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ANALYTIC IN A POLYDISC FUNCTIONS OF BOUNDED L-INDEX IN JOINT VARIABLES

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A concept of boundedness of **L**-index in joint variables (see in Bordulyak M.T. *The space of entire in* $\mathbb{C}^n$ *functions of bounded L-index,* Mat. Stud. **4** (1995), 53–58; Bandura A. I., Bordulyak M. T., Skaskiv O. B. *Sufficient conditions of boundedness of L-index in joint variables,* Mat. Stud. **45** (2016), 12–26) is generalised for an analytic function in a polydisc. We proved criteria of boundedness of **L**-index in joint variables which describe local behaviour of partial derivative. Some improvements of known sufficient conditions of boundednees of **L**-index in joint variables are obtained.

1. Introduction. B. Lepson ([1]) laid the foundation of theory of entire functions of bounded index. In the paper [2] it was introduced the class of *entire functions of bounded* **L**-*index in joint variables* with $\mathbf{L}(z) = (l_1(z), \ldots, l_n(z))$, $z = (z_1, z_2, \ldots, z_n) \in \mathbb{C}^n$. It was a generalization of the previous concept of entire function of bounded **L**-index in joint variables with $\mathbf{L}(z) = (l_1(z_1), \ldots, l_n(z_n))$, supposed by M. T. Bordulyak and M. M. Sheremeta ([3, 4]). Meanwhile there are known papers of S. N. Strochyk, M. M. Sheremeta, V. O. Kushnir ([5, 6]), devoted to l-index of ananalytic function in a disc or in an arbitrary domain $G \subset \mathbb{C}$. Their investigations are particularized in a monograph of M. M. Sheremeta ([7]) where a bibliography on this topic is given. However, they only considered functions of one complex variable. To the best of our knowledge there are only two papers about analytic functions of bounded index [8, 9] in some domain $\Omega \subset \mathbb{C}^n$ $(n \geq 2)$. In [8] J. Gopala Krishna and S. M. Shah introduced a concept of an analytic function of bounded index for $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n_+$ in a domain (a nonempty connected open set) $\Omega \subset \mathbb{C}^n$ $(n \in \mathbb{N})$. If $\mathbf{L}(z) \equiv (\frac{1}{\alpha_1}, \ldots, \frac{1}{\alpha_n})$ and $\Omega = \mathbb{C}^n$ then Bordulyak-Sheremeta's definition ([3, 4]) coincides with Krishna-Shah's definition. Besides, an analytic in a domain function of bounded index by Krishna and Shah is entire. It follows from necessary condition of l-index boundedness for analytic functions in the unit disc ([7, Th.3.3, p. 71]): $\int_0^r l(t)dt \to \infty$ as $r \to 1$.

In the other above-mentioned preprint [9] authors proposed a generalization of the concept of analytic in a domain function of bounded index, which was introduced by J. G. Krishna and S. M. Shah. We used slice functions to explore properties of analytic in the unit ball functions of bounded L-index in direction. This approach is well studied for entire functions in [10]–[15].

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For analytic functions in the unit ball we proved necessary and sufficient conditions of boundedness of L-index in direction for analytic functions, got sufficient conditions of boundedness of L-index in direction for analytic solutions of PDE, and estimated growth of the functions, etc. Thus, the method of slices fits as for entire functions in $\mathbb{C}^n$ as for analytic functions in a ball.

Besides a ball, an important geometric object in $\mathbb{C}^n$ is a polydisc. Above we noted that an analytic function of bounded index by Krishna and Shah is entire. On the other hand, there was not a flexible definition of the conception of bounded index for analytic functions of several variables (in particular, by the approaches of M. M. Sheremera, M. T. Bordulyak, M. Salmassi, F. Nuray, R. Patterson, B. C. Chakraborty, see [3, 4], [16]–[18], [19]–[21]). Thus, necessity arises *to introduce and to investigate analytic in polydisc functions of bounded* **L**-*index in joint variables.*

2. Main definitions and notation. We need some standard notation. Denote $\mathbb{R}_+ = [0,+\infty)$, $\mathbf{0} = (0,\dots,0) \in \mathbb{R}^n_+$, $\mathbf{e} = (1,\dots,1) \in \mathbb{R}^n_+$, $R = (r_1,\dots,r_n) \in \mathbb{R}^n_+$, $z = (z_1,\dots,z_n) \in \mathbb{C}^n$. For $A = (a_1,\dots,a_n) \in \mathbb{R}^n$, $B = (b_1,\dots,b_n) \in \mathbb{R}^n$ we will use formal notations without violation of the existence of these expressions $AB = (a_1b_1,\cdots,a_nb_n)$, $A/B = (a_1/b_1,\dots,a_n/b_n)$, $A^B = a_1^{b_1}a_2^{b_2}\cdot\ldots\cdot a_n^{b_n}$. The notation $A < B$ means that $a_j < b_j$, $j \in \{1,\dots,n\}$; the relation $A \le B$ is defined similarly. For $K = (k_1,\dots,k_n) \in \mathbb{Z}^n_+$ denote $\|K\| = k_1+\dots+k_n$, $K! = k_1!\cdot\ldots\cdot k_n!$.

The polydisc $\{z \in \mathbb{C}^n\colon\ |z_j - z_j^0| < r_j,\ j = 1,\dots,n\}$ is denoted by $\mathbb{D}^n(z^0,R)$, its skeleton $\{z \in \mathbb{C}^n\colon\ |z_j - z_j^0| = r_j,\ j = 1,\dots,n\}$ is denoted by $\mathbb{T}^n(z^0,R)$, and the closed polydisc $\{z \in \mathbb{C}^n\colon\ |z_j - z_j^0| \le r_j,\ j = 1,\dots,n\}$ is denoted by $\mathbb{D}^n[z^0,R]$, $\mathbb{D}^n = \mathbb{D}^n(\mathbf{0},\mathbf{1})$, $\mathbb{D} = \{z \in \mathbb{C}\colon\ |z| < 1\}$. For $K = (k_1,\dots,k_n) \in \mathbb{Z}^n_+$ and the partial derivatives of function $F(z) = F(z_1,\dots,z_n)$ we use the notation

$$F^{(K)}(z) = \frac{\partial^{\|K\|}F}{\partial z^K} = \frac{\partial^{k_1+\dots+k_n}f}{\partial z_1^{k_1}\dots\partial z_n^{k_n}}.$$

Let $\mathbf{L}(z) = (l_1(z),\dots,l_n(z))$, where $l_j(z)\colon \mathbb{D}^n \to \mathbb{R}_+$ is a continuous function such that

$$(\forall z \in \mathbb{D}^n)\colon\ l_j(z) > \beta/(1-|z_j|),\ j \in \{1,\dots,n\},$$

where $\beta > 1$ is some constant, $\boldsymbol{\beta} := (\beta,\dots,\beta) \in \mathbb{R}^n$. S. N. Strochyk, M. M. Sheremeta, V. O. Kushnir ([5]–[7]) imposed a similar condition for function $l\colon \mathbb{D} \to \mathbb{R}_+$ and $l\colon G \to \mathbb{R}_+$, where G is an arbitrary domain in $\mathbb{C}$.

An analytic function $F\colon \mathbb{D}^n \to \mathbb{C}$ is called a function of *bounded* **L**-*index (in joint variables),* if there exists $n_0 \in \mathbb{Z}_+$ such that for all $z \in \mathbb{D}^n$ and for all $J \in \mathbb{Z}^n_+$

$$\frac{|F^{(J)}(z)|}{J!\mathbf{L}^J(z)} \le \max\left\{\frac{|F^{(K)}(z)|}{K!\mathbf{L}^K(z)}\colon\ K \in \mathbb{Z}^n_+,\ \|K\| \le n_0\right\}. \tag{1}$$

The least such integer n_0 is called the **L**-*index in joint variables of the function* F and is denoted by $N(F,\mathbf{L},\mathbb{D}^n) = n_0$. It is an analog of the definition of an entire function of bounded **L**-index in joint variables in $\mathbb{C}^n$ (see [2]–[4]).

By $Q^n(\mathbb{D}^n)$ we denote the class of functions **L** which satisfy the condition

$$(\forall r_j \in [0,\beta],\ j \in \{1,\dots,n\})\colon\ 0 < \lambda_{1,j}(R) \le \lambda_{2,j}(R) < \infty,$$

where

$$\lambda_{1,j}(R) = \inf_{z^0\in\mathbb{D}^n} \inf\left\{\frac{l_j(z)}{l_j(z^0)} \colon z\in\mathbb{D}^n\left[z^0, R/\mathbf{L}(z^0)\right]\right\}, \tag{2}$$

$$\lambda_{2,j}(R) = \sup_{z^0\in\mathbb{D}^n} \sup\left\{\frac{l_j(z)}{l_j(z^0)} \colon z\in\mathbb{D}^n\left[z^0, R/\mathbf{L}(z^0)\right]\right\}. \tag{3}$$

Example 1. The function $F(z)=\exp\{\frac{1}{(1-z_1)(1-z_2)}\}$ has bounded $\mathbf{L}$-index in joint variables with $\mathbf{L}(z)=\big(\frac{1}{(1-|z_1|)^2(1-|z_2|)}, \frac{1}{(1-|z_1|)(1-|z_2|)^2}\big)$ and $N(F,\mathbf{L},\mathbb{D}^n)=0$.

3. Behaviour of derivatives of function of bounded L-index in joint variables. The following theorem is basic in the theory of functions of bounded index. It was necessary to prove more usable criteria of index boundedness which describe a behaviour of maximum modulus on a disc or behaviour of the logarithmic derivative (see [7, 11]). Denote $\mathcal{B}=(0,\beta]$ and $\mathcal{B}^n=(0,\beta]\times\ldots\times(0,\beta]$, where $\times$ stands for the Cartesian product.

Theorem 1. *Let $\mathbf{L}\in Q^n(\mathbb{D}^n)$. An analytic in $\mathbb{D}^n$ function F has bounded $\mathbf{L}$-index in joint variables if and only if for each $R\in\mathcal{B}^n$ there exist $n_0\in\mathbb{Z}_+$, $p_0>0$ such that for every $z^0\in\mathbb{D}^n$ there exists $K^0\in\mathbb{Z}^n_+$, $\|K^0\|\le n_0$, and*

$$\max\left\{\frac{1}{K!}\cdot\frac{|F^{(K)}(z)|}{\mathbf{L}^K(z)} \colon \|K\|\le n_0,\ z\in\mathbb{D}^n\left[z^0, R/\mathbf{L}(z^0)\right]\right\} \le \frac{p_0}{K^0!}\cdot\frac{|F^{(K^0)}(z^0)|}{\mathbf{L}^{K^0}(z^0)}. \tag{4}$$

Proof. Let F be of bounded $\mathbf{L}$-index in joint variables with $N=N(F,\mathbf{L},\mathbb{D}^n)<\infty$. For every $r_j\in(0,\beta]$, $j\in\{1,\ldots,n\}$ we put

$$q=q(R)=\lfloor 2(N+1)\|R\|\prod_{j=1}^n(\lambda_{1,j}(R))^{-N}(\lambda_{2,j}(R))^{N+1}\rfloor+1$$

where $\lfloor x\rfloor$ is the entire part of the real number x, i.e. it is the floor function. For $p\in\{0,\ldots,q\}$ and $z^0\in\mathbb{D}^n$ we denote

$$S_p(z^0,R)=\max\left\{\frac{|F^{(K)}(z)|}{K!\mathbf{L}^K(z)} \colon \|K\|\le N, z\in\mathbb{D}^n\left[z^0, \frac{pR}{q\mathbf{L}(z^0)}\right]\right\},$$

$$S^*_p(z^0,R)=\max\left\{\frac{|F^{(K)}(z)|}{K!\mathbf{L}^K(z^0)} \colon \|K\|\le N, z\in\mathbb{D}^n\left[z^0, \frac{pR}{q\mathbf{L}(z^0)}\right]\right\}.$$

Using (2) and the inclusion $\mathbb{D}^n\left[z^0,\frac{pR}{q\mathbf{L}(z^0)}\right]\subset\mathbb{D}^n\left[z^0,\frac{R}{\mathbf{L}(z^0)}\right]$, we have

$$S_p(z^0,R)=\max\left\{\frac{|F^{(K)}(z)|}{K!\mathbf{L}^K(z)}\frac{\mathbf{L}^K(z^0)}{\mathbf{L}^K(z^0)} \colon \|K\|\le N, z\in\mathbb{D}^n\left[z^0, \frac{pR}{q\mathbf{L}(z^0)}\right]\right\}\le$$
$$\le S^*_p(z^0,R)\max\left\{\prod_{j=1}^n\frac{l_j^N(z^0)}{l_j^N(z)} \colon z\in\mathbb{D}^n\left[z^0, \frac{pR}{q\mathbf{L}(z^0)}\right]\right\}\le S^*_p(z^0,R)\prod_{j=1}^n(\lambda_{1,j}(R))^{-N}.$$

Then using (3), we obtain

$$S^*_p(z^0,R)=\max\left\{\frac{|F^{(K)}(z)|}{K!\mathbf{L}^K(z)}\frac{\mathbf{L}^K(z)}{\mathbf{L}^K(z^0)} \colon \|K\|\le N, z\in\mathbb{D}^n\left[z^0, \frac{pR}{q\mathbf{L}(z^0)}\right]\right\}\le$$

$$\leq \max\left\{\frac{|F^{(K)}(z)|}{K!\mathbf{L}^K(z)}(\Lambda_2(R))^K\colon \|K\|\leq N, z\in\mathbb{D}^n\left[z^0, \frac{pR}{q\mathbf{L}(z^0)}\right]\right\}\leq S_p(z^0,R)\prod_{j=1}^n(\lambda_{2,j}(R))^N. \quad (5)$$

Let $K^{(p)}$ with $\|K^{(p)}\|\leq N$ and $z^{(p)}\in\mathbb{D}^n\left[z^0, \frac{pR}{q\mathbf{L}(z^0)}\right]$ be such that

$$S_p^*(z^0,R)=\frac{|F^{(K^{(p)})}(z^{(p)})|}{K^{(p)}!\mathbf{L}^{K^{(p)}}(z^0)}. \quad (6)$$

Since, by the maximum principle, $z^{(p)}\in\mathbb{T}^n(z^0, \frac{pR}{q\mathbf{L}(z^0)})$, we have $z^{(p)}\neq z^0$. We choose

$$\widetilde{z}_j^{(p)}=z_j^0+\frac{p-1}{p}(z_j^{(p)}-z_j^0).$$

Then for every $j\in\{1,\dots,n\}$ we have that

$$|\widetilde{z}_j^{(p)}-z_j^0|=\frac{p-1}{p}|z_j^{(p)}-z_j^0|=\frac{p-1}{p}\frac{pr_j}{ql_j(z^0)}, \quad (7)$$

$$|\widetilde{z}_j^{(p)}-z_j^{(p)}|=|z_j^0+\frac{p-1}{p}(z_j^{(p)}-z_j^0)-z_j^{(p)}|=\frac{1}{p}|z_j^0-z_j^{(p)}|=\frac{1}{p}\frac{pr_j}{ql_j(z^0)}=\frac{r_j}{ql_j(z^0)}. \quad (8)$$

By (7) we obtain $\widetilde{z}^{(p)}\in\mathbb{D}^n\left[z^0, \frac{(p-1)R}{q(R)\mathbf{L}(z^0)}\right]$ and

$$S_{p-1}^*(z^0,R)\geq\frac{|F^{(K^{(p)})}(\widetilde{z}^{(p)})|}{K^{(p)}!\mathbf{L}^{K^{(p)}}(z^0)}.$$

From (6) it follows that

$$0\leq S_p^*(z^0,R)-S_{p-1}^*(z^0,R)\leq\frac{|F^{(K^{(p)})}(z^{(p)})|-|F^{(K^{(p)})}(\widetilde{z}^{(p)})|}{K^{(p)}!\mathbf{L}^{K^{(p)}}(z^0)}=$$

$$=\frac{1}{K^{(p)}!\mathbf{L}^{K^{(p)}}(z^0)}\int_0^1\frac{d}{dt}|F^{(K^{(p)})}(\widetilde{z}^{(p)}+t(z^{(p)}-\widetilde{z}^{(p)}))|dt\leq$$

$$\leq\frac{1}{K^{(p)}!\mathbf{L}^{K^{(p)}}(z^0)}\int_0^1\sum_{j=1}^n|z_j^{(p)}-z_{*j}^{(p)}|\left|\frac{\partial^{\|K^{(p)}\|+1}F}{\partial z_1^{k_1^{(p)}}\dots\partial z_j^{k_j^{(p)}+1}\dots\partial z_n^{k_n^{(p)}}}(\widetilde{z}^{(p)}+t(z^{(p)}-\widetilde{z}^{(p)}))\right|dt=$$

$$=\frac{1}{K^{(p)}!\mathbf{L}^{K^{(p)}}(z^0)}\sum_{j=1}^n|z_j^{(p)}-z_{*j}^{(p)}|\left|\frac{\partial^{\|K^{(p)}\|+1}F}{\partial z_1^{k_1^{(p)}}\dots\partial z_j^{k_j^{(m)}+1}\dots\partial z_n^{k_n^{(p)}}}(\widetilde{z}^{(p)}+t^*(z^{(p)}-\widetilde{z}^{(p)}))\right|, \quad (9)$$

where $0\leq t^*\leq 1, \widetilde{z}^{(p)}+t^*(z^{(p)}-\widetilde{z}^{(p)})\in\mathbb{D}^n(z^0, \frac{pR}{q\mathbf{L}(z^0)})$. For $z\in\mathbb{D}^n(z^0, \frac{pR}{q\mathbf{L}(z^0)})$ and $J\in\mathbb{Z}_+^n$, $\|J\|\leq N+1$ we have

$$\frac{|F^{(J)}(z)|\mathbf{L}^J(z)}{J!\mathbf{L}^J(z^0)\mathbf{L}^J(z)}\leq(\Lambda_2(R))^J\max\left\{\frac{|F^{(K)}(z)|}{K!\mathbf{L}^K(z)}\colon\|K\|\leq N\right\}\leq\prod_{j=1}^n(\lambda_{2,j}(R))^{N+1}(\lambda_{1,j}(R))^{-N}\times$$

$$\times\max\left\{\frac{|F^{(K)}(z)|}{K!\mathbf{L}^K(z^0)}\colon\|K\|\leq N\right\}\leq\prod_{j=1}^n(\lambda_{2,j}(R))^{N+1}(\lambda_{1,j}(R))^{-N}S_p^*(z^0,R).$$

From (9) and (8) we obtain

$$0 \le S_p^*(z^0,R) - S_{p-1}^*(z^0,R) \le$$
$$\le \prod_{j=1}^n (\lambda_{2,j}(R))^{N+1}(\lambda_{1,j}(R))^{-N} S_p^*(z^0,R)\sum_{j=1}^n (k_j^{(p)}+1)l_j(z^0)|z_j^{(p)} - \widetilde{z}_j^{(p)}| =$$
$$= \prod_{j=1}^n (\lambda_{2,j}(R))^{N+1}(\lambda_{1,j}(R))^{-N} \frac{S_p^*(z^0,R)}{q(R)}\sum_{j=1}^n (k_j^{(p)}+1)r_j \le$$
$$\le \prod_{j=1}^n (\lambda_{2,j}(R))^{N+1}(\lambda_{1,j}(R))^{-N} \frac{S_p^*(z^0,R)}{q(R)}(N+1)\|R\| \le \frac{1}{2}S_p^*(z^0,R).$$

This inequality implies $S_p^*(z^0,R) \le 2S_{p-1}^*(z^0,R)$, and in view of inequalities (5) and (6) we have

$$S_p(z^0,R) \le 2\prod_{j=1}^n (\lambda_{1,j}(R))^{-N} S_{p-1}^*(z^0,R) \le 2\prod_{j=1}^n (\lambda_{1,j}(R))^{-N}(\lambda_{2,j}(R))^N S_{p-1}(z^0,R).$$

Therefore,

$$\max\left\{\frac{|F^{(K)}(z)|}{K!\mathbf{L}^K(z)} \colon \|K\| \le N, z \in \mathbb{D}^n\left[z^0, \frac{pR}{q\mathbf{L}(z^0)}\right]\right\} = S_q(z^0,R) \le$$
$$\le 2\prod_{j=1}^n (\lambda_{1,j}(R))^{-N}(\lambda_{2,j}(R))^N S_{q-1}(z^0,R) \le \ldots \le (2\prod_{j=1}^n (\lambda_{1,j}(R))^{-N}(\lambda_{2,j}(R))^N)^q S_0(z^0,R) =$$
$$= \left(2\prod_{j=1}^n (\lambda_{1,j}(R))^{-N}(\lambda_{2,j}(R))^N\right)^q \max\left\{\frac{|F^{(K)}(z^0)|}{K!\mathbf{L}^K(z^0)} \colon \|K\| \le N\right\}. \tag{10}$$

From (10) we obtain inequality (4) with $p_0 = (2\prod_{j=1}^n (\lambda_{1,j}(R))^{-N}(\lambda_{2,j}(R))^N)^q$ and some K^0 with $\|K^0\| \le N$. The necessity of condition (4) is proved.

Now we prove the sufficiency. Suppose that for every $R \in \mathcal{B}^n$ there exist $n_0 \in \mathbb{Z}_+$, $p_0 > 1$ such that for all $z_0 \in \mathbb{D}^n$ and some $K^0 \in \mathbb{Z}_+^n$, $\|K^0\| \le n_0$, the inequality (4) holds.

We write Cauchy's formula as following $\forall z^0 \in \mathbb{D}^n$ $\forall k \in \mathbb{Z}_+^n$ $\forall s \in \mathbb{Z}_+^n$

$$\frac{F^{(K+S)}(z^0)}{S!} = \frac{1}{(2\pi i)^n}\int_{\mathbb{T}^n\left(z^0,\frac{R}{\mathbf{L}(z^0)}\right)} \frac{F^{(K)}(z)}{(z-z^0)^{S+\mathbf{e}}}dz.$$

Therefore, applying (4), we deduce

$$\frac{|F^{(K+S)}(z^0)|}{S!} \le \frac{1}{(2\pi)^n}\int\limits_{\mathbb{T}^n\left(z^0,\frac{R}{\mathbf{L}(z^0)}\right)} \frac{|F^{(K)}(z)|}{|z-z^0|^{S+\mathbf{e}}}|dz| \le \int\limits_{\mathbb{T}^n\left(z^0,\frac{R}{\mathbf{L}(z^0)}\right)} |F^{(K)}(z)|\frac{\mathbf{L}^{S+\mathbf{e}}(z^0)}{(2\pi)^n R^{S+\mathbf{e}}}|dz| \le$$
$$\le \int\limits_{\mathbb{T}^n\left(z^0,\frac{R}{\mathbf{L}(z^0)}\right)} |F^{(K^0)}(z^0)|\frac{K!p_0\prod_{j=1}^n \lambda_{2,j}^{n_0}(R)\mathbf{L}^{S+K+\mathbf{e}}(z^0)}{(2\pi)^n K^0! R^{S+\mathbf{e}}\mathbf{L}^{K^0}(z^0)}|dz| =$$
$$= |F^{(K^0)}(z^0)|\frac{K!p_0\prod_{j=1}^n \lambda_{2,j}^{n_0}(R)\mathbf{L}^{S+K}(z^0)}{K^0! R^S\mathbf{L}^{K^0}(z^0)}.$$

This implies

$$\frac{|F^{(K+S)}(z^0)|}{(K+S)!\mathbf{L}^{S+K}(z^0)} \leq \frac{\prod_{j=1}^n \lambda_{2,j}^{n_0}(R) p_0 K! S!}{(K+S)! R^S} \frac{|F^{(K^0)}(z^0)|}{K^0! \mathbf{L}^{K^0}(z^0)}. \tag{11}$$

Obviously,

$$\frac{K!S!}{(K+S)!} = \frac{s_1!}{(k_1+1)\cdot \ldots \cdot (k_1+s_1)} \cdots \frac{s_n!}{(k_n+1)\cdot \ldots \cdot (k_n+s_n)} \leq 1.$$

We choose $r_j \in (1, \beta]$, $j \in \{1, \ldots, n\}$. Hence, $\frac{p_0 \prod_{j=1}^n \lambda_{2,j}^{n_0}(R)}{R^S} \to 0$ as $\|S\| \to +\infty$. Thus, there exists s_0 such that for all $S \in \mathbb{Z}_+^n$ with $\|S\| \geq s_0$ the inequality

$$\frac{p_0 K! S! \prod_{j=1}^n \lambda_{2,j}^{n_0}(R)}{(K+S)! R^S} \leq 1$$

holds. Inequality (11) yields $\frac{|F^{(K+S)}(z^0)|}{(K+S)!\mathbf{L}^{K+S}(z^0)} \leq \frac{|F^{(K^0)}(z^0)|}{K^0!\mathbf{L}^{K^0}(z^0)}$. This means that for every $j \in \mathbb{Z}_+^n$

$$\frac{|F^{(J)}(z^0)|}{J!\mathbf{L}^J(z^0)} \leq \max\left\{ \frac{|F^{(K)}(z^0)|}{K!\mathbf{L}^K(z^0)} \colon \|K\| \leq s_0 + n_0 \right\}$$

where s_0 and n_0 are independent of z_0. Therefore, the function F has bounded $\mathbf{L}$-index in joint variables with $N(F, \mathbf{L}, \mathbb{D}^n) \leq s_0 + n_0$. □

Theorem 2. *Let $\mathbf{L} \in Q^n(\mathbb{D}^n)$. In order that an analytic in $\mathbb{D}^n$ function F be of bounded $\mathbf{L}$-index in joint variables it is necessary that for every $R \in \mathcal{B}^n$ $\exists n_0 \in \mathbb{Z}_+$ $\exists p \geq 1$ $\forall z^0 \in \mathbb{D}^n$ $\exists K^0 \in \mathbb{Z}_+^n$, $\|K^0\| \leq n_0$, and*

$$\max\left\{ |F^{(K^0)}(z)| \colon z \in \mathbb{D}^n\left[z^0, R/\mathbf{L}(z^0)\right] \right\} \leq p|F^{(K^0)}(z^0)| \tag{12}$$

and it is sufficient that for every $R \in \mathcal{B}^n$ $\exists n_0 \in \mathbb{Z}_+$ $\exists p \geq 1$ $\forall z^0 \in \mathbb{D}^n$ $\forall j \in \{1, \ldots, n\}$ $\exists K_j^0 = (0, \ldots, 0, \underbrace{k_j^0}_{j\text{-th place}}, 0, \ldots, 0)$ such that $k_j^0 \leq n_0$ and

$$\max\left\{ |F^{(K_j^0)}(z)| \colon z \in \mathbb{D}^n\left[z^0, R/\mathbf{L}(z^0)\right] \right\} \leq p|F^{(K_j^0)}(z^0)| \ \forall j \in \{1, \ldots, n\}, \tag{13}$$

Proof. The proof of Theorem 1 implies that the inequality (4) is true for some K^0. Therefore, we have

$$\frac{p_0}{K^0!} \frac{|F^{(K^0)}(z^0)|}{\mathbf{L}^{K^0}(z^0)} \geq \max\left\{ \frac{1}{K^0!} \frac{|F^{(K^0)}(z)|}{\mathbf{L}^{K^0}(z)} \colon z \in \mathbb{D}^n\left[z^0, R/\mathbf{L}(z^0)\right] \right\} =$$
$$= \max\left\{ \frac{|F^{(K^0)}(z)|}{K^0!} \frac{\mathbf{L}^{K^0}(z^0)}{\mathbf{L}^{K^0}(z^0)\mathbf{L}^{K^0}(z)} \colon z \in \mathbb{D}^n\left[z^0, R/\mathbf{L}(z^0)\right] \right\} \geq$$
$$\geq \max\left\{ \frac{|F^{(K^0)}(z)|}{K^0!} \frac{\prod_{j=1}^n (\lambda_{2,j}(R))^{-n_0}}{\mathbf{L}^{K^0}(z^0)} \colon z \in \mathbb{D}^n\left[z^0, R/\mathbf{L}(z^0)\right] \right\}.$$

This inequality implies

$$\frac{p_0\prod_{j=1}^n(\lambda_{2,j}(R))^{n_0}}{K^0!}\frac{|F^{(K^0)}(z^0)|}{\mathbf{L}^{K^0}(z^0)}\geq \max\left\{\frac{1}{K^0!}\frac{|F^{(K^0)}(z)|}{\mathbf{L}^{K^0}(z^0)}\colon z\in\mathbb{D}^n\left[z^0,R/\mathbf{L}(z^0)\right]\right\}. \qquad (14)$$

From (14) we obtain inequality (12) with $p=p_0\prod_{j=1}^n(\lambda_{2,j}(R))^{n_0}$. The necessity of condition (12) is proved.

Now we prove the sufficiency of (13). Suppose that for every $R\in\mathcal{B}^n$ $\exists n_0\in\mathbb{Z}_+, p>1$ such that $\forall z_0\in\mathbb{D}^n$ and some $K_J^0\in\mathbb{Z}_+^n$ with $k_j^0\leq n_0$ the inequality (13) holds.

We write Cauchy's formula as following $\forall z^0\in\mathbb{D}^n$ $\forall s\in\mathbb{Z}_+^n$

$$\frac{F^{(K_J^0+S)}(z^0)}{S!}=\frac{1}{(2\pi i)^2}\int\limits_{\mathbb{T}^n(z^0,R/\mathbf{L}(z^0))}\frac{F^{(K_J^0)}(z)}{(z-z^0)^{S+\mathbf{e}}}dz.$$

This yields

$$\frac{|F^{(K_j^0+S)}(z^0)|}{S!}\leq\frac{1}{(2\pi)^2}\int\limits_{\mathbb{T}^n(z^0,R/\mathbf{L}(z^0))}\frac{|F^{(K_j^0)}(z)|}{|z-z^0|^{S+\mathbf{e}}}|dz|\leq$$

$$\leq\frac{1}{(2\pi)^n}\int\limits_{\mathbb{T}^n(z^0,R/\mathbf{L}(z^0))}\max\{|F^{(K_j^0)}(z)|\colon z\in\mathbb{D}^n\}\frac{\mathbf{L}^{S+\mathbf{e}}(z^0)}{R^{S+\mathbf{e}}}|dz|=$$

$$=\max\{|F^{(K_j^0)}(z)|\colon z\in\mathbb{D}^n\}\frac{\mathbf{L}^S(z^0)}{R^S}.$$

Now we put $R=\boldsymbol{\beta}$ and use (13)

$$\frac{|F^{(K_j^0+S)}(z^0)|}{S!}\leq\frac{\mathbf{L}^S(z^0)}{\beta^{\|S\|}}\max\{|F^{(K_j^0)}(z)|\colon z\in\mathbb{D}^n\}\leq\frac{p\mathbf{L}^S(z^0)}{\beta^{\|S\|}}|F^{(K_j^0)}(z^0)|. \qquad (15)$$

We choose $S\in\mathbb{Z}_+^n$ such that $\|S\|\geq s_0$, where $\frac{p}{\beta^{s_0}}\leq 1$. Therefore (15) implies that for all $j\in\{1,\ldots,n\}$ and $k_j^0\leq n_0$

$$\frac{|F^{(K_j^0+S)}(z^0)|}{\mathbf{L}^{K_j^0+S}(z^0)(K_j^0+S)!}\leq\frac{p}{\beta^{\|S\|}}\frac{S!K_j^0!}{(S+K_j^0)!}\frac{|F^{(K_j^0)}(z^0)|}{\mathbf{L}^{K_j^0}(z^0)K_j^0!}\leq\frac{|F^{(K_j^0)}(z^0)|}{\mathbf{L}^{K_j^0}(z^0)K_j^0!}.$$

Consequently, $N(F,\mathbf{L},\mathbb{D}^n)\leq n_0+s_0$. □

Remark 1. Inequality (12) is a necessary and sufficient condition of boundedness of l-index for functions of one variable [7, 6, 22]. But it is unknown whether this condition is sufficient condition of boundedness of $\mathbf{L}$-index in joint variables. Our restrictions (13) are corresponding multidimensional sufficient conditions. Moreover, assumptions (13) provide boundedness of l_j-index in the directions $\mathbf{e}_j$ (see definition and properties for entire functions in [10, 11]). As a matter of fact, we implicitly deduce the assertion similar to Theorem 6 in [2]. The theorem states that if an entire function F in $\mathbb{C}^n$ has bounded l_j-index in a direction e_j for every $j\in\{1,...,n\}$, then F is of bounded $\mathbf{L}$-index in joint variables, where $\mathbf{L}=(l_1,\ldots,l_n)$, $\mathbf{e}_j=(0,\ldots,0,\underbrace{1}_{j\text{–th place}},0,\ldots,0)$.

Denote $\widetilde{\mathbf{L}}(z)=(\widetilde{l}_1(z),\ldots,\widetilde{l}_n(z))$. $\mathbf{L}\asymp\widetilde{\mathbf{L}}$ means that there exist $\Theta_1=(\theta_{1,j},\ldots,\theta_{1,n})\in\mathbb{R}^n_+$, $\Theta_2=(\theta_{2,j},\ldots,\theta_{2,n})\in\mathbb{R}^n_+$ such that $\forall z\in\mathbb{D}^n$ $\theta_{1,j}\widetilde{l}_j(z)\le l_j(z)\le\theta_{2,j}\widetilde{l}_j(z)$, $j\in\{1,\ldots,n\}$.

Theorem 3. *Let $\mathbf{L}\in Q^n(\mathbb{D}^n)$ and $\mathbf{L}\asymp\widetilde{\mathbf{L}}$. An analytic function F in $\mathbb{D}^n$ has bounded $\widetilde{\mathbf{L}}$-index in joint variables if and only if it has bounded $\mathbf{L}$-index in joint variables.*

Proof. It is easy to prove that if $\mathbf{L}\in Q^n(\mathbb{D}^n)$ and $\mathbf{L}\asymp\widetilde{\mathbf{L}}$ then $\widetilde{\mathbf{L}}\in Q^n(\mathbb{D}^n)$.

Let $N(F,\widetilde{\mathbf{L}},\mathbb{D}^n)=\widetilde{n}_0<+\infty$. Then by Theorem 1 for every $\widetilde{R}=(\widetilde{r}_1,\ldots,\widetilde{r}_n)\in\mathcal{B}^n$ there exists $\widetilde{p}\ge 1$ such that for each $z^0\in\mathbb{D}^n$ and some K^0 with $\|K^0\|\le\widetilde{n}_0$, the inequality (4) holds with $\widetilde{\mathbf{L}}$ and $\widetilde{R}$ instead of $\mathbf{L}$ and R. Hence

$$\frac{\widetilde{p}}{K^0!}\frac{|F^{(K^0)}(z^0)|}{\mathbf{L}^{K^0}(z^0)}=\frac{\widetilde{p}}{K^0!}\frac{\Theta_2^{K^0}|F^{(K^0)}(z^0)|}{\Theta_2^{K^0}\mathbf{L}^{K^0}(z^0)}\ge\frac{\widetilde{p}}{K^0!}\frac{|F^{(K^0)}(z^0)|}{\Theta_2^{K^0}\widetilde{\mathbf{L}}^{K^0}(z^0)}\ge$$
$$\ge\frac{1}{\Theta_2^{K^0}}\max\left\{\frac{|F^{(K)}(z)|}{K!\widetilde{L}^K(z)}\colon\|K\|\le\widetilde{n}_0, z\in\mathbb{D}^n\left[z^0,\widetilde{R}/\widetilde{\mathbf{L}}(z)\right]\right\}\ge$$
$$\ge\frac{1}{\Theta_2^{K^0}}\max\left\{\frac{\Theta_1^K|F^{(K)}(z)|}{K!L^K(z)}\colon\|K\|\le\widetilde{n}_0, z\in\mathbb{D}^n\left[z^0,\widetilde{R}/\widetilde{\mathbf{L}}(z)\right]\right\}\ge$$
$$\ge\frac{\min\limits_{0\le\|K\|\le n_0}\{\Theta_1^K\}}{\Theta_2^{K^0}}\max\left\{\frac{|F^{(K)}(z)|}{K!L^K(z)}\colon\|K\|\le\widetilde{n}_0, z\in\mathbb{D}^n\left[z^0,\widetilde{R}/\widetilde{\mathbf{L}}(z)\right]\right\}.$$

In view of Theorem 1 we obtain that the function F has bounded $\mathbf{L}$-index. □

Theorem 4. *Let $\mathbf{L}\in Q^n(\mathbb{D}^n)$. An analytic function F in $\mathbb{D}^n$ has bounded $\mathbf{L}$-index in joint variables if and only if there exist $R\in\mathcal{B}^n$, $n_0\in\mathbb{Z}_+$, $p_0>1$ such that for each $z^0\in\mathbb{D}^n(z^0,R)$ and for some $K^0\in\mathbb{Z}^n_+$ with $\|K^0\|\le n_0$ the inequality (4) holds.*

Proof. The necessity of this theorem follows from the necessity of Theorem 1. We prove the sufficiency. The proof of Theorem 1 with $R=\boldsymbol{\beta}$ implies that $N(F,\mathbf{L},\mathbb{D}^n)<+\infty$.

Let $\mathbf{L}^*(z)=\frac{R_0\mathbf{L}(z)}{R}$, $R^0=\boldsymbol{\beta}$. In general case from validity of (4) for F and $\mathbf{L}$ with $R=(r_1,\ldots,r_n)$, $r_j<\beta$, $j\in\{1,\ldots,n\}$ we obtain

$$\max\left\{\frac{|F^{(K)}(z)|}{K!(R_0\mathbf{L}(z)/R)^K}\colon\|K\|\le n_0, z\in\mathbb{D}^n\left[z^0,R_0/\mathbf{L}^*(z^0)\right]\right\}\le$$
$$\le\max\left\{\frac{|F^{(K)}(z)|}{K!\mathbf{L}^K(z)}\colon\|K\|\le n_0, z\in\mathbb{D}^n\left[z^0,R/\mathbf{L}(z^0)\right]\right\}\le$$
$$\le\frac{p_0}{K^0!}\frac{|F^{(K^0)}(z^0)|}{\mathbf{L}^{K^0}(z^0)}=\frac{\beta^{\|K^0\|}p_0}{R^{K^0}K^0!}\frac{|F^{(K^0)}(z)|}{(R_0\mathbf{L}(z)/R)^{K^0}}<\frac{p_0\beta^{nn_0}}{\prod_{j=1}^n r_j^{n_0}}\frac{|F^{(K^0)}(z)|}{K^0!(\mathbf{L}^*(z))^{K^0}}.$$

i. e. (4) holds for $F,\mathbf{L}^*$ and $R_0=(\beta,\ldots,\beta)$. Now as above we apply Theorem 1 to the function $F(z)$ and $\mathbf{L}^*(z)=R_0\mathbf{L}(z)/R$. This implies that F is of bounded $\mathbf{L}^*$-index in joint variables. Therefore, by Theorem 3 the function F has bounded $\mathbf{L}$-index in joint variables. □

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