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CONTINUAL APPROXIMATE SOLUTION OF THE BOLTZMANN EQUATION WITH ARBITRARY DENSITY

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The new explicit approximate solution of the non-linear Boltzmann equation was constructed. It has the form of the continual distribution in the case of global Maxwellians with arbitrary density. We obtained some sufficient conditions which minimized the uniform-integral remainder and pure integral remainder between the left- and the right-hand sides of this equation.

1. Introduction. The kinetic Boltzmann equation is one of the central equations in classical mechanics of many-particle systems. For the model of hard spheres it has the form ([1]–[3])

$$D(f) = Q(f, f), \tag{1}$$

$$D(f) = \frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x}, \tag{2}$$

$$Q(f, f) = \frac{d^2}{2}\int_{\mathbb{R}^3} dv_1 \int_{\Sigma} d\alpha|(v - v_1, \alpha)|[f(t, v_1', x)f(t, v', x) - f(t, v_1, x)f(t, v, x)], \tag{3}$$

where $f(t, v, x)$ is the distribution we want to find, $\partial f/\partial x$ is its spatial gradient, $t \in \mathbb{R}^1$ is time, $x = (x^1, x^2, x^3) \in \mathbb{R}^3$ is the position, $v = (v^1, v^2, v^3) \in \mathbb{R}^3$ is the molecule velocity, $d > 0$ is its diameter, $\alpha \in \Sigma$, where Σ is the unit sphere in $\mathbb{R}^3$, v and v_1 are the molecule velocities before collision, v' and v_1' are those after collision, which are defined by formulae

$$v' = v - \alpha(v - v_1, \alpha), \quad v_1' = v + \alpha(v - v_1, \alpha), \tag{4}$$

Now we do not know any exact solution of the equation (1)–(3) except global and local Maxwellians ([1]–[3], some other exact solutions were obtained only in the case of Maxwellian molecules and for some of their generalizations [4]–[7]).

That is why there was a question of the search of explicit approximate solutions of this integro-differential equation and satisfying it with arbitrary accuracy.

In subsequent papers, bimodal distributions including both global and local Maxwellians of various particular forms describing screw ([8], [9]), tornado-like ([10], [11]), and other equilibrium gas states were studied. Some multimodal distributions were also studied in [12].

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In the paper [13] a new approach to the search for explicit approximate solutions of the Boltzmann equation was proposed, namely the continual form of the distribution function. Such distributions generalize bimodal distributions obtained earlier. It was assumed that the mass velocity of the global Maxwellian does not take fixed discrete values but becomes an arbitrary parameter taking any values in $\mathbb{R}^3$. In this paper we propose that the density also takes arbitrary values.

2. The problem setting. As, for instance, in [13], we will also consider a distribution function of the form

$$f = \int_{\mathbb{R}^3} du \int_0^{+\infty} d\rho\, \varphi(t,x,u,\rho) M(v,u,\rho), \tag{5}$$

which contains the global Maxwellian

$$M(v,u,\rho) = \rho \left(\frac{\beta}{\pi}\right)^{\frac{3}{2}} e^{-\beta(v-u)^2}, \tag{6}$$

where ρ is the density, u is the mass velocity and an inverse temperature β is arbitrary non-negative constant.

It is assumed that the coefficient function $\varphi(t,x,u,\rho)$ is non-negative and belong to $C^1(\mathbb{R}^4)$ with t,x and to $C(\mathbb{R}^3 \times \mathbb{R}^1_+)$ with u,ρ. It is required to find $\varphi(t,x,u,\rho)$ and the behavior of all parameters so that the uniform-integral (mixed) or pure integral remainder ([14]), i.e. the functionals of the form

$$\Delta = \sup_{(t,x)\in\mathbb{R}^4} \int_{\mathbb{R}^3} |D(f) - Q(f,f)| dv, \tag{7}$$

$$\Delta_1 = \int_{\mathbb{R}^1} dt \int_{\mathbb{R}^3} dx \int_{\mathbb{R}^3} |D(f) - Q(f,f)| dv, \tag{8}$$

become vanishingly small.

Let us introduce such the the new denotation as

$$\psi(t,x,u,\rho) = \varphi(t,x,u,\rho) \cdot \rho, \tag{9}$$

then the distribution function will have the next form:

$$f = \int_{\mathbb{R}^3} du \int_0^{+\infty} d\rho\, \psi(t,x,u,\rho) \widetilde{M}(v,u,\rho), \tag{10}$$

where

$$\widetilde{M}(v,u,\rho) = \left(\frac{\beta}{\pi}\right)^{\frac{3}{2}} e^{-\beta(v-u)^2}. \tag{11}$$

In Section 3 the solution of this problem is constructed, which is the direct generalization of continual approximate solutions of [13]. In this construction we use the "mixed" (uniform-integral) remainder.

In Section 4 the new continual solution is found, where the error is "pure integral".

3. Minimization of the uniform-integral remainder Δ.

Theorem 1. *Let conditions (10) and (11) be valid. Let the following functions:* ψ, $|\frac{\partial\psi}{\partial t}|$, $|\frac{\partial\psi}{\partial x}|$ *be bounded with respect to* t, x *on* $\mathbb{R}^7 \times \mathbb{R}^1_+$, *and the quantities*

$$\psi,\ |u|\psi,\ \frac{\partial\psi}{\partial t},\ \frac{\partial\psi}{\partial x},\ u\frac{\partial\psi}{\partial x} \in L_1(\mathbb{R}^3 \times \mathbb{R}^1_+) \tag{12}$$

in the variable u *and* ρ *uniformly in* t, x *on* $\mathbb{R}^4$.

Then the remainder Δ *in (7) is correctly defined (i.e. the finite integral and the finite supremum in (7) exist), and there is a value* Δ' *that*

$$\Delta \le \Delta', \tag{13}$$

with

$$\lim_{\beta\to+\infty} \Delta' = \sup_{(t,x)\in\mathbb{R}^4} \left[\int_{\mathbb{R}^3} du \int_{\mathbb{R}^1_+} d\rho \left| \frac{\partial\psi}{\partial t} + u\frac{\partial\psi}{\partial x} \right| + \right.$$

$$\left. +2\pi d^2 \int_{\mathbb{R}^6} du_1 du_2 \int_{\mathbb{R}^2_+} d\rho_1 d\rho_2 \psi(t,x,u_1,\rho_1)\psi(t,x,u_2,\rho_2)|u_1 - u_2| \right]. \tag{14}$$

To prove Theorem 1 we need the following Lemma ([8]), which provides a sufficient condition for the continuity of the special supremum of multivariate function taken with respect to some of the variables.

Lemma 1. *Suppose the following conditions:*
1) $\forall z \in Z$, *the function* $g(y,z)$ *is bounded on* Y;
2) $g(y,z)$ *is continuous by* z *uniformly with respect to* y, *well then*

$$\forall z_0 \in Z, \forall \varepsilon > 0, \exists \delta > 0, \forall y \in Y, \forall z \in Z,\ |z - z_0| < \delta \Rightarrow |g(y,z) - g(y,z_0)| < \varepsilon$$

are valid for the function $g(y,z) : Y \times Z \to \mathbb{R}^1; Y \in \mathbb{R}^p; Z \in \mathbb{R}^q$.

Then the function $l(z) = \sup_{y\in Y} |g(y,z)|$ *is continuous on the variable* $z \in Z$.

Proof of Theorem 1. Substituting expression (10) in Botzmann equation (1)–(3), we obtain

$$D(f) = \int_{\mathbb{R}^3} du \int_0^{+\infty} d\rho \left(\frac{\partial\psi}{\partial t} + v\frac{\partial\psi}{\partial x} \right) \widetilde{M}(v,u,\rho). \tag{15}$$

$$Q(f,f) = \frac{d^2}{2} \int_{\mathbb{R}^3} dv_1 \int_{\Sigma} d\alpha |(v - v_1, \alpha)| \times$$

$$\times \left[\int_{\mathbb{R}^3} du_1 \int_0^{+\infty} d\rho_1 \psi(t,x,u_1,\rho_1) \widetilde{M}(v_1', u_1, \rho_1) \int_{\mathbb{R}^3} du_2 \int_0^{+\infty} d\rho_2 \psi(t,x,u_2,\rho_2) \widetilde{M}(v', u_2, \rho_2) - \right.$$

$$\left. - \int_{\mathbb{R}^3} du_1 \int_0^{+\infty} d\rho_1 \psi(t,x,u_1) \widetilde{M}(v_1,u_1,\rho_1) \int_{\mathbb{R}^3} du_2 \int_0^{+\infty} d\rho_2 \psi(t,x,u_2,\rho_2) \widetilde{M}(v,u_2,\rho_2) \right]. \tag{16}$$

According to (15) and (16) we rewrite expression (7) as

$$\Delta = \sup_{(t,x)\in\mathbb{R}^4} \int_{\mathbb{R}^3} \left| \int_{\mathbb{R}^3} du \int_0^{+\infty} d\rho \left(\frac{\partial\psi}{\partial t} + v\frac{\partial\psi}{\partial x} \right) \widetilde{M}(v,u,\rho) - \frac{d^2}{2} \int_{\mathbb{R}^3} dv_1 \int_{\Sigma} d\alpha |(v - v_1, \alpha)| \times \right.$$

$$\times \int_{\mathbb{R}^6} du_1 du_2 \int_{\mathbb{R}^2_+} d\rho_1 d\rho_2 \psi(t,x,u_1,\rho_1)\psi(t,x,u_2,\rho_2)\big[\widetilde{M}(v_1',u_1,\rho_1)\widetilde{M}(v',u_2,\rho_2)-$$

$$-\widetilde{M}(v_1,u_1,\rho_1)\widetilde{M}(v,u_2,\rho_2)\big]\Bigg| dv \le \sup_{(t,x)\in\mathbb{R}^4} \int_{\mathbb{R}^3}\Bigg[\int_{\mathbb{R}^3} du \int_0^{+\infty} d\rho \left|\frac{\partial\psi}{\partial t} + v\frac{\partial\psi}{\partial x}\right| \widetilde{M}(v,u,\rho)+$$

$$+\frac{d^2}{2}\int_{\mathbb{R}^3} dv_1 \int_\Sigma d\alpha |(v-v_1,\alpha)| \int_{\mathbb{R}^6} du_1 du_2 \int_{\mathbb{R}^2_+} d\rho_1 d\rho_2 \psi(t,x,u_1,\rho_1)\psi(t,x,u_2,\rho_2)\times$$

$$\times\left|\widetilde{M}(v_1',u_1,\rho_1)\widetilde{M}(v',u_2,\rho_2) - \widetilde{M}(v_1,u_1,\rho_1)\widetilde{M}(v,u_2,\rho_2)\right|\Bigg] dv \le$$

$$\le \sup_{(t,x)\in\mathbb{R}^4} \int_{\mathbb{R}^3}\Bigg[\int_{\mathbb{R}^3} du \int_0^{+\infty} d\rho \left|\frac{\partial\psi}{\partial t} + v\frac{\partial\psi}{\partial x}\right| \widetilde{M}(v,u,\rho) + \frac{d^2}{2}\int_{\mathbb{R}^3} dv_1 \int_\Sigma d\alpha|(v-v_1,\alpha)|\times$$

$$\times \int_{\mathbb{R}^6} du_1 du_2 \int_{\mathbb{R}^2_+} d\rho_1 d\rho_2 \psi(t,x,u_1,\rho_1)\psi(t,x,u_2,\rho_2)\Big[\widetilde{M}(v_1',u_1,\rho_1)\widetilde{M}(v',u_2,\rho_2)+$$

$$+\widetilde{M}(v_1,u_1,\rho_1)\widetilde{M}(v,u_2,\rho_2)\Big]\Bigg] dv. \tag{17}$$

Therefore,

$$\Delta' = \sup_{(t,x)\in\mathbb{R}^4}\Bigg[\int_{\mathbb{R}^6} dv du \int_0^{+\infty} d\rho \left|\frac{\partial\psi}{\partial t} + v\frac{\partial\psi}{\partial x}\right| \widetilde{M}(v,u,\rho)+$$

$$+\frac{d^2}{2}\int_{\mathbb{R}^3} dv \int_{\mathbb{R}^3} dv_1 \int_\Sigma d\alpha|(v-v_1,\alpha)| \int_{\mathbb{R}^6} du_1 du_2 \int_{\mathbb{R}^2_+} d\rho_1 d\rho_2 \psi(t,x,u_1,\rho_1)\psi(t,x,u_2,\rho_2)\times$$

$$\times\Big[\widetilde{M}(v_1',u_1,\rho_1)\widetilde{M}(v',u_2,\rho_2) + \widetilde{M}(v_1,u_1,\rho_1)\widetilde{M}(v,u_2,\rho_2)\Big]\Bigg] =$$

$$= \sup_{(t,x)\in\mathbb{R}^4}\Bigg[\int_{\mathbb{R}^3} du \int_{\mathbb{R}^1_+} d\rho \int_{\mathbb{R}^3} dv \left|\frac{\partial\psi}{\partial t} + v\frac{\partial\psi}{\partial x}\right| \widetilde{M}(v,u,\rho) + \frac{d^2}{2}\int_{\mathbb{R}^6} du_1 du_2 \int_{\mathbb{R}^2_+} d\rho_1 d\rho_2\times$$

$$\times\psi(t,x,u_1,\rho_1)\psi(t,x,u_2,\rho_2)\int_{\mathbb{R}^6} dv dv_1 \int_\Sigma d\alpha|(v-v_1,\alpha)|\times$$

$$\times\Big[\widetilde{M}(v_1',u_1,\rho_1)\widetilde{M}(v',u_2,\rho_2) + \widetilde{M}(v_1,u_1,\rho_1)\widetilde{M}(v,u_2,\rho_2)\Big]\Bigg].$$

$$\Delta' = \sup_{(t,x)\in\mathbb{R}^4}\Bigg[\int_{\mathbb{R}^3} du \int_{\mathbb{R}^1_+} d\rho \int_{\mathbb{R}^3} dv \left|\frac{\partial\psi}{\partial t} + v\frac{\partial\psi}{\partial x}\right| \widetilde{M}(v,u,\rho)+$$

$$+\int_{\mathbb{R}^6} du_1 du_2 \int_{\mathbb{R}^2_+} d\rho_1 d\rho_2 \psi(t,x,u_1,\rho_1)\psi(t,x,u_2,\rho_2)\int_{\mathbb{R}^3}\Big[G(\widetilde{M_1},\widetilde{M_2}) + \widetilde{M_2}L(\widetilde{M_1})\Big] dv\Bigg], \tag{18}$$

where, as usual, we introduce the notion of "gain"(G) and "loss"(L) parts of the collision integral [1,2]:

$$G(f,g) = \frac{d^2}{2}\int_{\mathbb{R}^3} dv_1 \int_\Sigma d\alpha|(v-v_1,\alpha)| f(t,v_1',x)g(t,v_1,x), \tag{19}$$

$$L(g) = \frac{d^2}{2} \int_{\mathbb{R}^3} dv_1 \int_{\Sigma} d\alpha |(v - v_1, \alpha)| g(t, v_1, x), \tag{20}$$

and we introduce the notation $\widetilde{M_i} = \widetilde{M}(v, u_i, \rho_i),\ i = 1, 2$. In (18), we also interchange the integration order and the validity of this procedure can be justified as follows.

In the first summand

1) the integrand in the first term is continuous;

2) $\int_{\mathbb{R}^1_+} \int_{\mathbb{R}^3} \left| \frac{\partial \varphi}{\partial t} + v \frac{\partial \varphi}{\partial x} \right| M(v, u, \rho) du$ converges uniformly in $\mathbb{R}^3 \times \mathbb{R}^1_+$ (by the Weierstrass theorem).

$$\left| \left| \frac{\partial \psi}{\partial t} + v \frac{\partial \psi}{\partial x} \right| \cdot \left(\frac{\beta}{\pi} \right)^{3/2} e^{-\beta(v-u)^2} \right| \leq \left(\frac{\beta}{\pi} \right)^{3/2} e^{-\beta(v-u)^2} \left(\left| \frac{\partial \psi}{\partial t} \right| + |v| \left| \frac{\partial \psi}{\partial x} \right| \right)$$

and is integrable by virtue of condition (12).

The integrand in the second summand is continuous by the theorem conditions, and the inner integral converges uniformly in u_1, u_2, ρ_1 and ρ_2 by the Weierstrass theorem because there is an integrating majorant. Hence, we can change the integration order here.

Because, as is known ([1])

$$\int_{\mathbb{R}^3} Q(f, g) dv = 0,$$

then

$$\int_{\mathbb{R}^3} G(\widetilde{M_1}, \widetilde{M_2}) dv = \int_{\mathbb{R}^3} \widetilde{M_1} L(\widetilde{M_2}) dv. \tag{21}$$

According to (21) the expression (18) can be simplified, and using the supremum property, we can write

$$\Delta' = \sup_{(t,x)\in\mathbb{R}^4} \left[\int_{\mathbb{R}^3} du \int_{\mathbb{R}^1_+} d\rho \int_{\mathbb{R}^3} dv \left| \frac{\partial \psi}{\partial t} + v \frac{\partial \psi}{\partial x} \right| \left(\frac{\beta}{\pi} \right)^{\frac{3}{2}} e^{-\beta(v-u)^2} \right] +$$

$$+ \sup_{(t,x)\in\mathbb{R}^4} \left[\int_{\mathbb{R}^6} du_1 du_2 \int_{\mathbb{R}^2_+} d\rho_1 d\rho_2\, \psi(t, x, u_1, \rho_1) \psi(t, x, u_2, \rho_2) \int_{\mathbb{R}^3} \left[\widetilde{M_1} L(\widetilde{M_2}) + \widetilde{M_2} L(\widetilde{M_1}) \right] dv \right].$$

If we introduce a change the variables

$$\sqrt{\beta}(v - u) = w; \quad v = \frac{w}{\sqrt{\beta}} + u,$$

we have

$$\Delta' = \sup_{(t,x)\in\mathbb{R}^4} \left[\pi^{-3/2} \int_{\mathbb{R}^3} du \int_{\mathbb{R}^1_+} d\rho \int_{\mathbb{R}^3} dw \left| \frac{\partial \psi}{\partial t} + \left(\frac{w}{\sqrt{\beta}} + u \right) \frac{\partial \psi}{\partial x} \right| e^{-w^2} \right] +$$

$$+ \sup_{(t,x)\in\mathbb{R}^4} \left[\int_{\mathbb{R}^6} du_1 du_2 \int_{\mathbb{R}^2_+} d\rho_1 d\rho_2\, \psi(t, x, u_1, \rho_1) \psi(t, x, u_2, \rho_2) \times \right.$$

$$\left. \times \int_{\mathbb{R}^3} dw \left[\widetilde{M_1} L(\widetilde{M_2}) + \widetilde{M_2} L(\widetilde{M_1}) \right] \right]. \tag{22}$$

We apply Lemma 1 to each supremum in equation (22) (similarly as in the proof [13]). Because the lemma conditions are satisfied for each of these supremums, the whole quantity

Δ' is continuous in γ on $\mathbb{R}^1_+$. So, we can pass to the limit with $\beta \to +\infty$, that is equivalent to the tending of γ to zero. To pass to the limit conveniently, we consider the expression for Δ' before the change of variables, namely

$$\Delta' = \sup_{(t,x)\in\mathbb{R}^4} \left[\int_{\mathbb{R}^3} du \int_{\mathbb{R}^1_+} d\rho \int_{\mathbb{R}^3} dv \left| \frac{\partial\psi}{\partial t} + v\frac{\partial\psi}{\partial x} \right| \widetilde{M}(v,u,\rho) \right] +$$
$$+ \sup_{(t,x)\in\mathbb{R}^4} \left[\int_{\mathbb{R}^6} du_1 du_2 \int_{\mathbb{R}^2_+} d\rho_1 d\rho_2 \, \psi(t,x,u_1,\rho_1)\psi(t,x,u_2,\rho_2) \int_{\mathbb{R}^3} \left[\widetilde{M}_1 L(\widetilde{M}_2) + \widetilde{M}_2 L(\widetilde{M}_1)\right] dv \right].$$

We use the fact that in the sense of distributions, is obvious that

$$\widetilde{M}(v,u,\rho) \xrightarrow[\beta\to+\infty]{} \delta(v-u). \tag{23}$$

Thus,

$$\int_{\mathbb{R}^3} \left| \frac{\partial\psi}{\partial t} + v\frac{\partial\psi}{\partial x} \right| \widetilde{M}(v,u,\rho) dv \xrightarrow[\beta\to+\infty]{} \int_{\mathbb{R}^3} \left| \frac{\partial\psi}{\partial t} + v\frac{\partial\psi}{\partial x} \right| \delta(v-u) dv \xrightarrow[\beta\to+\infty]{} \left| \frac{\partial\psi}{\partial t} + u\frac{\partial\psi}{\partial x} \right|,$$
$$\int_{\mathbb{R}^3} \widetilde{M}_1 L(\widetilde{M}_2) dv \xrightarrow[\beta\to+\infty]{} \int_{\mathbb{R}^3} \widetilde{M}(v,u_1,\rho_1) d^2\pi |v-u_2| dv \xrightarrow[\beta\to+\infty]{} d^2\pi|u_1-u_2|, \tag{24}$$
$$\int_{\mathbb{R}^3} \widetilde{M}_2 L(\widetilde{M}_1) dv \xrightarrow[\beta\to+\infty]{} \int_{\mathbb{R}^3} \widetilde{M}(v,u_2,\rho_2) d^2\pi |v-u_1| dv \xrightarrow[\beta\to+\infty]{} d^2\pi|u_1-u_2|,$$

and by virtue of these equations, we obtain (13). □

Now, with the help of the obtained expression for the limit with $\beta \to +\infty$, we can find some sufficient conditions for tending of the remainder Δ to zero, which it is convenient to formulate in the form of a corollary of Theorem 1

Corollary 1. *Let all the assumptions of the Theorem 1 be valid. Then the statement*

$$\Delta \to 0 \tag{25}$$

holds true, if the function ψ defined in (9) has the form

$$\psi(t,x,u,\rho) = C(x-ut) \left(\frac{P}{\pi}\right)^{3/2} e^{-P(u-u_0)^2} h(\rho), \tag{26}$$

where C is any smooth, positive and bounded function together with all its derivatives, $u_0 \in \mathbb{R}^3$ is an arbitrary fixed vector, $h(\rho)$ is a continuous, non-negative function from $L_1(\mathbb{R}^1_+)$ with root if $\rho = 0$ no less than the first degree, and $P \to +\infty$.

Proof. We use limit expression (14) and substitute expression (26) in it. The integrand of the first term then vanishes, $C'(x-ut)(-u) + uC'(x-ut) = 0$.

We consider the integral in the second summand (let $\exists M > 0$: $|C| \le M$):

$$\int_{\mathbb{R}^6} du_1 du_2 \int_{\mathbb{R}^2_+} C(x-u_1 t) \left(\frac{P}{\pi}\right)^{3/2} e^{-P(u_1-u_0)^2} h(\rho_1) C(x-u_2 t) \times$$
$$\times \left(\frac{P}{\pi}\right)^{3/2} e^{-P(u_2-u_0)^2} h(\rho_2) |u_1-u_2| d\rho_1 d\rho_2 \le$$

$$\leq M^2\left(\frac{P}{\pi}\right)^3\int_{\mathbb{R}^6}h(\rho_1)h(\rho_2)\int_{\mathbb{R}^6}e^{-P(u_1-u_0)^2}e^{-P(u_2-u_0)^2}|u_1-u_2|du_1du_2d\rho_1d\rho_2=$$

$$=\left[\begin{matrix}\sqrt{P}(u_1-u_0)=w_1; & u_1=\frac{w_1}{\sqrt{P}}+u_0\\ \sqrt{P}(u_2-u_0)=w_2; u_2=\frac{w_2}{\sqrt{P}}+u_0 & \end{matrix}\right]=$$

$$=M^2\left(\frac{P}{\pi}\right)^3\frac{1}{P^3}\int_{\mathbb{R}^6}h(\rho_1)h(\rho_2)\int_{\mathbb{R}^3}e^{-w_2{}^2}\times$$

$$\times\int_{\mathbb{R}^3}e^{-w_1{}^2}\left|\frac{w_1}{\sqrt{P}}+u_0-\frac{w_2}{\sqrt{P}}-u_0\right|dw_1dw_2d\rho_1d\rho_2\leq$$

$$\leq\frac{M^2}{\pi^3}\int_{\mathbb{R}^6}h(\rho_1)h(\rho_2)\int_{\mathbb{R}^3}e^{-w_2{}^2}\int_{\mathbb{R}^3}e^{-w_1{}^2}\frac{|w_1|}{\sqrt{P}}dw_1dw_2d\rho_1d\rho_2+$$

$$+\frac{M^2}{\pi^3}\int_{\mathbb{R}^6}h(\rho_1)h(\rho_2)\int_{\mathbb{R}^3}e^{-w_1{}^2}\int_{\mathbb{R}^3}e^{-w_2{}^2}\frac{|w_2|}{\sqrt{P}}dw_2dw_1d\rho_1d\rho_2=$$

$$=\frac{M^2}{\pi^3}\left[\pi^{3/2}\frac{2\pi}{\sqrt{P}}+\pi^{3/2}\frac{2\pi}{\sqrt{P}}\right]\int_{\mathbb{R}^3}h(\rho_2)\int_{\mathbb{R}^3}h(\rho_1)d\rho_1d\rho_2=$$

$$=\frac{4M^2}{\sqrt{P\pi}}\int_{\mathbb{R}^3}h(\rho_2)\int_{\mathbb{R}^3}h(\rho_1)d\rho_1d\rho_2\xrightarrow[P\to+\infty]{}0.$$

Here, we use the Euler-Poisson integral and the values of integrals calculated above. Thus, thanks to the corollary conditions, the integral in the second summand converges and it tends to zero as $P\to+\infty$. So, equality to zero of the expression (14) is proved. □

Remark 1. Relation (25) also holds at a fixed P in (26) under the additional condition $d\to 0$ (the near-Knudsen gas).

Remark 2. In expression (26), we can obviously take $C(x-ut)$ instead of the first factor $C\big([u\times x]\big)$, and take other δ-functions instead of the second factor.

Remark 3. The physical meaning of the obtained results has an abstract mathematical sense, and estimate (13) and the limit in expression (14) ensure the further arbitrary smallness of the uniform-integral remainder between the parts of the Botzmann equation Δ for the given coefficient functions and at a sufficiently small absolute temperature, which only means assuming that the thermal constituent of the molecule velocities is small when an arbitrary value of the mass velocity and density of a flow is preserved.

4. Minimization of the pure integral remainder Δ_1.

Theorem 2. *Let conditions (10) and (11) be valid. Let the following functions belong to* $L_1(\mathbb{R}^7\times\mathbb{R}^1_+)$

$$\psi,\ |u|\psi,\ \left|\frac{\partial\psi}{\partial t}\right|,\ \left|\frac{\partial\psi}{\partial x}\right|,\ u\frac{\partial\psi}{\partial x} \tag{27}$$

in the variable t, x, u *and* ρ.

Then the integral Δ_1 *from (8) converges and there exists a value* Δ_1' *such that*

$$\Delta_1\leq\Delta_1', \tag{28}$$

where

$$\lim_{\beta\to+\infty}\Delta_1'=\int_{R^1}dt\int_{R^3}dx\left[\int_{\mathbb{R}^3}du\int_{\mathbb{R}^1_+}d\rho\left|\frac{\partial\psi}{\partial t}+u\frac{\partial\psi}{\partial x}\right|+\right.$$

$$+2\pi d^2\int_{\mathbb{R}^6}du_1du_2\int_{\mathbb{R}^2_+}d\rho_1d\rho_2\psi(t,x,u_1,\rho_1)\psi(t,x,u_2,\rho_2)|u_1-u_2|\Bigg]. \tag{29}$$

Proof. According to (15) and (16) in the proof of Theorem 1 we rewrite expression (8) as

$$\Delta_1=\int_{R^1}dt\int_{R^3}dx\int_{\mathbb{R}^3}\Bigg|\int_{\mathbb{R}^3}du\int_0^{+\infty}d\rho\left(\frac{\partial\psi}{\partial t}+v\frac{\partial\psi}{\partial x}\right)\widetilde{M}(v,u,\rho)-$$

$$-\frac{d^2}{2}\int_{\mathbb{R}^3}dv_1\int_\Sigma d\alpha|(v-v_1,\alpha)|\times$$

$$\times\int_{\mathbb{R}^6}du_1du_2\int_{\mathbb{R}^2_+}d\rho_1d\rho_2\psi(t,x,u_1,\rho_1)\psi(t,x,u_2,\rho_2)\big[\widetilde{M}(v_1',u_1,\rho_1)\widetilde{M}(v',u_2,\rho_2)-$$

$$-\widetilde{M}(v_1,u_1,\rho_1)\widetilde{M}(v,u_2,\rho_2)\big]\Bigg|dv\le\int_{R^1}dt\int_{R^3}dx\int_{\mathbb{R}^3}\Bigg[\int_{\mathbb{R}^3}du\int_0^{+\infty}d\rho\left|\frac{\partial\psi}{\partial t}+v\frac{\partial\psi}{\partial x}\right|\widetilde{M}(v,u,\rho)+$$

$$+\frac{d^2}{2}\int_{\mathbb{R}^3}dv_1\int_\Sigma d\alpha|(v-v_1,\alpha)|\int_{\mathbb{R}^6}du_1du_2\int_{\mathbb{R}^2_+}d\rho_1d\rho_2\psi(t,x,u_1,\rho_1)\psi(t,x,u_2,\rho_2)\times$$

$$\times\big|\widetilde{M}(v_1',u_1,\rho_1)\widetilde{M}(v',u_2,\rho_2)-\widetilde{M}(v_1,u_1,\rho_1)\widetilde{M}(v,u_2,\rho_2)\big|\Bigg]dv\le$$

$$\le\int_{R^1}dt\int_{R^3}dx\int_{\mathbb{R}^3}\Bigg[\int_{\mathbb{R}^3}du\int_0^{+\infty}d\rho\left|\frac{\partial\psi}{\partial t}+v\frac{\partial\psi}{\partial x}\right|\widetilde{M}(v,u,\rho)+\frac{d^2}{2}\int_{\mathbb{R}^3}dv_1\int_\Sigma d\alpha|(v-v_1,\alpha)|\times$$

$$\times\int_{\mathbb{R}^6}du_1du_2\int_{\mathbb{R}^2_+}d\rho_1d\rho_2\psi(t,x,u_1,\rho_1)\psi(t,x,u_2,\rho_2)\Big[\widetilde{M}(v_1',u_1,\rho_1)\widetilde{M}(v',u_2,\rho_2)+$$

$$+\widetilde{M}(v_1,u_1,\rho_1)\widetilde{M}(v,u_2,\rho_2)\Big]\Bigg]dv. \tag{30}$$

Hence,

$$\Delta_1'=\int_{R^1}dt\int_{R^3}dx\Bigg[\int_{\mathbb{R}^6}dvdu\int_0^{+\infty}d\rho\left|\frac{\partial\psi}{\partial t}+v\frac{\partial\psi}{\partial x}\right|\widetilde{M}(v,u,\rho)+$$

$$+\frac{d^2}{2}\int_{\mathbb{R}^3}dv\int_{\mathbb{R}^3}dv_1\int_\Sigma d\alpha|(v-v_1,\alpha)|\int_{\mathbb{R}^6}du_1du_2\int_{\mathbb{R}^2_+}d\rho_1d\rho_2\psi(t,x,u_1,\rho_1)\psi(t,x,u_2,\rho_2)\times$$

$$\times\Big[\widetilde{M}(v_1',u_1,\rho_1)\widetilde{M}(v',u_2,\rho_2)+\widetilde{M}(v_1,u_1,\rho_1)\widetilde{M}(v,u_2,\rho_2)\Big]\Bigg]=$$

$$=\int_{R^1}dt\int_{R^3}dx\Bigg[\int_{\mathbb{R}^3}du\int_{\mathbb{R}^1_+}d\rho\int_{\mathbb{R}^3}dv\left|\frac{\partial\psi}{\partial t}+v\frac{\partial\psi}{\partial x}\right|\widetilde{M}(v,u,\rho)+\frac{d^2}{2}\int_{\mathbb{R}^6}du_1du_2\int_{\mathbb{R}^2_+}d\rho_1d\rho_2\times$$

$$\times\psi(t,x,u_1,\rho_1)\psi(t,x,u_2,\rho_2)\int_{\mathbb{R}^6}dvdv_1\int_\Sigma d\alpha|(v-v_1,\alpha)|\times$$

$$\times\Big[\widetilde{M}(v_1',u_1,\rho_1)\widetilde{M}(v',u_2,\rho_2)+\widetilde{M}(v_1,u_1,\rho_1)\widetilde{M}(v,u_2,\rho_2)\Big]\Bigg]=$$

$$=\int_{R^1}dt\int_{R^3}dx\Bigg[\int_{\mathbb{R}^3}du\int_{\mathbb{R}^1_+}d\rho\int_{\mathbb{R}^3}dv\left|\frac{\partial\psi}{\partial t}+v\frac{\partial\psi}{\partial x}\right|\widetilde{M}(v,u,\rho)+$$

$$+\int_{\mathbb{R}^6} du_1 du_2 \int_{\mathbb{R}^2_+} d\rho_1 d\rho_2 \psi(t,x,u_1,\rho_1)\psi(t,x,u_2,\rho_2) \int_{\mathbb{R}^3} \left[G(\widetilde{M_1},\widetilde{M_2}) + \widetilde{M_2} L(\widetilde{M_1})\right] dv \Bigg], \quad (31)$$

where, as in earlier, we used the notion (19) and (20) for the parts of the collision integral. In (31), we also interchange the integration order (similarly as in the proof of Theorem 1).

According to (21) we rewrite the expression (31) as

$$\Delta_1' = \int_{\mathbb{R}^1} dt \int_{\mathbb{R}^3} dx \int_{\mathbb{R}^3} du \int_{\mathbb{R}^1_+} d\rho \int_{\mathbb{R}^3} dv \left| \frac{\partial \psi}{\partial t} + v \frac{\partial \psi}{\partial x} \right| \left(\frac{\beta}{\pi}\right)^{\frac{3}{2}} e^{-\beta(v-u)^2} +$$
$$+ \int_{\mathbb{R}^1} dt \int_{\mathbb{R}^3} dx \int_{\mathbb{R}^6} du_1 du_2 \int_{\mathbb{R}^2_+} d\rho_1 d\rho_2 \, \psi(t,x,u_1,\rho_1)\psi(t,x,u_2,\rho_2) \times$$
$$\times \int_{\mathbb{R}^3} \left[\widetilde{M_1} L(\widetilde{M_2}) + \widetilde{M_2} L(\widetilde{M_1})\right] dv.$$

If we introduce the notations

$$\sqrt{\beta}(v-u) = w; \quad v = \frac{w}{\sqrt{\beta}} + u,$$

we can write

$$\Delta_1' = \pi^{-3/2} \int_{\mathbb{R}^1} dt \int_{\mathbb{R}^3} dx \int_{\mathbb{R}^3} du \int_{\mathbb{R}^1_+} d\rho \int_{\mathbb{R}^3} dw \left| \frac{\partial \psi}{\partial t} + \left(\frac{w}{\sqrt{\beta}} + u\right) \frac{\partial \psi}{\partial x} \right| e^{-w^2} +$$
$$+ \int_{\mathbb{R}^1} dt \int_{\mathbb{R}^3} dx \int_{\mathbb{R}^6} du_1 du_2 \int_{\mathbb{R}^2_+} d\rho_1 d\rho_2 \, \psi(t,x,u_1,\rho_1)\psi(t,x,u_2,\rho_2) \times$$
$$\times \int_{\mathbb{R}^3} dw \left[\widetilde{M_1} L(\widetilde{M_2}) + \widetilde{M_2} L(\widetilde{M_1})\right]. \quad (32)$$

The integrand functions in both summandss of the expression Δ_1' are continuous in the variables t, x, u, ρ and β due conditions (27) of the theorem. Next, the integral (32) converges uniformly with respect to the variable β on any compact because of (27) once more and the present of the factor e^{-w^2}. Hence, the value Δ_1' is continuous in β and we can pass to the limit as $\beta \to +\infty$. To pass to the limit it is convenient to consider the expression for Δ' before the change of variables. From (23) and (24) we obtain (29). □

Based on the obtained expression for the limit as $\beta \to +\infty$ in (29), we can find a sufficient condition for the value Δ_1 to tend to zero, which we formulate as a corollary of Theorem 2.

At first, let us introduce some definitions.

Definition 1. Let G be the domain in $\mathbb{R}^n$ such that the number of components of connectivity of the intersection of G with any straight line parallel to some co-ordinate axes, are finite. Denote by G_δ ($\delta > 0$) the δ-*exaggeration* of G (terminology of [15]), i.e. the set of all points, the distance from which to G is not greater than δ.

If $n = 4$ and the co-ordinates are denoted as t, x^k ($k = 1,2,3$), we will denote by G^x the projection of G on the hyperplane $t = 0$, and by G^k the projection of G on the hyperplane $x^k = 0$ ($k = 1,2,3$).

Definition 2. Let $G \subset \mathbb{R}^4$; $\delta > 0$. We will call as *"δ-plateau"* upon the domain G such a function $\varphi_\delta(G,t,x) \in \mathfrak{C}^1(\mathbb{R}^4)$ for which the following conditions are valid:

$$\varphi_\delta(G,t,x) = \begin{cases} 1, & (t,x) \in G, \\ 0, & (t,x) \in \mathbb{R}^4 \backslash G_\delta, \\ 0 \le \varphi_\delta \le 1, & (t,x) \in G_\delta \backslash G, \end{cases} \tag{33}$$

and, in addition to that, on any straight line which is parallel to some co-ordinate axis, φ_δ has no more than a finite number of strict extremums. One can find constructive description of functions, which satisfy (33) and belong to $\mathfrak{C}^x$, for example, in [16].

Corollary 2. *Let all the assumptions of Theorem 2 be valid. Then the statement*

$$\Delta_1 \to 0 \tag{34}$$

holds true, if the function ψ defined in (9) has the form

$$\psi(t,x,u,\rho) = g(t,x) \left(\frac{P}{\pi}\right)^{3/2} e^{-P(u-u_0)^2} h(\rho), \tag{35}$$

where $g(t,x)$ is of the form of finite "plateaus" ([12]), such that the measure of projections of the sets $\operatorname{supp} g(t,x)$ *on the hyperplane $t = 0$ (i.e. configuration space $\mathbb{R}^3$) tends to zero (in the case, when g are independent of t) and*

$$u^k \cdot (\operatorname{supp} g)_k \to 0, \qquad k = 1,2,3, \tag{36}$$

where G_k denotes the projection of any set $G \subset \mathbb{R}^4$ on the hyperplane $x^k = 0$, $u_0 \in \mathbb{R}^3$ is an arbitrary fixed vector, $h(\rho)$ is continuous, non-negative function from $L_1(\mathbb{R}^1_+)$ with root if $\rho = 0$ of at least the first degree, and $P \to +\infty$.

Proof. We use limit expression (29) and substitute expression (35) in it. The integrand of the first term tends to zero (as shown in [12]). And, by assumption of the corollary, the integral in the second summand converges and it tends to zero. The corollary is proved. □

Thus, in this paper we managed to generalize some the results obtained in [13]. Now the continual approximate solution of the Boltzmann equation (1)–(3) with arbitrary density is constructed.

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