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NEW CHROMATIC NUMBERS IN OPEN PROBLEMS

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Given a graph G and a natural number r, we use all possible r-colorings of $V(G)$ and $E(G)$ to introduce chromatic path numbers and chromatic diameters of G. Replacing colorings of $E(G)$ with orientations, we define directed path numbers and directed diameter of G. We formulate 7 open problems.

Let G be a finite connected graph (with the set of vertices $V(G)$ and the set of edges $E(G)$) endowed with the path metric d ($d(u,v)$ is the length of a shortest path between u and v). A path $u_0, u_1, \dots, u_n$ is *geodesic* if $d(u_0, v_n) = n$. The $diam(G)$ is the maximal distance between two vertices of G.

For a graph G and a natural number r, we define

r-$path_E(G)$:=maximal p such that, for every r-coloring of $E(G)$, there is an edge-monochrome path of length p;

r-$path_V(G)$:=maximal p such that, for every r-coloring of $V(G)$, there is a vertex-monochrome path of length p;

r-$diam_E(G)$:=maximal p such that, for every r-coloring of $E(G)$, there is an edge-monochrome geodesic path of length p;

r-$diam_V(G)$:=maximal p such that, for every r-coloring of $V(G)$, there is a vertex-monochrome geodesic path of length p.

Replacing colorings of $E(G)$ with orientations, we get

$\overrightarrow{path}(G)$:=maximal p such that, for any orientation of $E(G)$, there is a directed path of length p;

$\overrightarrow{diam}(G)$:=maximal p such that, for any orientation of $E(G)$, there is a directed geodesic path of length p.

We recall that the *chromatic number* $\chi(G)$ is the minimal r such that there is an r-coloring of $V(G)$ with no monochrome incident vertices.

If $r \geq \chi(G)$ then r-$path_V(G) = 0$.

Each r-coloring of $V(G)$ define the natural $\frac{r(r+1)}{2}$-coloring of $E(G)$: each edge is colored in colors of its ends. If $\frac{r(r+1)}{2} \geq \chi(G)$ then $\frac{r(r+1)}{2}$-$path_E(G) = 1$.

It seems, there are no direct correlation between $path_V(G)$ and $path_E(G)$.

For a graph G, the *line graph* L_G is a graph with the set of vertices $E(G)$ in which two vertices are incident if and only if corresponding edges are incident in G. Clearly, r-$path_V(L_G) \geq r$-$path_E(G)$.

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Every r-coloring $\gamma\colon V(G) \longrightarrow \{1,\dots,r\}$ define on orientation of $E(G)$ by the rule: an edge $\{u,v\}$ is oriented as $\overrightarrow{uv}$ if and only if $\gamma(a) < \gamma(b)$. It follows that $\overrightarrow{path}(G) \le \chi(G) - 1$.

1. By [4], for any natural numbers n and r, there exists a graph G of $diam(G) = n$ such that $\overrightarrow{diam}(G) = n$, $r\text{-}diam_V(G) = n-1$ and $r\text{-}diam_E(G) \ge \frac{n}{r}$. Moreover, there exists a graph H such that, for any r-coloring of $V(H)$, r-coloring and orientation of $E(H)$, there is a directed, vertex-monochrome and edge-monochrome geodesic path of length n.

By [1], $\overrightarrow{diam}(G) = 1$ if and only if G is a comparability (=transitively orientable) graph. It is well known (Wikipedia) that G is a comparability graph if and only if each cycle of odd length has a triangulated chord. We refer to [1] for some bounds on $|V(G)|$ provided that $\overrightarrow{diam}(G) = n$.

2. For a graph G, we have

- $r\text{-}path_V(G) = 0$ if and only if $\chi(G) = r$;
- $2\text{-}path_E(G) = 1$ if and only if G is either cycle or interval;
- $\overrightarrow{path}(G) = 1$ if and only if G is bipartite.

Problem 1. *Characterize all G of $2\text{-}path_V(G) = 1$.*

Problem 2. *Characterize all G of $2\text{-}path_E(G) = 2$.*

Problem 3. *Characterize all G of $\overrightarrow{path}(G) = 2$.*

3. If $2\text{-}diam_E(G) = 1$ then G has no induced triods (in particular, if G is a tree then G is an interval) and each cycle of odd length has a triangulated chord, so G is a comparability graph.

Problem 4. *Characterize all graphs G of $2\text{-}diam_E(G) = 1$.*

If G is either even cycle or tree then $2\text{-}diam_V(G) = 1$, but $2\text{-}diam_V(P_5) = 2$, P_5 is the 5-gonal pyramid. Thus, comparability does not imply $2\text{-}diam_V(G) = 1$ and vise versa.

Problem 5. *Characterize all graphs G of $2\text{-}diam_V\ (G) = 1$.*

4. Between the chromatic path numbers and chromatic diameters, we insert the induced versions

r-$\mathbf{path_E}(G)$:=maximal p such that, for any r-coloring of $E(G)$, there is an edge-monochrome induced path of length p;

r-$\mathbf{path_V}(G)$:=maximal p such that, for any r-coloring of $V(G)$, there is a vertex-monochrome induced path of length p;

$\overrightarrow{\mathbf{path}}(G)$:=maximal p such that, for any orientation of $E(G)$, there is a directed induced path of length p.

5. For a graph G and a natural number r, we use the standard notation (see [2])

$R(G,r)$:=maximal t such that, under any r-coloring of edges of the complete graph K_t, there is a monochrome copy of G.

If $p = r\text{-}path_E(K_m)$ then $R(I_p, r) \le m$, I_p is the interval of length p. Clearly, $r\text{-}path_V(K_m) \ge \frac{m}{r}$.

Under any orientation of $E(K_m)$, by Rado theorem, there is a directed path of length m, so $\overrightarrow{path}(K_m) = m$.

By Theorem 3 from [3], $2|G| \geq (\overrightarrow{\mathbf{path}}(G))^2$.

6. A subgraph G of a graph H is *isometrically embedded* if $d_G(u,v) = d_H(u,v)$ for any vertices u, v of G.

The constructions of isometric embeddings in [1],[4] use Cartesian products of complete graphs.

We recall that *Cartesian product* $G = G_1 \times \ldots \times G_n$ is a graph with the set of vertices $V(G) = V(G_1) \times \ldots \times V(G_n)$ and the set of edges $E(G) = \bigcup_{i \leq n} E_i(G)$ defined by the rule: for $u = (u_1, \ldots, u_n)$, $v = (v_1, \ldots, v_n)$, $uv \in E_i(G) \iff u_i v_i \in E_i(G)$, $u_k = v_k$, $k \in \{1, \ldots, k-1, k+1, \ldots, n\}$.

Problem 6. *Evaluate* $path(K_n^m)$ *and* $\mathbf{path}(K_n^m)$.

Problem 7. *Evaluate* $\overrightarrow{diam}(K_n^m)$.

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