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A FOURTH ORDER FINITE DIFFERENCE METHOD FOR A NONLINEAR HELMHOLTZ TYPE BOUNDARY VALUE PROBLEMS IN PDES

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In this paper we present a fourth order finite difference method for solving nonlinear Helmholtz type elliptic boundary value problems in two dimensions subject to the Dirichlet boundary conditions. The present fourth order method is based on the approximation of derivative by finite difference method and Helmholtz equation. The truncation error and convergence analysis are presented for the proposed method. We present numerical experiments to demonstrate the efficiency and accuracy of the method.

1. Introduction. It is well known that to solve the Helmholtz equation $\Delta u + K^2u = f(x,y)$ for a higher wave number K and achieve higher order accuracy in numerical solution is difficult. A scholarly work and discussion on numerical solutions of the Helmholtz equation can be found in [1, 2, 3] and references there in. In the present paper we do not focus on exactly the Helmholtz equation, but the treatment of the problem is the same as to that of the Helmholtz equation.

Let us consider the nonlinear elliptic problem

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + K^2u = f(x,y,u,u_x,u_y), \qquad a \le x, y \le b \tag{1}$$

in the square region $\Omega = \{(x,y) : a \le x \le b, a \le y \le b\}$ with the boundary conditions

$$u(x,y) = g(x,y) \quad \text{on} \quad \partial\Omega, \tag{2}$$

where $\partial\Omega$ is the boundary of Ω. We propose a finite difference method for finding its numerical solution. We assume the existence and uniqueness of a solution $u(x,y)$ for the problem (1) under a following assumption:

(i) f is continuous,

(ii) $\frac{\partial f}{\partial u} \ge 0$,

(iii) $|\frac{\partial f}{\partial u_x}| \le q_1$,

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(iv) $|\frac{\partial f}{\partial u_x}| \leq q_2$,

where q_1 and q_2 are positive constants. Also let us assume that $u(x, y) \in C^6$, the set of all functions of x and y with continuous derivatives up to order 6 in the region Ω. Further any specific assumption on $f(x, y, u, u_x, u_y)$, to ensure existence and uniqueness will not be considered.

Solving the Helmholtz equations numerically by the finite difference method is attractive and have drawn attention of the many researchers [4, 5, 6]. To obtain more accurate numerical result for Helmholtz equation recently the sixth order finite difference method was applied in [5, 7]. Having seen results of this method for a solution of such special boundary value problems, we are motivated and challenged to investigate what will happen if a similar idea is used to derive a method for numerical solution of nonlinear boundary value problems (1).

So in this paper we develop a fourth order finite difference method capable for solving problems (1) numerically. The derivation of the method depends on difference approximation of the derivative on discrete mesh points in the region of interest and the Helmholtz equation. We analyse the performance of the method in reference with the different values of wave number K in solving model problems.

The present work is organized as follows. In Section 2: we present the fourth order finite difference approximation for elliptic problems (1). A finite difference method that exactly satisfies the boundary conditions. A derivation of the present method is discussed in Section 3. A local truncation error and convergence of the proposed method is discussed in Section 4 and Section 5. Finally, the application of the proposed method presented together with illustrative numerical results has been produced to show the efficiency and accuracy of the method in Section 6. A conclusion and discussion on the performance of the method are presented in Section 7.

2. The finite difference method. Consider the square domain $\Omega = [a, b] \times [a, b]$ for the solution of problem (1). Let $h = (b - a)/(N + 1)$ be the uniform mesh size in the $\mathbf{x}$ and $\mathbf{y}$ directions of the Cartesian coordinate system parallel to the coordinate axes. Generate mesh points (x_i, y_j), $x_i = a + ih, i = 0, 1, 2, \ldots, N + 1$ and $y_j = a + jh, , i = 0, 1, 2, \ldots, N + 1$. Let us denote the interior central mesh point (x_i, y_j) by (i, j). Consider other mesh points $(i \pm 1, j), (i, j \pm 1)$ and $(i \pm 1, j \pm 1)$ neighbouring to the central mesh point (i, j). These nine points together constitute a compact cell. So using these notations, we can rewrite problem (1) at mesh points (i, j) as follows,

$$u_{xxi,j} + u_{yyi,j} + K^2 u_{i,j} = f(x_i, y_j, u_{i,j}, u_{xi,j}, u_{yi,j}) \tag{3}$$

Here after let us further simplify the notation and denote $f(x_i, y_j, u_{i,j}, u_{xi,j}, u_{yi,j})$ as $f_{i,j}$. If the forcing function in problem (1) is $f(x, y)$, the fourth-order finite difference method for the problem given in [7] is

$$\begin{gathered}\frac{2}{3}(u_{i+1,j} + u_{i,j+1} + u_{i-1,j} + u_{i,j-1}) + \frac{1}{6}(u_{i+1,j+1} + u_{i-1,j+1} + u_{i-1,j-1} + u_{i+1,j-1}) + \\ + \Big(\frac{-10}{3} + h^2K^2\Big(1 - \frac{h^2K^2}{12}\Big)\Big)u_{i,j} = h^2\Big(\frac{2}{3} - \frac{h^2K^2}{12}\Big)f_{i,j} + \frac{h^2}{12}(f_{i+1,j} + f_{i,j+1} + f_{i-1,j} + f_{i,j-1}).\end{gathered} \tag{4}$$

Let us denote the exact and approximate values of the solution of (1) at mesh point (i, j) by u_{ij} and u_{ij}, respectively. We define following approximations:

$$\overline{u}_{xi,j} = \frac{u_{i+1,j} - u_{i-1,j}}{2h}, \tag{5}$$

$$\overline{u}_{xi\pm1,j} = \frac{\pm 3u_{i\pm1,j} \mp 4u_{i,j} \pm u_{i\mp1,j}}{2h}, \tag{6}$$

$$\overline{u}_{xi,j\pm1} = \frac{u_{i+1,j\pm1} - u_{i-1,j\pm1}}{2h}, \tag{7}$$

$$\overline{u}_{xxi,j\pm1} = \frac{u_{i+1,j\pm1} - 2u_{i,j\pm1} + u_{i-1,j\pm1}}{h^2}, \tag{8}$$

$$\overline{u}_{yi,j} = \frac{u_{i,j+1} - u_{i,j-1}}{2h}, \tag{9}$$

$$\overline{u}_{yi,j\pm1} = \frac{\pm 3u_{i,j\pm1} \mp 4u_{i,j} \pm u_{i,j\mp1}}{2h}, \tag{10}$$

$$\overline{u}_{yi\pm1,j} = \frac{u_{i\pm1,j+1} - u_{i\pm1,j-1}}{2h}, \tag{11}$$

$$\overline{u}_{yyi\pm1,j} = \frac{u_{i\pm1,j+1} - 2u_{i\pm1,j} + u_{i\pm1,j-1}}{h^2}. \tag{12}$$

Define $\overline{f}_{i\pm1,j} = f(x_{i\pm1}, y_j, u_{i\pm1,j}, \overline{u}_{xi\pm1,j}, \overline{u}_{yi\pm1,j})$, $\overline{f}_{i,j\pm1} = f(x_i, y_{j\pm1}, u_{i,j\pm1}, \overline{u}_{xi,j\pm1}, \overline{u}_{yi,j\pm1})$

$$\overline{\overline{u}}_{xi,j} = \overline{u}_{xi,j} + a_1(\overline{u}_{xi+1,j} + \overline{u}_{xi-1,j}) + a_2 h(\overline{u}_{yyi+1,j} - \overline{u}_{yyi-1,j}) + \frac{h}{K^2} a_3(\overline{f}_{i+1,j} - \overline{f}_{i-1,j}), \tag{13}$$

$$\overline{\overline{u}}_{yi,j} = \overline{u}_{yi,j} + a_4(\overline{u}_{yi,j+1} + \overline{u}_{yi,j-1}) + a_5 h(\overline{u}_{xxi,j+1} - \overline{u}_{xxi,j-1}) + \frac{h}{K^2} a_6(\overline{f}_{i,j+1} - \overline{f}_{i,j-1}), \tag{14}$$

where a_m, $m = 1, 2, \ldots, 6$, are free parameters to be determined. Finally, define

$$\overline{\overline{f}}_{i,j} = f(x_i, y_j, u_{i,j}, \overline{\overline{u}}_{xi,j}, \overline{\overline{u}}_{yi,j}).$$

Following the ideas from [7], we propose a nine points fourth order finite difference method for the problem (1) as follows,

$$\begin{aligned} &\frac{2}{3}(u_{i+1,j} + u_{i,j+1} + u_{i-1,j} + u_{i,j-1}) + \frac{1}{6}(u_{i+1,j+1} + u_{i-1,j+1} + u_{i-1,j-1} + u_{i+1,j-1}) + \\ &\quad + \Big(\frac{-10}{3} + h^2K^2\Big(1 - \frac{h^2K^2}{12}\Big)\Big)u_{i,j} = h^2\Big(\frac{2}{3} - \frac{h^2K^2}{12}\Big)\overline{\overline{f}}_{i,j} + \\ &\quad + \frac{h^2}{12}(\overline{f}_{i+1,j} + \overline{f}_{i,j+1} + \overline{f}_{i-1,j} + \overline{f}_{i,j-1}) \ \forall\ i, j = 1, 2, \ldots, N. \end{aligned} \tag{15}$$

3. Derivation of the method. To develop a fourth order finite difference method for problem (1), we apply the ideas from [8, 9, 10]. Let us define the notation we use in derivation of the method, i.e. $\frac{\partial^{p+q} f}{\partial x^p \partial y^q} = f^{(p,q)}$. Using the Taylor series expansion around the point $(i\pm1, j)$, from (6) and (11) we have

$$\overline{u}_{xi\pm1,j} = u_{xi\pm1,j} - \frac{h^2}{3}u_{i,j}^{(3,0)} \mp \frac{h^3}{12}u_{i,j}^{(4,0)} + O(h^4). \tag{16}$$

From (16) we obtain

$$\overline{u}_{xi+1,j} + \overline{u}_{xi-1,j} = 2u_{i,j}^{(1,0)} + \frac{h^2}{3}u_{i,j}^{(3,0)} + O(h^4), \tag{17}$$

$$\overline{u}_{yi\pm1,j} = u_{yi\pm1,j} + \frac{h^2}{6}u_{i,j}^{(0,3)} \pm \frac{h^3}{6}u_{i,j}^{(1,3)} + O(h^4). \tag{18}$$

Define $G_{i\pm1,m} = (\frac{\partial f}{\partial u_x})_{i\pm1,j}$,$H_{i\pm1,m} = (\frac{\partial f}{\partial u_y})_{i\pm1,j}$ and from (16)–(18), we find

$$\overline{f}_{i\pm1,j} = f_{i\pm1,j} + \Big(-\frac{h^2}{3}u_{i,j}^{(3,0)} \mp \frac{h^3}{12}u_{i,j}^{(4,0)}\Big)G_{i,j} \mp \frac{h^3}{3}u_{i,j}^{(3,0)}G_{i,j}^{(1,0)}+ \\ +\Big(\frac{h^2}{6}u_{i,j}^{(0,3)} \pm h^3u_{i,j}^{(1,3)}\Big)H_{i,j} \pm h^3u_{i,j}^{(0,3)}H_{i,j}^{(1,0)} + O(h^4). \quad (19)$$

Similarly we can obtain

$$\overline{f}_{i,j\pm1} = f_{i,j\pm1} + \Big(\frac{h^2}{6}u_{i,j}^{(3,0)} \pm h^3u_{i,j}^{(3,1)}\Big)G_{i,j} \pm h^3u_{i,j}^{(3,0)}G_{i,j}^{(0,1)}+ \\ +\Big(- \frac{h^2}{3}u_{i,j}^{(0,3)} \mp \frac{h^3}{12}u_{i,j}^{(0,4)}\Big)H_{i,j} \mp \frac{h^3}{3}u_{i,j}^{(0,3)}H_{i,j}^{(0,1)} + O(h^4). \quad (20)$$

Thus from (19), we have

$$\overline{f}_{i+1,j} - \overline{f}_{i-1,j} = f_{i+1,j} - f_{i-1,j} - \frac{h^3}{6}u_{i,j}^{(4,0)}Gi,j - \frac{2h^3}{3}u_{i,j}^{(3,0)}G_{i,j}^{(1,0)}+ \\ +2h^3(u_{i,j}^{(1,3)}H_{i,j} + u_{i,j}^{(0,3)}H_{i,j}^{(1,0)}) + O(h^4) = 2hf_{i,j}^{(1,0)} + \frac{h^3}{3}f_{i,j}^{(3,0)}- \\ -\frac{h^3}{6}u_{i,j}^{(4,0)}Gi,j - \frac{2h^3}{3}u_{i,j}^{(3,0)}G_{i,j}^{(1,0)} + 2h^3(u_{i,j}^{(1,3)}H_{i,j} + u_{i,j}^{(0,3)}H_{i,j}^{(1,0)}) + O(h^4) = \\ = 2h(u_{i,j}^{(3,0)} + u_{i,j}^{(1,2)} + K^2u_{i,j}^{(1,0)}) + \frac{h^3}{3}f_{i,j}^{(3,0)} - \frac{h^3}{6}u_{i,j}^{(4,0)}G_{i,j} - \frac{2h^3}{3}u_{i,j}^{(3,0)}G_{i,j}^{(1,0)}+ \\ +2h^3(u_{i,j}^{(1,3)}H_{i,j} + u_{i,j}^{(0,3)}H_{i,j}^{(1,0)}) + O(h^4). \quad (21)$$

From (12), we have

$$\overline{u}_{yyi\pm1,j} = u_{i\pm1,j}^{(0,2)} + \frac{h^2}{12}u_{i,j}^{(0,4)} \pm \frac{h^3}{12}u_{i,j}^{(1,4)} + O(h^4). \quad (22)$$

Thus from (22),

$$\overline{u}_{yyi+1,j} - \overline{u}_{yyi-1,j} = 2hu_{i,j}^{(1,2)} + \frac{h^3}{3}\Big(u_{i,j}^{(3,2)} + \frac{1}{2}u_{i,j}^{(1,4)}\Big)+O(h^4). \quad (23)$$

From (5),

$$\overline{u}_{xi,j} = u_{i,j}^{(1,0)} + \frac{h^2}{6}u_{i,j}^{(3,0)} + O(h^4). \quad (24)$$

Substituting the values from (17), (21), (23) and (24) into (13), we have

$$\overline{\overline{u}}_{xi,j} = (1+2a_1+2a_3h^2)u_{i,j}^{(1,0)}+\frac{h^2}{6}\Big(1+2a_1+\frac{12}{K^2}a_3\Big)u_{i,j}^{(3,0)}+2h^2\Big(a_2+\frac{1}{K^2}a_3\Big)u_{i,j}^{(3,0)}+O(h^4). \quad (25)$$

By similar method and from (14), we have

$$\overline{\overline{u}}_{yi,j} = (1+2a_4+2a_6h^2)u_{i,j}^{(0,1)}+\frac{h^2}{6}\Big(1+2a_4+\frac{12}{K^2}a_6\Big)u_{i,j}^{(0,3)}+2h^2\Big(a_5+\frac{1}{K^2}a_6\Big)u_{i,j}^{(0,3)}+O(h^4). \quad (26)$$

Let us assume that $a_1 + a_3h^2 = 0$ and $a_4 + a_6h^2 = 0$. Thus from (25) and (26), we have

$$\overline{\overline{f}}_{i,j} = f_{i,j} + \Big(\frac{h^2}{6}\Big(1 + 2a_1 + \frac{12}{K^2}a_3\Big)u_{i,j}^{(3,0)} + 2h^2\Big(a_2 + \frac{1}{K^2}a_3\Big)u_{i,j}^{(3,0)}\Big)G_{i,j}+$$

$$+\Big(\frac{h^2}{6}\Big(1+2a_4+\frac{12}{K^2}a_6\Big)u_{i,j}^{(0,3)}+2h^2\Big(a_5+\frac{1}{K^2}a_6\Big)u_{i,j}^{(0,3)}\Big)H_{i,j}+O(h^4). \tag{27}$$

So $(\frac{2}{3}-\frac{h^2K^2}{12})\overline{\overline{f}}_{i,j}+\frac{1}{12}(\overline{f}_{i+1,j}+\overline{f}_{i,j+1}+\overline{f}_{i-1,j}+\overline{f}_{i,j-1})$ will provide $O(h^4)$ approximation for $(\frac{2}{3}-\frac{h^2K^2}{12})f_{i,j}+\frac{1}{12}(f_{i+1,j}+f_{i,j+1}+f_{i-1,j}+f_{i,j-1})$ if

$$a_1+a_3h^2=0,\ a_4+a_6h^2=0,\ 1+2a_1+\frac{12}{K^2}a_3=\frac{2}{8-h^2K^2},\ 1+2a_4+\frac{12}{K^2}a_6=\frac{2}{8-h^2K^2},$$
$$a_2+\frac{1}{K^2}a_3=0,\ a_5+\frac{1}{K^2}a_6=0. \tag{28}$$

Solving system of equations (28) for $a_i, i=1,2,..,6$, we obtain

$$a_2=a_5=\frac{a_1}{h^2K^2}=\frac{a_4}{h^2K^2}=\frac{a_3}{-K^2}=\frac{a_6}{-K^2}=\frac{1}{16-2h^2K^2}, \tag{29}$$

provided $h^2K^2\neq 8$. Thus if $h^2K^2\neq 8$, then for the values of parameters (29) we have the fourth order finite difference method (15) for a numerical solution of problem (1). If we write the system of equations given by (15) at each mesh point, we will obtain either linear or nonlinear system of equations which depends on the forcing function f.

4. Local truncation error. In this section, we consider the local truncation error associated to the proposed difference method (15). Let the local truncation error in (15) defined as in [11] be defined by $T_{i,j}$ at the mesh point (i,j), $\forall i,j=1,2,\dots,N$,

$$T_{i,j}=\Big(\frac{h^2K^2}{12}-\frac{2}{3}\Big)\overline{\overline{f}}_{i,j}-\frac{1}{12}(\overline{f}_{i+1,j}+\overline{f}_{i,j+1}+\overline{f}_{i-1,j}+\overline{f}_{i,j-1})+$$
$$+\Big[\Big(\frac{-10}{3h^2}+K^2\Big(1-\frac{h^2K^2}{12}\Big)\Big)u_{i,j}+$$
$$+\frac{1}{6h^2}(u_{i+1,j+1}+u_{i-1,j+1}+u_{i-1,j-1}+u_{i+1,j-1})+\frac{2}{3h^2}(u_{i+1,j}+u_{i,j+1}+u_{i-1,j}+u_{i,j-1})\Big].$$

From (16)–(29), we conclude that

$$T_{i,j}=\Big(\frac{h^2K^2}{12}-\frac{2}{3}\Big)f_{i,j}-\frac{1}{12}(f_{i+1,j}+f_{i,j+1}+f_{i-1,j}+f_{i,j-1})+$$
$$+\Big[\frac{2}{3h^2}(u_{i+1,j}+u_{i,j+1}+u_{i-1,j}+u_{i,j-1})+\frac{1}{6h^2}(u_{i+1,j+1}+u_{i-1,j+1}+u_{i-1,j-1}+u_{i+1,j-1})+$$
$$+\Big(K^2\Big(1-\frac{h^2K^2}{12}\Big)-\frac{10}{3h^2}\Big)u_{i,j}\Big]+O(h^4). \tag{30}$$

Write the right hand side of (30) in the Taylor series around the mesh point (i,j) and simplify,

$$T_{i,j}=\frac{h^4}{720}(5(u_{i,j}^{(4,2)}+u_{i,j}^{(2,4)})-3(u_{i,j}^{(6,0)}+u_{i,j}^{(0,6)})-5K^2(u_{i,j}^{(4,0)}+u_{i,j}^{(0,4)}))+O(h^4). \tag{31}$$

Thus the local truncation error in (15) is of the form $O(h^4)$.

5. Convergence of the method. We next discuss the convergence of the method and under suitable conditions prove that the order of the convergence is $O(h^4)$. For each $i,j=1(1)N$, let us define

$$\phi_{i,j}=h^2\Big(\frac{2}{3}-\frac{h^2K^2}{12}\Big)\overline{\overline{f}}_{i,j}+\frac{h^2}{12}(\overline{f}_{i+1,j}+\overline{f}_{i,j+1}+\overline{f}_{i-1,j}+\overline{f}_{i,j-1})+\text{Boundary Values}$$

and error function $E_{i,j} = u_{i,j} - U_{i,j},\ 1 \le i, j \le N$. Let define matrix $\mathbf{S}_{N^2\times 1}$,

$$\mathbf{S} = [S_{1,1}, S_{2,1}, ..., S_{N,1}, S_{1,2}, S_{2,2}, ..., S_{N,2}, ..., S_{1,N}, S_{1,N}, ..., S_{N,N}]^T.$$

The difference method (15) represents a system of nonlinear equations in the unknown $u_{i,j}, 1 \le i, j \le N$. Using the notation $\mathbf{S}$ for the $\boldsymbol{\phi}$, $\mathbf{u}$, $\mathbf{U}$ and $\mathbf{T}$, we can write (15) in matrix form as given below,

$$\mathbf{Du} + \boldsymbol{\phi}(\mathbf{u}) = \mathbf{0}, \tag{32}$$

where $\mathbf{D}$=$\mathbf{[BAB]}_{N^2\times N^2}$ is a tri block diagonal matrix, in which the tri diagonal matrics are $A = [-\frac{2}{3}, \frac{10}{3} - h^2K^2(1 - \frac{h^2K^2}{12}), -\frac{2}{3}]_{N\times N}$ and $B = [-\frac{1}{6}, -\frac{2}{3}, -\frac{1}{6}]_{N\times N}$. If $\mathbf{U}$ is an exact solution of (15), then we can write (15) in a matrix form as given below,

$$\mathbf{DU} + \boldsymbol{\phi}(\mathbf{U}) + \mathbf{T} = \mathbf{0}, \tag{33}$$

where $\mathbf{T}$ is the truncation error matrix and each element $\mathbf{T}$ is of $O(h^4)$. Let us define

$$\begin{aligned}
\overline{F}_{i\pm1,j} &= f(x_{i\pm1}, y_j, U_{i\pm1,j}, \overline{U}_{xi\pm1,j}, \overline{U}_{yi\pm1,j}), \quad \overline{f}_{i\pm1,j} = f(x_{i\pm1}, y_j, u_{i\pm1,j}, \overline{u}_{xi\pm1,j}, \overline{u}_{yi\pm1,j}),\\
\overline{F}_{i,j\pm1} &= f(x_i, y_{j\pm1}, U_{i,j\pm1}, \overline{U}_{xi,j\pm1}, \overline{U}_{yi,j\pm1}), \quad \overline{f}_{i,j\pm1} = f(x_i, y_{j\pm1}, u_{i,j\pm1}, \overline{u}_{xi,j\pm1}, \overline{u}_{yi,j\pm1}),\\
\overline{\overline{F}}_{i,j} &= f(x_i, y_j, U_{i,j}, \overline{\overline{U}}_{xi,j}, \overline{\overline{U}}_{yi,j}), \quad \overline{\overline{f}}_{i,j} = f(x_i, y_j, u_{i,j}, \overline{\overline{u}}_{xi,j}, \overline{\overline{u}}_{yi,j}).
\end{aligned}$$

After linearization of $\overline{f}_{i\pm1,j}$, we have

$$\overline{f}_{i\pm1,j} - \overline{F}_{i\pm1,j} = (u_{i\pm1,j} - U_{i\pm1,j})I_{i\pm1,j} + (\overline{u}_{xi\pm1,j} - \overline{U}_{xi\pm1,j})G_{i\pm1,j} + (\overline{u}_{yi\pm1,j} - \overline{U}_{yi\pm1,j})H_{i\pm1,j}, \tag{34}$$

where $I = \frac{\partial f}{\partial U}$, $G = \frac{\partial f}{\partial \overline{U}_x}$ and $H = \frac{\partial f}{\partial \overline{U}_y}$. Similarly we linearize and write for $\overline{f}_{i,j\pm1}$ and $\overline{\overline{f}}_{i,j}$,

$$\overline{f}_{i,j\pm1} - \overline{F}_{i,j\pm1} = (u_{i,j\pm1} - U_{i,j\pm1})I_{i,j\pm1} + (\overline{u}_{xi,j\pm1} - \overline{U}_{xi,j\pm1})G_{i,j\pm1} + (\overline{u}_{yi,j\pm1} - \overline{U}_{yi,j\pm1})H_{i,j\pm1}. \tag{35}$$

$$\overline{\overline{f}}_{i,j} - \overline{\overline{F}}_{i,j} = (u_{i,j} - U_{i,j})I_{i,j} + (\overline{\overline{u}}_{xi,j} - \overline{\overline{U}}_{xi,j})G^1_{i,j} + (\overline{\overline{u}}_{yi,j} - \overline{\overline{U}}_{yi,j})H^1_{i,j}, \tag{36}$$

where $G^1 = \frac{\partial f}{\partial \overline{\overline{U}}_x}$ and $H^1 = \frac{\partial f}{\partial \overline{\overline{U}}_y}$. By the Taylor series expansions of $G_{i\pm1,j}$, $G_{i,j\pm1}$, $H_{i\pm1,j}$, $H_{i,j\pm1}$, $I_{i\pm1,j}$ and $I_{i,j\pm1}$ at the central mesh point (i, j) and from (32)–(33), we get the matrix equation

$$\boldsymbol{\phi}(\mathbf{u}) - \boldsymbol{\phi}(\mathbf{U}) = \mathbf{PE}, \tag{37}$$

where $\mathbf{P}$= $(P_{l,m})_{N^2\times N^2}$ is the tri block diagonal matrix defined as,

$$\begin{aligned}
P_{(m-1)N+l,(m-1)N+l} &= \frac{h^4K^4}{12} + \frac{h^2(8 - h^2K^2)}{12}\Big\{I_{l,m} - \frac{4}{K^2}a_3(G^2_{l,m} + H^2_{l,m})\Big\} -\\
&\quad - \frac{h^2}{3}(G^{(1,0)}_{l,m} + H^{(0,1)}_{l,m}), \quad [l = 1(1)N, m = 1(1)N].\\
P_{(m-1)N+l,(m-1)N+l+1} &= \frac{h^2(8 - h^2K^2)}{12}\Big(\frac{1}{2h} + \frac{1}{h}(a_4 - 2a_5) +\\
&\quad + \frac{1}{K^2}a_6(2H_{l,m} + h(I_{l,m} + H^{(0,1)}_{l,m}))\Big)H_{l,m} +
\end{aligned}$$

$$+\frac{h^2}{12}\Big(I_{l,m}+2H^{(0,1)}_{l,m}+h\Big(I^{(0,1)}_{l,m}+\frac{1}{2}H^{(0,2)}_{l,m}\Big)+\frac{1}{h}H_{l,m}\Big),\quad [l=1(1)N, m=1(1)N-1].$$

$$P_{(m-1)N+l,(m-1)N+l-1}=$$
$$=\frac{h^2(8-h^2K^2)}{12}\Big\{\frac{1}{K^2}a_3H_{l,m}G_{l,m}+\frac{h}{2K^2}a_3(G_{l,m}H^{(1,0)}_{l,m}+G^{(0,1)}_{l,m}H_{l,m})-\frac{1}{h}a_2(G_{l,m}+H_{l,m})\Big\}-$$
$$-\frac{1}{24}h(H_{l,m}+G_{l,m})+\frac{1}{24}h^2(H^{(1,0)}_{l,m}+G^{(0,1)}_{l,m}),\quad [l=2(1)N, m=1(1)N].$$

$$P_{(m-1)N+l,mN+l}=$$
$$=\frac{h^2(8-h^2K^2)}{12}\Big(\frac{1}{2h}+\frac{1}{h}(a_1-2a_2)+\frac{1}{K^2}a_3(2G_{l,m}+h(I_{l,m}+G^{(1,0)}_{l,m}))\Big)G_{l,m}+$$
$$+\frac{h^2}{12}\Big(I_{l,m}+2G^{(0,1)}_{l,m}+h\Big(I^{(1,0)}_{l,m}+\frac{1}{2}G^{(2,0)}_{l,m}\Big)+\frac{1}{h}G_{l,m}\Big),\quad [l=1(1)N, m=1(1)N-1].$$

$$P_{(m-1)N+l,mN+l+1}=$$
$$=\frac{h^2(8-h^2K^2)}{12}\Big\{\frac{1}{K^2}a_3H_{l,m}G_{l,m}+\frac{h}{2K^2}a_3(G_{l,m}H^{(1,0)}_{l,m}+G^{(0,1)}_{l,m}H_{l,m})+$$
$$+\frac{1}{h}a_2(G_{l,m}+H_{l,m})\Big\}+\frac{h}{24}(H_{l,m}+G_{l,m})+\frac{h^2}{24}(H^{(1,0)}_{l,m}+G^{(0,1)}_{l,m}),\quad [l=1(1)N, m=1(1)N-1].$$

$$P_{(m-1)N+l,(m-1)N+l+1}=$$
$$=\frac{h^2(8-h^2K^2)}{12}\Big\{-\frac{1}{K^2}a_3H_{l,m}G_{l,m}-\frac{h}{2K^2}a_3(G_{l,m}H^{(1,0)}_{l,m}-G^{(0,1)}_{l,m}H_{l,m})+\frac{1}{h}a_2(G_{l,m}-H_{l,m})\Big\}-$$
$$-\frac{h}{24}(H_{l,m}-G_{l,m})-\frac{h^2}{24}(H^{(1,0)}_{l,m}+G^{(0,1)}_{l,m}),\quad [l=2(1)N, m=1(1)N-1].$$

$$P_{(m-1)N+l,(m-2)N+l-1}=$$
$$=\frac{h^2(8-h^2K^2)}{12}\Big\{-\frac{1}{K^2}a_3H_{l,m}G_{l,m}+\frac{h}{2K^2}a_3(G_{l,m}H^{(1,0)}_{l,m}-G^{(0,1)}_{l,m}H_{l,m})-\frac{1}{h}a_2(G_{l,m}-H_{l,m})\Big\}+$$
$$+\frac{h}{24}(H_{l,m}-G_{l,m})-\frac{h^2}{24}(H^{(1,0)}_{l,m}+G^{(0,1)}_{l,m}),\ [l=2(1)N, m=1(1)N-1].$$

$$P_{(m-1)N+l,(m-1)N+l-1}=$$
$$\frac{h^2(8-h^2K^2)}{12}\Big(-\frac{1}{2h}-\frac{1}{h}(a_1-2a_2)+\frac{1}{K^2}a_3(2G_{l,m}-h(I_{l,m}+G^{(0,1)}_{l,m}))\Big)G_{l,m}+$$
$$+\frac{h^2}{12}(I_{l,m}+2G^{(0,1)}_{l,m}-h\Big(I^{(0,1)}_{l,m}+\frac{1}{2}G^{(0,2)}_{l,m}\Big)-\frac{1}{h}G_{l,m}),\quad [l=2(1)N, m=1(1)N-1].$$

$$P_{(m-1)N+l,mN+l-1}=\frac{h^2(8-h^2K^2)}{12}\Big(-\frac{1}{2h}-\frac{1}{h}(a_4-2a_5)+$$
$$+\frac{1}{K^2}a_6(2H_{l,m}-h(I_{l,m}+H^{(0,1)}_{l,m}))\Big)H_{l,m}+$$
$$+\frac{h^2}{12}\Big(I_{l,m}+2H^{(0,1)}_{l,m}-h\Big(I^{(0,1)}_{l,m}+\frac{1}{2}H^{(0,2)}_{l,m}-\frac{1}{h}H_{l,m}\Big)\Big),\quad [l=2(1)N, m=1(1)N-1].$$

Let there are no roundoff errors in solution of difference equations (15), so from (32), (33) and (37) we get the error equations,

$$\mathbf{JE}=\mathbf{T} \tag{38}$$

where $\mathbf{J}=(\mathbf{D}+\mathbf{P})$ and $\mathbf{J}=(J_{i,j})_{N^2\times N^2}$ is defined as follows

$$J_{(j-1)N+i,(j-1)N+i}=\frac{10}{3}-h^2K^2+\frac{h^4K^4}{12}+\frac{h^2(8-h^2K^2)}{12}\Big\{I_{i,j}-\frac{4}{K^2}a_3(G^2_{i,j}+H^2_{i,j})\Big\}-$$

$$-\frac{h^2}{3}(G^{(1,0)}_{i,j}+H^{(0,1)}_{i,j}),\quad [i=1(1)N, j=1(1)N].$$

$$J_{(j-1)N+i,(j-1)N+i+1}=$$

$$=-\frac{2}{3}+\frac{h^2(8-h^2K^2)}{12}\Big(\frac{1}{2h}+\frac{1}{h}(a_4-2a_5)+\frac{1}{K^2}a_6(2H_{i,j}+h(I_{i,j}+H^{(0,1)}_{i,j}))\Big)H_{i,j}+$$
$$+\frac{h^2}{12}\Big(I_{i,j}+2H^{(0,1)}_{i,j}+h\Big(I^{(0,1)}_{i,j}+\frac{1}{2}H^{(0,2)}_{i,j}\Big)+\frac{1}{h}H_{i,j}\Big),\quad [i=1(1)N, j=1(1)N-1].$$

$$J_{(j-1)N+i,(j-1)N+i-1}=$$

$$=-\frac{1}{6}+\frac{h^2(8-h^2K^2)}{12}\Big\{\frac{1}{K^2}a_3H_{i,j}G_{i,j}+\frac{h}{2K^2}a_3(G_{i,j}H^{(1,0)}_{i,j}+G^{(0,1)}_{i,j}H_{i,j})-$$
$$-\frac{1}{h}a_2(G_{i,j}+H_{i,j})\Big\}-\frac{h}{24}(H_{i,j}+G_{i,j})+\frac{h^2}{24}(H^{(1,0)}_{i,j}+G^{(0,1)}_{i,j}),\quad [i=2(1)N, j=1(1)N].$$

$$J_{(j-1)N+i,jN+i}=$$

$$=-\frac{2}{3}+\frac{h^2(8-h^2K^2)}{12}\Big(\frac{1}{2h}+\frac{1}{h}(a_1-2a_2)+\frac{1}{K^2}a_3(2G_{i,j}+h(I_{i,j}+G^{(1,0)}_{i,j}))\Big)G_{i,j}+$$
$$+\frac{h^2}{12}\Big(I_{i,j}+2G^{(0,1)}_{i,j}+h\Big(I^{(1,0)}_{i,j}+\frac{1}{2}G^{(2,0)}_{i,j}\Big)+\frac{1}{h}G_{i,j}\Big),\quad [i=1(1)N, j=1(1)N-1].$$

$$J_{(j-1)N+i,jN+i+1}=$$

$$=-\frac{1}{6}+\frac{h^2(8-h^2K^2)}{12}\Big\{\frac{1}{K^2}a_3H_{i,j}G_{i,j}+\frac{h}{2K^2}a_3(G_{i,j}H^{(1,0)}_{i,j}+G^{(0,1)}_{i,j}H_{i,j})+$$
$$+\frac{1}{h}a_2(G_{i,j}+H_{i,j})\Big\}+\frac{h}{24}(H_{i,j}+G_{i,j})+\frac{h^2}{24}(H^{(1,0)}_{i,j}+G^{(0,1)}_{i,j}),\quad [i=1(1)N, j=1(1)N-1].$$

$$J_{(j-1)N+i,(j-1)N+i+1}=-\frac{1}{6}+\frac{h^2(8-h^2K^2)}{12}\Big\{-\frac{1}{K^2}a_3H_{i,j}G_{i,j}-\frac{h}{2}a_3(G_{i,j}H^{(1,0)}_{i,j}-G^{(0,1)}_{i,j}H_{i,j})+$$
$$+\frac{1}{h}a_2(G_{i,j}-H_{i,j})\Big\}-\frac{h}{24}(H_{i,j}-G_{i,j})-\frac{h^2}{24}(H^{(1,0)}_{i,j}+G^{(0,1)}_{i,j}),\quad [i=2(1)N, j=1(1)N-1].$$

$$J_{(j-1)N+i,(j-2)N+i-1}=$$

$$=-\frac{1}{6}+\frac{h^2(8-h^2K^2)}{12}\Big\{-\frac{1}{K^2}a_3H_{i,j}G_{i,j}+\frac{h}{2K^2}a_3(G_{i,j}H^{(1,0)}_{i,j}-G^{(0,1)}_{i,j}H_{i,j})$$
$$-\frac{1}{h}a_2(G_{i,j}-H_{i,j})\Big\}+\frac{h}{24}(H_{i,j}-G_{i,j})-\frac{h^2}{24}(H^{(1,0)}_{i,j}+G^{(0,1)}_{i,j}),\quad [i=2(1)N, j=1(1)N-1].$$

$$J_{(j-1)N+i,(j-1)N+i-1}=$$

$$=\frac{h^2(8-h^2K^2)}{12}\Big(-\frac{1}{2h}-\frac{1}{h}(a_1-2a_2)+\frac{1}{K^2}a_3(2G_{i,j}-h(I_{i,j}+G^{(0,1)}_{i,j}))\Big)G_{i,j}-$$
$$-\frac{2}{3}+\frac{h^2}{12}(I_{i,j}+2G^{(0,1)}_{i,j}-h\Big(I^{(0,1)}_{i,j}+\frac{1}{2}G^{(0,2)}_{i,j}\Big)-\frac{1}{h}G_{i,j}),\quad [i=2(1)N, j=1(1)N-1].$$

$$J_{(j-1)N+i,jN+i-1}=$$

$$=\frac{-2}{3}+\frac{h^2(8-h^2K^2)}{12}\Big(-\frac{1}{2h}-\frac{1}{h}(a_4-2a_5)+\frac{1}{K^2}a_6(2H_{i,j}-h(I_{i,j}+H^{(0,1)}_{i,j}))\Big)H_{i,j}$$
$$+\frac{h^2}{12}\Big(I_{i,j}+2H^{(0,1)}_{i,j}-h\Big(I^{(0,1)}_{i,j}+\frac{1}{2}H^{(0,2)}_{i,j}-\frac{1}{h}H_{i,j}\Big)\Big),\quad [i=2(1)N, j=1(1)N-1].$$

Define the sets $I_0=\{I_{i\pm1,j}, I_{i,j\pm1}, I_{i,j}\}$, $G_0=\{G_{i\pm1,j}, G_{i,j\pm1}, G^1_{i,j}\}$, $H_0=\{H_{i\pm1,j}, H_{i,j\pm1}, H^1_{i,j}\}$, $D_x=\{G_{xi,j}, H_{xi,j}\}$, $D_y=\{G_{yi,j}, H_{yi,j}\}$.

Let $\overline{\Omega} = \partial\Omega \cup \Omega$ and

$$I_* = \min_{(x,y)\in\overline{\Omega}} \frac{\partial f}{\partial u}, \quad I^* = \max_{(x,y)\in\overline{\Omega}} \frac{\partial f}{\partial u}.$$

Then $I^* \geq t \geq I_*,\ \forall t \in I_0$. Let us assume that

$$q_1 > |G_0| > 0, \quad q_2 > |H_0| > 0, \quad q_3 > |D_x| > 0, \quad q_4 > |D_y| > 0,$$

$\max|\{q_1, q_2\}| < \frac{27}{8}$ and $|q_1 + q_2| < 4$. It is easy to verify for sufficiently small h that $\mathbf{J}$ has positive diagonal and negative non diagonal elements. Thus $\mathbf{J}$ is a row diagonally dominant matrix. Let $\mathbf{J}$ be the adjacency matrix of some graph G_J. We may easily prove that the graph G_J is strongly connected. From this fact it follows that the adjacency matrix $\mathbf{J}$ is irreducible [13]. By the row sum criterion it follows that $\mathbf{J}$ is monotone [12]. Thus there exists the positive $\mathbf{J}^{-1}$. Thus from (38), we have

$$\|\mathbf{E}\|_\infty \leq \|\mathbf{J}^{-1}\|_\infty \|\mathbf{T}\|_\infty. \tag{39}$$

For the bounds of $(\mathbf{J})^{-1}$, let $\mathbf{R} = [R_1, R_2, ..., R_{N^2}]^T$ denote the row sums of $\mathbf{J} = (J_{i,j})_{N^2\times N^2}$. It is easy to verify for sufficiently small h,

$$R_1 = \frac{11}{6} + hK^2\Big(\frac{1}{2} + a_1 - 2a_2\Big) + h\Big(\Big(K^2(a_1 - 2a_2) + \frac{1}{8}\Big)G_{1,1} + \frac{1}{8}H_{1,1}\Big), \tag{40}$$

$R_1 > \frac{11}{6}$, if $2 \leq h^2K^2 < 8$. Similarly we can define bounds for $R_k, k = 1, \dots N^2$. Let us define ([14, 15]),

$$R_i(\mathbf{J}) = 2(|J_{i,i}|) - \sum_{i=1}^{N^2} |J_{i,j}|, \quad j = 1, 2, \dots, N^2,$$

and $R_*(\mathbf{J}) = \min_{1\leq i\leq N^2} R_i(\mathbf{J})$. Let us assume that $R_i(\mathbf{J}) \geq 0$, then

$$\|\mathbf{J}^{-1}\|_\infty \leq \frac{1}{R_*(\mathbf{J})}. \tag{41}$$

Thus from (39) and (40), we have

$$\|\mathbf{E}\|_\infty \leq \frac{1}{R_*(\mathbf{J})}\|\mathbf{T}\|_\infty. \tag{42}$$

With the help of (31) and (42), for sufficiently small h, we have

$$\|\mathbf{E}\| \leq O(h^4). \tag{43}$$

Thus the proposed difference method (15) converges and the order of the convergence is four.

6. Numerical experiments. We have tested the validity and accuracy of the proposed finite difference method on uniform mesh on linear and nonlinear model problems. We have considered nonlinear and linear Helmholtz type equations with the Dirichlet boundary conditions in Cartesian coordinate system regardless the magnitude of the wave number K. We consider a square as the region of integration which covered with a uniform sub square mesh of side h. To solve the resulting system of equations which we got after discretization of the problems, we have used the Gauss-Seidel and the Newton-Raphson methods for linear and nonlinear equations respectively. In the tables, we have shown the maximum absolute error MAE at discrete mesh points, computed for different values of N, using the following formula,

$$MAE = \max_{1 \le i,j \le N} |u(x_i, y_j) - u_{i,j}|,$$

where $u_{i,j}$ and $u(x_i, y_j)$ are the value computed by method (15) and the exact value respectively. All computations were made on the Windows 2007 Ultimate operating system in the GNU FORTRAN environment version 99 compiler (2.95 of gcc) on Intel Core i3-2330M, 2.20 Ghz PC. The stopping condition for iteration was either error of order 10^{-10} or number of iterations 10^5.

Problem 1. Consider the nonlinear equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + K^2 u = u\Big(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial x}\Big) + f(x,y), \quad 0 < x, y < 1,$$

where $u(x; y)$ is subject to the boundary conditions on all sides of the unit square. The right hand side of $f(x, y)$ is given such that the exact solution is $u(x, y) = \exp(K(x + y))$. The resulting system of nonlinear equations is solved by the Newton Raphson method by guessing zero as initial solution. In table 1, we have presented the computed value of MAE for different values of N and K for the constructed solution.

Table 1. Maximum absolute errors in $u(x, y) = \exp(K(x + y))$ for Problem 1.

	MAE			
	N			
K	4	8	16	32
.80	.22828579e-2	.54621696e-3	.13017654e-3	.24795532e-4
.60	.10316372e-2	.26750565e-3	.60319901e-4	.61988831e-5
.40	.25475025e-3	.67353249e-4	.16927719e-4	.95367432e-6
.20	.24914742e-4	.60796738e-5	.35762787e-6	.40531158e-5

Problem 2. Consider the linear model problem: to find a solution of the equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + (K^2 - 1)u = 0, \qquad 0 < x, y < \pi,$$

satisfying the boundary conditions on all sides of the unit square. The exact solution is $u(x, y) = \exp(x)\sin(Ky)$. In Table 2, we have presented the computed MAE for different values of N and K for the considered exact solution.

Table 2. Maximum absolute errors in $u(x, y) = \exp(x)\sin(Ky)$ for Problem 2.

	MAE		
	N		
K	8	16	32
1.6	.75311661e-2	.40674210e-3	.53405762e-4
.8	.92601776e-3	.36239624e-4	.38146973e-5
.4	.32663345e-3	.17642975e-4	.28610229e-5
.2	.13065338e-3	.73909760e-5	.57220459e-5

Problem 3. Consider the problem of the nonlinear convection equation ([6]): to find a solution of

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + K^2 u = u^2 + f(x, y), \qquad 0 < x, y < 1,$$

that satisfies the boundary conditions on all sides of the unit square. The function $f(x, y)$ is given such that the exact solution is $u(x, y) = \exp(-Kx)\sin(y)$. In Table 3, we have presented the computed MAE for different values of N and K for the considered exact solution.

Table 3. Maximum absolute errors in $u(x, y) = \exp(-Kx)\sin(y)$ for Problem 3.

	MAE			
	N			
K	4	8	16	32
.8	.11911988e-3	.76889992e-5	.41723251e-6	.11920929e-6
.4	.35423040e-3	.22351742e-4	.11920929e-5	.11920929e-6
.2	.42659044e-3	.26106834e-4	.14901161e-5	.11324883e-5
.1	.46092272e-3	.28192997e-4	.21457672e-5	.14901161e-6

7. Conclusion. In this paper we have applied the finite difference method for numerical solution of Helmholtz type PDEs. We have described the fourth order and the Dirichlet boundary conditions incorporated into the method in natural way. Under appropriate assumptions, we have discussed the convergence of the proposed method. We found that in one model problem, the proposed method converges, but the order of convergence depends on the wave number K, and for some value of K it is not in agreement with the estimated order. Thus we conclude that the order of the method in numerical experiment may depend on the approximations used in experiment and the wave number K. Over all the results we obtained in numerical experiment with model problems are in good agreement to the estimated order of the method. The numerical experiments show that this proposed method has a good numerical stability in model problems. Our future work will be to develop the numerical method independent of K to solve boundary value problems and improve the computational performance; work in this specific direction is in progress.

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