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LINEAR FUNCTIONALS IN SPACE OF ENTIRE FUNCTIONS OF FINITE ORDER AND LESS THE GIVEN TYPE

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We consider the linear topology space of entire functions of a proximate order and normal type less than the given type with respect to the proximate order. We obtain a form of continuous linear functional on this space.

1. We introduce the necessary definitions. A function $\varrho(r)$, defined on the ray $(0,\infty)$ and satisfying the Lipschitz condition on any segment $[a,b]\subset(0,\infty)$, that satisfies the conditions

$$\lim_{r\to\infty}\varrho(r)=\rho\geq 0,\ \text{and}\ \lim_{r\to\infty} r\varrho'_+(r)\ln r=0$$

is called a *proximate order.*

A detailed exposition of properties of a proximate order can be found in [1, 2]. In this paper we use the notation $V(r)=r^{\varrho(r)}$. We will assume that $V(r)$ is an increasing function on $(0,\infty)$ and $\lim\limits_{r\to+0}V(r)=0$.

We will say that a proximate order $\varrho_1(r)$ is less than the proximate order $\varrho(r)$ if

$$\lim_{r\to\infty}\varrho_1(r)=\rho_1<\rho=\lim_{r\to\infty}\varrho(r)\,.$$

We now formulate a simple property of a proximate order that we shall need frequently [1, (2), p.33].

For $r\to\infty$ and $0<a\leq k\leq b<\infty$ the asymptotic inequality

$$(1-\varepsilon)k^\rho V(r)<V(kr)<(1+\varepsilon)k^\rho V(r) \tag{1}$$

holds uniformly in k.

Let $M_f(r)=\max\limits_{|z|=r}|f(z)|$. If for an entire function $f(z)$ the quantity

$$\sigma_f=\limsup_{r\to\infty}\frac{\log M_f(r)}{V(r)}$$

is different from zero and infinity, then $\varrho(r)$ is called of a *proximate order* of the *entire function $f(z)$* and σ_f is called the *type of the function $f(z)$ with respect to the proximate order $\varrho(r)$.*

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Let $\varrho(r)$ be a proximate order, $\lim_{r\to\infty}\varrho(r)=\rho>0$. A single valued function $f(z)$ of the complex variable z is said to belong to the space $[\varrho(r),\sigma)$ if $f(z)$ has the order less than $\varrho(r)$ or equal $\varrho(r)$ but in this second case the type is less than σ.

Thus $f\in[\varrho(r),\sigma)$ if and only if there exist $C_f>0$ and $\sigma_f>0$, $\sigma_f<\sigma$, such that $|f(z)|\le C_f e^{\sigma_f V(|z|)}$ for all $z\in\mathbb{C}$.

A sequence of functions $\{f_n(z)\}$ from $[\varrho(r),\sigma)$ *converges in the sense of* $[\varrho(r),\sigma)$ if (i) it converges uniformly on compacts, (ii) there exists $\beta<\sigma$ such that

$$|f_n(z)|<\exp[\beta V(|z|)],\quad |z|>r_0(\beta),\ n\ge 1,$$

where $r_0(\beta)$ does not depend on n.

For a suitable $C(\beta)$, which does not depend on n, for all z

$$|f_n(z)|<C(\beta)\exp[\beta V(|z|)]\quad (n\ge 1)\,. \tag{2}$$

The space $[\varrho(r),\beta)$ is a linear topology space with the sequential topology.

We introduce the function $\varphi(t)$ to be the unique solution of the equation $t=V(r)$. So

$$\varphi(V(t))=t\,. \tag{3}$$

The following theorem on relation of the type of the function to the rate of decrease of its Taylor coefficients is true.

Theorem 1. ([1, Theorem 2', p.42]) *The type σ_f of an entire function $f(z)=\sum\limits_{n=0}^{\infty}c_nz^n$ with a proximate order $\varrho(r)$ $(\rho>0)$ is given by the equation*

$$\limsup_{n\to\infty}\varphi(n)\sqrt[n]{|c_n|}=(e\sigma_f\rho)^{1/\rho}\,. \tag{4}$$

Let $\rho>0$. Set

$$d_n=\frac{(e\sigma\rho)^{n/\rho}}{(\varphi(n))^n},\quad n\ge 1,\ d_0=1\,.$$

For a function $f(z)=\sum\limits_{n=0}^{\infty}c_nz^n\in[\varrho(r),\sigma)$, we associate the function

$$F(z)=\sum_{n=0}^{\infty}b_nz^n,\quad b_n=\frac{c_n}{d_n},\ n\ge 0. \tag{5}$$

It is analytic in the closed disk $|z|\le 1$. Indeed, we have by (4) $\limsup\limits_{n\to\infty}\varphi(n)\sqrt[n]{|c_n|}<(e\sigma\rho)^{1/\rho}$, hence $\varphi(n)\sqrt[n]{|c_n|}\le(e\sigma_1\rho)^{1/\rho}$, $\sigma_f\sigma_1<\sigma$, $n>n_0$, and

$$|b_n|<\left(\frac{\sigma_1}{\sigma}\right)^{n/\rho},\quad n\ge n_0\,.$$

Since $\sigma_1<\sigma$, we obtain

$$\limsup_{n\to\infty}\sqrt[n]{|b_n|}<1$$

and the series (5) converges in the disk $|z|<r$, $r>1$.

Conversely, for any analytic function $F(z)$ in the disk $|z| \le 1$ it corresponds the function $f(z)$ from $[\varrho(r), \sigma)$.

If a function $f(z)$ of $[\varrho(r), \sigma)$ corresponds to the function $F(z)$ as indicated above we write

$$f(z) \sim F(z)\,.$$

It is obvious, that if $f(z) \in [\varrho(r), \sigma)$ then $f(\lambda z) \sim F(\lambda z)$ in the sense of $[\varrho(r), \lambda^\rho \sigma)$, where $\lambda > 0$ is a parameter, and if $f_n(z) \sim F_n(z)$ $(n = 1, 2, \ldots, m)$ then

$$\sum_{n=1}^{m} a_n f_n(z) \sim \sum_{n=1}^{m} a_n F_n(z)\,.$$

In the present paper we prove two theorems.

Theorem 2. *A sequence $\{f_n(z)\}$ of functions from $[\varrho(r), \sigma)$ converges in the sense of $[\varrho(r), \sigma)$ if and only if there exists $r > 1$ such that the sequence $\{F_n(z)\}$ $(f_n(z) \sim F_n(z))$ converges uniformly inside the disk $|z| \le r$ and $\{F_n(z)\}$ are analytic in this disk.*

Theorem 3. *Every continuous linear functional l on the space $[\varrho(r), \sigma)$ has the form*

$$l(f) = \sum_{n=0}^{\infty} a_n c_n, \quad f(z) = \sum_{n=0}^{\infty} c_n z^n\,, \tag{6}$$

where the quantities a_n satisfy

$$\limsup_{n\to\infty} \varphi^{-1}(n)\, \sqrt[n]{|a_n|} \le (e\sigma\rho)^{-1/\rho}\,. \tag{7}$$

The case $\varrho(r) \equiv \rho > 0$ was considered by A.F. Leont'ev [3, Theorem 1.1.7 and 1.1.10]. The case of the space $[\varrho(r), \sigma]$ of entire functions of type less than or equal σ with respect to the proximate order $\varrho(r)$ was consider in [4].

2. We now prove Theorem 2. Let

$$f_k(z) = \sum_{n=0}^{\infty} c_n^{(k)} z^n, \quad F_k(z) = \sum_{n=0}^{\infty} b_n^{(k)} z^n,\ k \ge 1,$$

and let the sequence $\{f_k(z)\}$ converge in the sense of $[\varrho(r), \sigma)$. By Cauchy's inequality and the condition (2)

$$|c_n^{(k)}| < C(\beta) \frac{\exp[\beta V(r)]}{r^n}, \quad |z| = r > 0\,,$$

for some β, $0 < \beta < \sigma$.

Put here $r = \dfrac{\varphi(n)}{(\beta\rho)^{1/\rho}}$, by (1) for and (3) we obtain

$$V\left(\frac{\varphi(n)}{(\beta\rho)^{1/\rho}}\right) \le \frac{(1+\varepsilon)n}{\beta\rho}, \quad n > n_\varepsilon\,.$$

This yields

$$|c_n^{(k)}| < \frac{C(\beta)(\beta\rho)^{n/\rho}}{(\varphi(n))^n} \exp\left[\beta V\left(\frac{\varphi(n)}{(\beta\rho)^{1/\rho}}\right)\right] \le$$
$$\le \frac{C_1(\beta,\varepsilon)(\beta\rho)^{n/\rho}}{(\varphi(n))^n} \exp\left[\frac{(1+\varepsilon)n}{\rho}\right],\ n \ge 0, k \ge 1.$$

It follows from (2), for $z = 0$,

$$|c_0^{(k)}| < C_2(\beta),\ k \geq 1.$$

Therefore

$$|b_n^{(k)}| = \frac{|c_n^{(k)}|}{d_n} < C_3(\beta, \varepsilon) \left(\frac{\beta e^{\varepsilon}}{\sigma}\right)^{n/\rho} \qquad (n \geq 0, k \geq 1)\,.$$

We take $r_0 < \left(\dfrac{\sigma}{\beta e^{\varepsilon}}\right)^{1/\rho}$. Since $\beta < \sigma$ we may take $\varepsilon > 0$ such that r_0 is greater than 1. For $|z| < r_0$,

$$|F_m(z) - F_k(z)| < \sum_{n=1}^{s} |b_n^{(m)} - b_n^{(k)}| r_0^n + 2C_3(\beta, \varepsilon) \sum_{n=s+1}^{\infty} \left(\frac{\beta e^{\varepsilon}}{\sigma}\right)^{n/\rho} r_0^n\,.$$

The series on the right-hand side converges. Given ε_1 we choose s so that the second addend is less than ε_1.

Uniform convergence $\{f_k(z)\}$ on compacts implies that for each fixed n the coefficient $c_n^{(k)}$ has a limit as $k \to \infty$. Then $b_n^{(k)}$ also has a limit as $k \to \infty$. That is why

$$\sum_{n=1}^{s} |b_n^{(m)} - b_n^{(k)}| r_0^n < \varepsilon_1$$

if m and k are sufficiently large. Finally, $|F_m(z) - F_k(z)| < 2\varepsilon_1$, $|z| < r_0$.

We now prove the second part of the theorem. Let the sequence $\{F_k(z)\}$ converge uniformly in the disk $|z| \leq r_0$, $r_0 > 1$ and all functions are analytic in this disk. Let $f_n(z) \sim F_n(z)$. Repeating the arguments of [3] and [4], from this estimate by Theorem 1, we get that the function $\sum_{n=0}^{\infty} \frac{(\sigma\, e\rho)^{n/\rho}}{(r_0\varphi(n))^n} z^n$ is an entire function of the proximate order $\varrho(r)$ and the type $\frac{\sigma}{r_0^{\rho}}$. This type is less than σ so $r_0 > 1$. Therefore the condition (2) holds.

Since the coefficient $c_n^{(k)}$ has a limit as $k \to \infty$ for each fixed n it is possible to prove similarly to the previous that the sequence $\{f_k(z)\}$ converges uniformly on compact sets. So this sequence converges in the sense of $[\varrho(r), \sigma)$.

Remark 1. If the sequence $\{f_k(z)\}$ converges to $f(z)$ in the sense of $[\varrho(r), \sigma)$, and $\{F_k(z)\}$ converges to $F(z)$ then $f(z) \sim F(z)$.

3. By Theorem 2 it can be argued that the space $[\varrho(r), \sigma)$ of functions $f(z)$ with convergence in the sense of $[\varrho(r), \sigma)$ is transform onto the space $A(\overline{D})$ of functions $F(z)$ analytic in the disk $\overline{D} = \{z : |z| \leq 1\}$. A sequence $\{F_k(z)\}$ of functions from $A(\overline{D})$ converges to $F(z)$ in the sense of $A(\overline{D})$ if there exists $r_0 > 1$ such that all $\{F_k(z)\}$ are analytic in the disk $D_0 = \{z : |z| \leq r_0\}$ and $\{F_k(z)\}$ converges to $F(z)$ by the norm $\|F\| = \max\limits_{z \in \overline{D}_0} |F(z)|$.

Next, using the form of a linear functional in the space $[\varrho(r), \sigma]$ ([4]), and the arguments of the proof of Theorem 1.1.10 [3], it is not difficult to establish the validity of Theorem 3.

We recall the following known facts ([3, p. 17–18]):

1) Let $F_n(z) \in A(\overline{D})$ $(n \geq 1)$. In order to a function $F_0(z) \in A(\overline{D})$ can be approximated with an arbitrary accuracy by linear combinations of functions $F_n(z)$ it is necessary and sufficient that the equalities

$$L(F_n) = 0, \quad n \geq 1, \tag{8}$$

where $L(F)$ is an arbitrary continuous linear functional on $A(\overline{D})$, imply $L(F_0) = 0$. In particular, the system of functions $\{F_n(z)\}$ is complete in $A(\overline{D})$ if and only if equalities (8) imply that for arbitrary L one has $L(F) = 0$ for any function $F(z) \in A(\overline{D})$.

2) Let M be a proper closed set in $A(\overline{D})$ and $F_0 \in A(\overline{D})$. Then there exists a functional $L(F)$ with the property: $L(F) = 0$ for all $F \in M$ but $L(F_0) \neq 0$.

Note that a closed set M in $A(\overline{D})$, corresponds to a closed set in the space $[\varrho(r), \sigma)$ by the relation $f(z) \sim F(z)$.

In virtue of these facts we obtain the following statements:

1) Let $f_n(z) \in [\varrho(r), \sigma)$ $(n \geq 1)$. In order to a function $f_0(z) \in [\varrho(r), \sigma)$ can be approximated with arbitrary accuracy by linear combinations of functions $f_n(z)$ (in the sense of $[\rho(r), \sigma)$) it is necessary and sufficient that equalities

$$l(f_n) = 0 \quad (n \geq 1)\,, \tag{9}$$

where $l(f)$ is an arbitrary continuous linear functional on $[\varrho(r), \sigma)$, imply $l(f_0) = 0$. In particular, the system of functions $\{f_n(z)\}$ is complete in $[\varrho(r), \sigma)$ if and only if equalities (9) imply $l(f) = 0$ for any function $f(z) \in [\varrho(r), \sigma)$.

2) Let N be a closed set in $[\varrho(r), \sigma)$ that does not coincide with the $[\varrho(r), \sigma)$ and $f_0(z) \in [\varrho(r), \sigma) \backslash N$. Then there exists a functional $l(f)$ with the property: $l(f) = 0$ for all $f(z) \in N$ but $l(f_0) \neq 0$.

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