\omega_1\|_{L^q(\Omega,\omega_1)}\|\varphi\|_{L^p(\Omega,\omega_1)}\|u_m-u\|_{L^p(\Omega,\omega_1)} \leq \|Gu_m-Gu\|_{L^{p'}(\Omega,\omega_1)}\|\varphi\|_{W_0^{1,p}(\Omega,\omega_1,\omega_2)}+$$

$$+\|g/\omega_1\|_{L^q(\Omega,\omega_1)}\|\varphi\|_{W_0^{1,p}(\Omega,\omega_1,\omega_2)}\|u_m-u\|_{W_0^{1,p}(\Omega,\omega_1,\omega_2)},$$

for all $\varphi \in W_0^{1,p}(\Omega,\omega_1,\omega_2)$. Hence,

$$|B(u_m,\varphi)-B(u,\varphi)| \leq |B_1(u_m,\varphi)-B_1(u,\varphi)| + |B_2(u_m,\varphi)-B_2(u,\varphi)| \leq$$

$$\leq \left[\sum_{j=1}^{n}\|F_ju_m-F_ju\|_{L^{p'}(\Omega,\omega_2)} + \|Gu_m-Gu\|_{L^{p'}(\Omega,\omega_1)}+\right.$$

$$\left.+\|g/\omega_1\|_{L^q(\Omega,\omega_1)}\|u_m-u\|_{W_0^{1,p}(\Omega,\omega_1,\omega_2)}\right]\|\varphi\|_{W_0^{1,p}(\Omega,\omega_1,\omega_2)}.$$

Then we obtain

$$\|Au_m-Au\|_* \leq \sum_{j=1}^{n}\|F_ju_m-F_ju\|_{L^{p'}(\Omega,\omega_2)}+$$

$$+\|Gu_m-Gu\|_{L^{p'}(\Omega,\omega_1)} + \|g/\omega_1\|_{L^q(\Omega,\omega_1)}\|u_m-u\|_{W_0^{1,p}(\Omega,\omega_1,\omega_2)}.$$

Therefore, $\|Au_m-Au\|_* \to 0$ as $m \to +\infty$ by (3) and (4). Hence, A is continuous and this implies that A is hemicontinuous.

By Theorem 3, the operator equation $Au = T$ has a unique solution $u \in W_0^{1,p}(\Omega,\omega_1,\omega_2)$ and it is the unique solution for problem (P).

Step 6. According to Definition 1, we have

$$B(u,u) = B_1(u,u) + B_2(u,u) = T(u). \tag{6}$$

Therefore, using Remark 1, (H7), we obtain

$$B_1(u,u) + B_2(u,u) = \int_\Omega \mathcal{A}(x,\nabla u)\cdot\nabla u\,\omega_2 dx + \int_\Omega b(x,u)u\,\omega_1 dx + \int_\Omega gu^2 dx \geq$$

$$\geq \int_\Omega \lambda|\nabla u|^p\omega_2 dx + \int_\Omega \Lambda|u|^p\omega_1 dx \geq \gamma\|u\|^p_{W_0^{1,p}(\Omega,\omega_1,\omega_2)},$$

where $\gamma = \min\{\lambda,\Lambda\} > 0$, and by (H9) we have

$$T(u) = \int_\Omega f_0 u\,dx + \sum_{j=1}^{n}\int_\Omega f_j D_j u\,dx \leq \|f_0/\omega_1\|_{L^{p'}(\Omega,\omega_1)}\|u\|_{L^p(\Omega,\omega_1)}+$$

$$+\sum_{j=1}^{n}\|f_j/\omega_2\|_{L^{p'}(\Omega)}\|D_j u\|_{L^p(\Omega,\omega_2)} \leq \left(\|f_0/\omega_1\|_{L^{p'}(\Omega,\omega_1)} + \sum_{j=1}^{n}\|f_j/\omega_2\|_{L^{p'}(\Omega,\omega_2)}\right)\|u\|_{W_0^{1,p}(\Omega,\omega_1,\omega_2)}.$$

Therefore, in (6), we obtain

$$\gamma\|u\|^p_{W_0^{1,p}(\Omega,\omega_1,\omega_2)} \leq \left(\|f_0/\omega_1\|_{L^{p'}(\Omega,\omega_1)} + \sum_{j=1}^{n}\|f_j/\omega_2\|_{L^{p'}(\Omega,\omega_2)}\right)\|u\|_{W_0^{1,p}(\Omega,\omega_1,\omega_2)}.$$

Since $p > 2$, we have

$$\|u\|_{W_0^{1,p}(\Omega,\omega_1,\omega_2)} \leq \frac{1}{\gamma^{p'/p}} \left(\left\| \frac{f_0}{\omega_1} \right\|_{L^{p'}(\Omega,\omega_1)} + \sum_{j=1}^{n} \left\| \frac{f_j}{\omega_2} \right\|_{L^{p'}(\Omega,\omega_2)} \right)^{p'/p}.$$

Proof of Corollary 1. If $u_1, u_2 \in W_0^{1,p}(\Omega,\omega_1,\omega_2)$ are solutions of (P_1) and (P_2) respectively, then

$$\int_\Omega \mathcal{A}(x,\nabla u_1) \cdot \nabla\varphi\omega_2 dx + \int_\Omega b(x,u_1)\varphi\omega_1 dx + \int_\Omega gu_1\varphi dx = \int_\Omega f_0\varphi dx + \sum_{j=1}^{n} \int_\Omega f_j D_j\varphi dx,$$

$$\int_\Omega \mathcal{A}(x,\nabla u_2) \cdot \nabla\varphi\omega_2 dx + \int_\Omega b(x,u_2)\varphi\omega_1 dx + \int_\Omega gu_2\varphi dx = \int_\Omega \tilde{f}_0\varphi dx + \sum_{j=1}^{n} \int_\Omega \tilde{f}_j D_j\varphi dx,$$

for all $\varphi \in W_0^{1,p}(\Omega,\omega_1,\omega_2)$. In particular, we have

$$\int_\Omega \left(\mathcal{A}(x,\nabla u_1) - \mathcal{A}(x,\nabla u_2) \right) \cdot \nabla(u_1 - u_2)\omega_2 dx + \int_\Omega (b(x,u_1) - b(x,u_2))(u_1 - u_2)\omega_1 dx +$$

$$+ \int_\Omega g(u_1 - u_2)^2 dx = \int_\Omega (f_0 - \tilde{f}_0)(u_1 - u_2)dx + \sum_{j=1}^{n} \int_\Omega (f_j - \tilde{f}_j)D_j(u_1 - u_2)dx. \quad (7)$$

for $\varphi = u_1 - u_2$.

(i) By (H2), we have

$$\int_\Omega \left(\mathcal{A}(x,\nabla u_1) - \mathcal{A}(x,\nabla u_2) \right) \cdot \nabla(u_1 - u_2)\omega_2 dx \geq \lambda \int_\Omega |\nabla(u_1 - u_2)|^p \omega_2 dx,$$

and by (H5), we obtain $\int_\Omega (b(x,u_1) - b(x,u_2))(u_1 - u_2)\omega_1 \geq \Lambda \int_\Omega |u_1 - u_2|^p \omega_1 dx$.

(ii) By (H7) we have $\int_\Omega g(u_1 - u_2)^2 dx \geq 0$.

Hence, in (7), if $\gamma = \min\{\lambda, \Lambda\}$, we obtain

$$\gamma\|u_1 - u_2\|_{W_0^{1,p}(\Omega,\omega_1,\omega_2)}^p \leq$$

$$\leq \left(\|(f_0 - \tilde{f}_0)/\omega_1\|_{L^{p'}(\Omega,\omega_1)} + \sum_{j=1}^{n} \|(f_j - \tilde{f}_j)/\omega_2\|_{L^{p'}(\Omega,\omega_2)} \right) \|u_1 - u_2\|_{W_0^{1,p}(\Omega,\omega_1,\omega_2)}.$$

Therefore,

$$\|u_1 - u_2\|_{W_0^{1,p}(\Omega,\omega_1,\omega_2)} \leq \frac{1}{\gamma^{p/p'}} \left(\|(f_0 - \tilde{f}_0)/\omega_1\|_{L^{p'}(\Omega,\omega_1)} + \sum_{j=1}^{n} \|(f_j - \tilde{f}_j)/\omega_2\|_{L^{p'}(\Omega,\omega_2)} \right)^{p/p'}.$$

Proof of Corollary 2. By Corollary 1 if u_m and u are solutions of (P_m) and (P) respectively, then

$$\|u_m - u\|_{W_0^{1,p}(\Omega,\omega_1,\omega_2)} \leq \frac{1}{\gamma^{p/p'}} \left(\|(f_{0m} - f_0)/\omega_1\|_{L^{p'}(\Omega,\omega_1)} + \sum_{j=1}^{n} \|(f_{jm} - f_j)/\omega_2\|_{L^{p'}(\Omega,\omega_2)} \right)^{p/p'}.$$

Since $\frac{f_{0m}}{\omega_1} \to \frac{f_0}{\omega_1}$ in $L^{p'}(\Omega,\omega_1)$, $\frac{f_{jm}}{\omega_2} \to \frac{f_j}{\omega_2}$ in $L^{p'}(\Omega,\omega_2)$ $(j = 1, ..., n)$, we obtain the convergence $u_m \to u$ in $W_0^{1,p}(\Omega,\omega_1,\omega_2)$.

Example 1. Let $\Omega = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$. We consider the weights functions $\omega_1(x,y) = (x^2 + y^2)^{-1/2}$ and $\omega_2(x,y) = (x^2 + y^2)^{-2/3}$ (ω_1 and ω_2 are A_3-weights, $p = 3$), and the functions

$$\mathcal{A} \colon \Omega \times \mathbb{R}^2 \to \mathbb{R}^2, \mathcal{A}((x,y),\xi) = h_2(x,y)|\xi|\xi,$$

$$b \colon \Omega \times \mathbb{R} \to \mathbb{R}, b((x,y),\eta) = \eta|\eta|\big(\cos^2(xy) + 1\big), \quad g \colon \Omega \to \mathbb{R}, g(x,y) = \frac{\sin^2(xy)}{(x^2 + y^2)^{1/12}},$$

with $\lambda = 1/2$, $\Lambda = 1/4$, $h_1(x,y) = \cos^2(xy) + 1$, $h_2(x,y) = 2e^{x^2+y^2}$ and $q = 3$. Let us consider the partial differential operator

$$Lu(x,y) = -\mathrm{div}\Big[\mathcal{A}((x,y),\nabla u)\omega_2(x,y)\Big] + b((x,y),u)\omega_1(x,y) + g(x,y)u.$$

By Theorem 1, the problem

$$(P) \begin{cases} Lu(x,y) = \dfrac{\cos(xy)}{(x^2 + y^2)} - \dfrac{\partial}{\partial x}\left(\dfrac{\sin(xy)}{(x^2 + y^2)^2}\right) - \dfrac{\partial}{\partial y}\left(\dfrac{\cos(xy)}{(x^2 + y^2)^2}\right) & \text{in } \Omega, \\ u(x,y) = 0 & \text{on } \partial\Omega, \end{cases}$$

has a unique solution $u \in W_0^{1,3}(\Omega, \omega_1, \omega_2)$.

Remark. To verify the conditions (H2) and (H5) we use the inequality

$$(|\xi|^{p-2}\xi - |\tilde\xi|^{p-2}\tilde\xi) \cdot (\xi - \tilde\xi) \geq \frac{1}{2^{p-1}}(|\xi| + |\tilde\xi|)^{p-2}|\xi - \tilde\xi|^2,$$

for every $\xi, \tilde\xi \in \mathbb{R}^n$ and $1 < p < \infty$ (see Proposition 17.3 in [1]). For $p = 3$, we have

$$(|\xi|\xi - |\tilde\xi|\tilde\xi) \cdot (\xi - \tilde\xi) \geq \frac{1}{4}(|\xi| + |\tilde\xi|)|\xi - \tilde\xi|^2,$$

and since $|\xi| + |\tilde\xi| \geq |\xi - \tilde\xi|$, we obtain $(|\xi|\xi - |\tilde\xi|\tilde\xi) \cdot (\xi - \tilde\xi) \geq \frac{1}{4}|\xi - \tilde\xi|^3$. Hence, using that $h_2(x,y) = 2e^{x^2+y^2} \geq 2$ in Ω, we have $(\mathcal{A}((x,y),\xi) - \mathcal{A}((x,y),\tilde\xi)) \cdot (\xi - \tilde\xi) \geq \frac{1}{2}|\xi - \tilde\xi|^3$. Analogously, we have

$$\left(b((x,y),\eta) - b((x,y),\tilde\eta)\right)(\eta - \tilde\eta) = (\cos^2(xy) + 1)(|\eta|\eta - |\tilde\eta|\tilde\eta)(\eta - \tilde\eta) \geq \frac{1}{4}|\eta - \tilde\eta|^3.$$

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