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ON THE TWO-MEMBER ASYMPTOTIC OF YOUNG CONJUGATED FUNCTIONS

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Let $P\colon (0;+\infty)\to+\infty$ with $p\not\equiv-\infty$. A connection between behavior of P and the growth of the function $Q(\sigma)=\sup\{P(t)+\sigma t\colon t\ge 0\}$ is in the term of the two-member generalized asymptotic.

1. Introduction. Let P be an arbitrary function different from $+\infty$ (it can achieve the value $-\infty$ but $P\not\equiv-\infty$) and let $Q(\sigma)=\sup\{P(t)+t\sigma\colon\ t\ge 0\}$. The functions P and Q are said to be *Young conjugated functions.* If we put $A=\sup\{\sigma\colon\ Q(\sigma)<+\infty\}$ then

$$A=-\varlimsup_{t\to+\infty}P(t)/t.$$

Indeed, if $\sigma<A$ then $P(t)+t,\sigma<Q(\sigma)$ and, thus, $P(t)/t<-\sigma+Q(\sigma)/t$ for all $t\ge 0$, whence

$$\varlimsup_{t\to+\infty}P(t)/t<-\sigma,$$

that is in view of the arbitrariness of σ we have

$$\varlimsup_{t\to+\infty}P(t)/t\le -A.$$

On the other hand, if $\sigma>A$ then $Q(\sigma)=+\infty$ and, therefore, for every $K\in(0,\,+\infty)$ there exists a sequence $(t_k)\uparrow+\infty$ such that $P(t_k)+\sigma t_k\ge K$, that is $P(t_k)/t_k\ge-\sigma+o(1)$ as $k\to\infty$, whence $\varlimsup\limits_{t\to+\infty}P(t)/t\ge -A$.

Suppose that $-\infty<A\le+\infty$ and by $\Omega(A)$ we denote the class of positive unbounded on $(-\infty,A)$ functions Φ such that the derivative Φ' is positive continuously differentiable and increasing to $+\infty$ on $(-\infty,A)$. From now on, we denote by φ the inverse function to Φ', and let $\Psi(x)=x-\Phi(x)/\Phi'(x)$ be the function associated with Φ in the sense of Newton. It is clear that the function φ is continuously differentiable and increasing to A on $(0,+\infty)$. The function Ψ is [1, p. 30; 2–3] continuously differentiable and increasing to A on $(-\infty,A)$. The following lemmas are proved in [3] and [1, p. 30–45].

Lemma 1. *Let $A\in(-\infty,+\infty]$ and $\Phi\in\Omega(A)$. In order that $Q(\sigma)\le\Phi(\sigma)$ for all $\sigma\in[\sigma_0,\,A)$, it is necessary and sufficient that* $\ln\,P(t)\le -t\Psi(\varphi(t))$ *for all* $t\ge t_0$.

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Lemma 2. *Let $A \in (-\infty, +\infty]$ and $\Phi \in \Omega(A)$. If $P(t_k) \geq -t_k \Psi(\varphi(t_k))$ for some sequence (t_k) of positive numbers increasing to $+\infty$ then for all $k \geq k_0$ and all $\sigma \in [\varphi(t_k), \varphi(t_{k+1})]$*

$$Q(\sigma) \geq \Phi(\sigma) + G_1(t_k, t_{k+1}, \Phi) - G_2(t_k, t_{k+1}, \Phi),$$

where ([1, p. 34; 4])

$$G_1(a, b, \Phi) =: \frac{ab}{b-a} \int_a^b \frac{\Phi(\varphi(qt))}{t^2}\, dt < G_2(a, b, \Phi) =: \Phi\left(\frac{1}{b-a}\int_a^b \varphi(qt)\ dt\right),$$

for $0 < a < b < +\infty$.

Lemma 3. *Let $A \in (-\infty, +\infty]$, $\Phi_* \in \Omega(A)$, $\Phi^* \in \Omega(A)$ and $\Phi_*(\sigma) \leq Q(\sigma) \leq \Phi^*(\sigma)$ for all $\sigma \in (\sigma_0, A)$. Then $P(t) \leq -t\Psi^*(\varphi^*(t))$ for all $t > t_0$ and there exists an increasing to $+\infty$ sequence (t_k) of positive numbers such that $P(t_k) \geq -t_k \Psi_*(\varphi_*(t_k))$ and*

$$G_1(t_k, t_{k+1}, \Phi^*) \geq \Phi_1(\varkappa_k(\Phi^*)), \quad \varkappa_k(\Phi) = \frac{1}{t_{k+1} - t_k} \int_{t_k}^{t_{k+1}} \varphi(t)\, dt,$$

where by Ψ_*, Ψ^*, φ_* and φ^* we denote the functions which correspond to Φ_* and Φ^*.

Using the lemmas, in [5] it is found conditions on P under which for example, Q has two-member exponential asymptotic

$$Q(\sigma) = Te^{\rho\sigma} + (1 + o(1))\tau e^{\rho\sigma} \quad (\sigma \to +\infty), \quad T > 0, 0 < \rho_1 < \rho < +\infty, \tau \in \mathbb{R},$$

Q has two-member exponential asymptotics

$$Q(\sigma) = T\sigma^p + (1 + o(1))\tau\sigma^{p_1} \quad (\sigma \to +\infty), \quad T > 0, p > 1, 0 < p_1 < p, \tau \in \mathbb{R},$$

or

$$Q(\sigma) = \frac{T}{|\sigma|^p} + \frac{(1 + o(1))\tau}{|\sigma|^{p_1}} \quad (\sigma \uparrow 0), \quad T > 0, 0 < p_1 < p < +\infty, \tau \in \mathbb{R}.$$

For $A = +\infty$ a general two-member asymptotic of Q is studied in [6–7].

Let L^0 be the class of positive continuously differentiable on $(0, +\infty)$ functions l such that $xl'(x) = O(l(x))$ as $x \to +\infty$. We remark that if $l \in L^0$ then $l((1 + o(1))x) = (1 + o(1))l(x)$ as $x \to +\infty$.

As in [6] we will say that a positive twice continuously differentiable increasing to $+\infty$ on $(-\infty, +\infty)$ function Φ_2 is subordinated to $\Phi_1 \in \Omega(+\infty)$ if $\Phi_2''(\sigma) = o(\Phi_1''(\sigma))$, $\Phi_2'(\sigma) = o(\sigma\Phi_1''(\sigma))$ as $\sigma \to +\infty$ and $\Phi_2'(\varphi_1) \in L^0$.

Theorem A ([6]). *Let $\Phi_1 \in \Omega(+\infty)$, $\varphi_1' \in L^0$, Φ_2 be subordinated to Φ_1 and $\tau \in \mathbb{R}$. In order that*

$$Q(\sigma) \leq \Phi_1(\sigma) + (1 + o(1))\tau\Phi_1(\sigma), \quad \sigma \to +\infty, \tag{1}$$

it is necessary and sufficient that

$$P(t) \leq -t\Psi_1(\varphi_1(t)) + (1 + o(1))\tau\Phi_2(\varphi_1(t)), \quad t \to +\infty. \tag{2}$$

Theorem B ([6]). *Let $\Phi_1 \in \Omega(+\infty)$, $\varphi_1' \in L^0$, Φ_2 be subordinated to Φ_1, $\tau \in \mathbb{R}$ and*

$$\Phi_j'(\sigma + O(\Phi_2'(\sigma)/\Phi_1''(\sigma))) = (1+o(1))\Phi_j'(\sigma) \quad (\sigma \to +\infty), \quad j = 1, 2. \tag{3}$$

If

$$P(t_n) \geq -t_n\Psi_1(\varphi_1(t_n)) + (1+o(1))\tau_n\Phi_2(\varphi_1(t_n)), \quad n \to +\infty, \tag{4}$$

for a some increasing to $+\infty$ sequence (t_n) such that

$$t_{n+1} = (1+o(1))t_n, \quad n \to \infty, \tag{5}$$

and

$$G_2(t_n, t_{n+1}, \Phi_1) - G_1(t_n, t_{n+1}, \Phi_1) = o(\Phi_2(\varphi_1(t_n))), \quad n \to \infty, \tag{6}$$

then

$$Q(\sigma) \geq \Phi_1(\sigma) + (1+o(1))\tau\Phi_1(\sigma), \quad \sigma \to +\infty. \tag{7}$$

Theorem C ([7]). *Let $\Phi_1 \in \Omega(+\infty)$, $\varphi_1' \in L^0$, Φ_2 be subordinated to Φ_1, $\tau \in \mathbb{R}$, the conditions (3) hold and (5) imply (6). Put*

$$\varkappa_n(\Phi_1) = \frac{1}{t_{n+1} - t_n}\int_{t_n}^{t_{n+1}} \varphi_1(t)dt, \ \xi_n = \frac{\Phi_2(\varphi_1(t_{n+1})) - \Phi_2(\varphi_1(t_{n+1}))}{t_{n+1} - t_n} \tag{8}$$

and suppose that

$$\xi_n\Phi_1'(\varkappa_n(\Phi_1') + O(\xi_n)) = o(G_2(t_n, t_{n+1}, \Phi_1)), \quad n \to \infty, \tag{9}$$

and

$$\Phi_2(\varkappa_n(\Phi_1') + O(\xi_n)) = o(G_2(t_n, t_{n+1}, \Phi_1)), \quad n \to \infty. \tag{10}$$

In order that

$$Q(\sigma) = \Phi_1(\sigma) + (1+o(1))\tau\Phi_1(\sigma), \quad \sigma \to +\infty, \tag{11}$$

it is necessary and sufficient that for every $\varepsilon > 0$:

1) for all $t \geq t_0 = t_0(\varepsilon)$

$$P(t) \leq -t\Psi_1(\varphi_1(t)) + (\tau + \varepsilon)\Phi_2(\varphi_1(t)); \tag{12}$$

2) there exists an increasing to $+\infty$ sequence (t_n) such that

$$P(t_n) \geq -t_n\Psi_1(\varphi_1(t_n)) + (\tau - \varepsilon))\Phi_2(\varphi_1(t_n)), \quad n \to +\infty, \tag{13}$$

and (6) holds.

In view of Theorem C the following problems arise ([7]).

1. For which function $\Phi_1 \in \Omega(+\infty)$ and subordinated to Φ_1 function Φ_2 do relations (9) and (10) hold for every sequence $(t_n) \uparrow +\infty$?

2. For which functions $\Phi_1 \in \Omega(+\infty)$ and Φ_2 does relation (5) imply (6)?

Here we will discuss the set of forth problems and will generalize Theorems 1–3 in the case of any $A \in (-\infty, +\infty]$. Obviously, the general case $A \in (-\infty, +\infty)$ can be reduced to the case $A = 0$ with $\sigma - A$ replacing σ.

2. Discussion of problems. The answer to the first problem is contained in the following proposition.

Proposition 1. *Let either $A = 0$ or $A = +\infty$, $\Phi_1 \in \Omega(A)$ and Φ_2 be a positive twice continuously differentiable function on $(-\infty, A)$ increasing to $+\infty$. Suppose that $\Phi_2'(\sigma) = o(\sigma\Phi_1''(\sigma))$, $\sigma\Phi_1'((1 + o(1))\sigma) = O(\Phi_1(\sigma))$ and $\Phi_2((1 + o(1))\sigma) = o(\Phi_1(\sigma))$ as $\sigma \uparrow A$. Then the relations (9) and (10) hold for every sequence $(t_n) \uparrow +\infty$.*

Proof. The condition $\Phi_2'(\sigma) = o(\sigma\Phi_1''(\sigma))$ as $\sigma \uparrow A$ implies the relation $\Phi_2'(\varphi_1(t)) = o(\varphi_1(t)\times \times\Phi_1''(\varphi_1(t)))$ as $t \to +\infty$. But $\Phi_1''(\varphi_1(t)) = 1/\varphi_1'(t)$. Therefore, $\Phi_2'(\varphi(t))\varphi'(t) = o(\varphi(t))$ as $t \to +\infty$ and, thus, in view of (8)

$$\xi_n = \frac{\Phi_2(\varphi_1(t_{n+1})) - \Phi_2(\varphi_1(t_n))}{t_{n+1} - t_n} = \frac{1}{t_{n+1} - t_n}\int_{t_n}^{t_{n+1}} \Phi_2'(\varphi_1(t))p_1'(t)dt = o(\kappa_n(\Phi_1), \quad n \to \infty.$$

Since $G_2(t_n, t_{n+1}, \Phi_1) = \Phi_1(\kappa_n(\Phi_1))$ and $\sigma\Phi_1'((1 + o(1))\sigma) = O(\Phi_1(\sigma))$ as $\sigma \uparrow A$ hence we obtain (9). From the condition $\Phi_2((1 + o(1))\sigma) = o(\Phi_1(\sigma))$ as $\sigma \uparrow A$ the relation (10) follows. □

The assumptions of Proposition 1 are satisfied if Φ_1 and Φ_2 are power functions, that is $\Phi_1(\sigma) = \sigma^p$ $(p > 1)$, $\Phi_2(\sigma) = \sigma^{p_1}$ $(0 < p_1 < p)$ by $A = +\infty$ and $\Phi_1(\sigma) = 1/|\sigma|^p$ $(p > 0)$, $\Phi_2(\sigma) = 1/|\sigma|^{p_1}$ $(0 < p_1 < p)$ by $A = 0$. The exponential functions $\Phi_1(\sigma) = \exp\{\rho\sigma\}$ $(\rho > 0)$, $\Phi_2(\sigma) = \exp\{\rho_1\sigma\}$ $(0 < \rho_1 < \rho)$ dissatisfy these condition. However, the following statement holds.

Proposition 2. *Let $A = +\infty$, $\Phi_1 \in \Omega(A)$ and Φ_2 be a positive continuously differentiable function on $(-\infty, A)$ increasing to $+\infty$. Suppose that $\Phi_2'(\sigma) = o(\Phi_1''(\sigma))$, $\Phi_1'(\sigma + o(1)) = O(\Phi_1(\sigma))$ and $\Phi_2(\sigma + o(1)) = O(\Phi_1(\sigma))$ as $\sigma \uparrow A$. Then the relations* (9) *and* (10) *hold for every sequence $(t_n) \uparrow +\infty$.*

Proof. The condition $\Phi_2'(\sigma) = o(\Phi_1''(\sigma))$ as $\sigma \uparrow A$ implies the relation $\Phi_2'(\varphi_1(t))\varphi_1'(t) \to 0$ as $t \to +\infty$ and, thus, $\xi_n \to 0$ as $n \to \infty$. Therefore, from the condition $\Phi_1'(\sigma+o(1)) = O(\Phi_1(\sigma))$ we obtain (9) and from the condition $\Phi_2(\sigma + o(1)) = o(\Phi_1(\sigma))$ as $\sigma \uparrow A$, we obtain (10). □

Now we consider the second problem. Suppose that $\Phi_2(\sigma) = o(\Phi_1(\sigma))$ as $\sigma \uparrow A$. Since $G_2(t_n, t_{n+1}, \Phi_1) \geq \Phi_1(\varphi_1(t_n))$ we have $\Phi_2(\varphi_1(t_n)) = o(G_2(t_n, t_{n+1}, \Phi_1))$ as $n \to \infty$ and, therefore, from (6) it follows that

$$G_2(t_n, t_{n+1}, \Phi_1) = (1 + o(1))G_1(t_n, t_{n+1}, \Phi_1), \quad n \to \infty. \tag{14}$$

By $\Omega^*(A)$ we denote the class of functions $\Phi \in \Omega(A)$ such that (14) implies (5). Then the following proposition is true.

Proposition 3. *Let either $A = 0$ or $A = +\infty$, $\Phi_1 \in \Omega^*(A)$, and Φ_2 be a positive continuous function on $(-\infty, A)$ increasing to $+\infty$ such that $\Phi_2(\sigma) = o(\Phi_1(\sigma))$ as $\sigma \uparrow A$. Then* (6) *implies* (5).

The following problem arises: *for which functions $\Phi_1 \in \Omega(A)$ does relation* (14) *implies* (5)*?*

We did not succeed to distinguish the general classes of functions for what (14) implies (5). However the class $\Omega^*(A)$ is nonempty. For example the following functionals belongs to this classes: an exponential function $\Phi(\sigma) = Te^{\rho\sigma}$ and a power function $\Phi(\sigma) = T\sigma^p$ for $\sigma \geq \sigma_0$

if $A = +\infty$ and a power function $\Phi(\sigma) = T|\sigma|^{-\rho}$ if $A = 0$, where $T > 0$, $\rho > 0$ and $p > 1$. We will show it when $T = \rho = 1$ and $p = 2$.

If $\Phi(\sigma) = e^\sigma$ then

$$G_1(t_n, t_{n+1}, \Phi) = \frac{t_n t_{n+1}}{t_{n+1} - t_n} \ln \frac{t_{n+1}}{t_n}, \quad G_2(t_n, t_{n+1}, \Phi) = \exp\left\{\frac{t_{n+1} \ln t_{n+1} - t_n \ln t_n}{t_{n+1} - t_n} - 1\right\}.$$

Therefore, if we put $t_{n+1} = (1 + \theta_n)t_n$ then

$$G_1(t_n, t_{n+1}, \Phi) = t_n \frac{(1 + \theta_n) \ln (1 + \theta_n)}{\theta_n}, \quad G_2(t_n, t_{n+1}, \Phi) = \frac{t_n}{e} \exp\left\{\frac{(1 + \theta_n) \ln (1 + \theta_n)}{\theta_n}\right\}$$

and, thus,

$$G_n =: \frac{G_1(t_n, t_{n+1}, \Phi)}{G_2(t_n, t_{n+1}, \Phi)} = e\eta_n e^{-\eta_n}, \quad \eta_n =: \frac{(1 + \theta_n) \ln (1 + \theta_n)}{\theta_n}.$$

If there exists a sequence $(\theta_{n_j}), \theta_{n_j} \to +\infty$ then $\eta_{n_j} \to +\infty$ and $G_{n_j} \to 0$ as $j \to \infty$. If $\theta_{n_j} \to \theta \in (0, +\infty)$ then $\eta_{n_j} \to \eta =: \frac{(1+\theta)\ln(1+\theta)}{\theta} > 0$ and $G_{n_j} \to e\eta e^{-\eta} < 1$ as $j \to \infty$. Therefore, from (14) it follows that $\theta_n \to 0$ as $n \to \infty$ and, thus, (5) holds.

Now, let $\Phi(\sigma) = \sigma^2$ for $\sigma \ge \sigma_0$ and $t_{n+1} = (1 + \theta_n)t_n$. Then $G_1(t_n, t_{n+1}, \Phi) = t_n t_{n+1}/4 = t_n(1 + \theta_n)/4$, $G_2(t_n, t_{n+1}, \Phi) = (t_n + t_{n+1})^2/16 = t_n^2(2 + \theta_n)^2/16$ and, thus, $G_n = 4(1 + \theta_n)/(2 + \theta_n)^2$, whence, as above, in view of (14) $\theta_n \to 0$ as $n \to \infty$ and, thus, (5) holds.

Finally, let $\Phi(\sigma) = 1/|\sigma|$. Then

$$G_1(t_n, t_{n+1}, \Phi) = \frac{2\sqrt{t_n t_{n+1}}}{\sqrt{t_n} + \sqrt{t_{n+1}}} = \frac{2\sqrt{t_n}\sqrt{1 + \theta_n}}{\sqrt{1 + \theta_n} + 1},$$
$$G_2(t_n, t_{n+1}, \Phi) = (\sqrt{t_n} + \sqrt{t_{n+1}})/2 = \sqrt{t_n}(\sqrt{1 + \theta_n} + 1)/2$$

and $G_n = 4\sqrt{1 + \theta_n}/(\sqrt{1 + \theta_n} + 1)^2$, whence, as above, in view of (14) $\theta_n \to 0$ as $n \to \infty$ and, thus, (5) holds.

3. Preliminary statements. Let $A \in (-\infty, +\infty]$ and $\Phi_1 \in \Omega(A)$. We will say that a positive twice continuously differentiable increasing to $+\infty$ on $(-\infty, A)$ a function Φ_2 is weakly subordinated to $\Phi_1 \in \Omega(+\infty)$ if $\Phi_2''(\sigma) = o(\Phi_1''(\sigma))$ as $\sigma \uparrow A$ and $\Phi_2'(\varphi_1) \in L^0$. We remark that $\Phi_2'(\varphi_1) \in L^0$ iff $\Phi_2''(\sigma)/\Phi_2'(\sigma) = O(\Phi_1''(\sigma)/\Phi_1'(\sigma))$ as $\sigma \uparrow A$.

Let $\tau \in \mathbb{R} \setminus \{0\}$ and either $A = 0$ or $A = +\infty$. Suppose that $\Phi_1 \in \Omega(A)$, $\varphi_1' \in L^0$ and Φ_2 is weakly subordinated to Φ_1. Since $\Phi_2''(\sigma) = o(\Phi_1''(\sigma))$ as $\sigma \uparrow A$, there exists a function $\Phi \in \Omega(A)$ such that

$$\Phi(\sigma) = \Phi_1(\sigma) + \tau\Phi_2(\sigma), \quad \sigma \in [\sigma_0(\tau), A). \tag{15}$$

Lemma 4. *For the function* (15) *the following asymptotic equalities are true*

$$\varphi(t) = \varphi_1(t) - (1 + o(1))\tau\Phi_2'(\varphi_1(t))\varphi_1'(t), \quad t \to +\infty, \tag{16}$$

and

$$t\Psi(\varphi(t)) = t\Psi_1(\varphi_1(t)) - (1 + o(1))\tau\Phi_2(\varphi_1(t)), \quad t \to +\infty. \tag{17}$$

Proof. Clearly, the inverse function φ to Φ satisfies the equation

$$\Phi'(\sigma) + \tau + \Phi_2'(\sigma) = t. \tag{18}$$

Since $\Phi_2'(\sigma) = o(\Phi_1'(\sigma))$ as $\sigma \uparrow A$, we look for a solution of (18) of the form

$$\varphi(t) = \varphi_1(t - \alpha(t)),\ \alpha(t) = o(t)\ (t \to +\infty). \tag{19}$$

Substituting (19) in (18) and taking into account the condition $\Phi_2'(\varphi_1) \in L^0$, we obtain

$$\alpha(t) = \tau + \Phi_2'(\varphi_1(t-\alpha(t))) = \tau\Phi_2'(\varphi_1((1+o(1))t)) = (1+o(1))\tau\Phi_2'(\varphi_1(t)), \quad t \to +\infty. \tag{20}$$

On the other hand, in view the condition $\varphi_1' \in L^0$ for some $\eta = \eta(t) \in [t - \alpha(t), t]$ we have

$$\varphi_1(t) - \varphi_1(t - \alpha(t)) = \varphi_1'(\eta)\alpha(t) = (1 + o(1))\varphi_1'(t)\alpha(t), \quad t \to +\infty.$$

Therefore, (19) and (20) imply (16).

Since $(t\Psi(\varphi(t)))' = \varphi(t)$ from (16) it follows that

$$t\Psi(\varphi(t)) - t_0\Psi(\varphi(t_0)) = t\Psi_1(\varphi_1(t)) - t_0\Psi_1(\varphi_1(t_0)) - (\tau+\varepsilon)(1+o(1))(\Phi_2'(\varphi_1(t)) - \Phi_2'(\varphi_1(t_0)))$$

as $t \to +\infty$. Since $\varphi_1(t) \uparrow A$ as $t \to +\infty$ and $\Phi_2(\sigma) \uparrow +\infty$ as $\sigma \uparrow A$, one has $\Phi_2(\varphi_1(t)) \uparrow +\infty$ as $t \to +\infty$. Therefore, the last inequality implies (17). □

Lemma 5. *Let $\Phi_1 \in \Omega(A)$, $\varphi_1' \in L^0$ and Φ_2 be weakly subordinated to Φ_1. Suppose that $\Phi_2'(\sigma) = o(\sigma\Phi_1''(\sigma))$ as $\sigma \uparrow A$, the conditions* (3) *as $\sigma \uparrow A$ and* (5) *hold. Then for the function* (15) *the following asymptotic equality is true*

$$G_2(t_n, t_{n+1}, \Phi) - G_1(t_n, t_{n+1}, \Phi) = G_2(t_n, t_{n+1}, \Phi_1) - G_1(t_n, t_{n+1}, \Phi_1) + o(\Phi_2(\varphi_1(t_n))) \tag{21}$$

as $n \to \infty$.

Proof. If we define $\kappa_n(\Phi_1)$ and ξ_n as in (8) then from (16) we obtain

$$\kappa_n(\Phi) = \frac{1}{t_{n+1} - t_n}\int_{t_n}^{t_{n+1}} \varphi(t)dt = \frac{1}{t_{n+1} - t_n}\int_{t_n}^{t_{n+1}} \varphi_1(t)dt -$$
$$-\frac{(1+o(1))\tau}{t_{n+1} - t_n}\int_{t_n}^{t_{n+1}} \tau\Phi_2'(\varphi_1(t))\varphi_1'(t)dt = \kappa_n(\Phi_1) - (1+o(1))\tau\xi_n, \quad n \to \infty.$$

The condition $\Phi_2'(\sigma) = o(\sigma\Phi_1''(\sigma))$ as $\sigma \uparrow A$ implies the relation $\xi_n = o(\kappa_n(\Phi_1))$ as $n \to \infty$. Therefore,

$$G_2(t_n, t_{n+1}, \Phi) = \Phi(\kappa_n(\Phi)) = \Phi_1(\kappa_n(\Phi)) + \tau\Phi_2(\kappa_n(\Phi) =$$
$$= \Phi_1(\kappa_n(\Phi_1) - (1+o(1))\tau\xi_n) + \tau\Phi_2(\kappa_n(\Phi_1) - (1+o(1))\tau\xi_n), \quad n \to \infty.$$

Since $\varphi_1'((1+o(1))x) = (1+o(1))\varphi_1'(x)$ as $x \to +\infty$ and $\Phi_1''(\varphi_1(x)) = 1/\varphi_1'(x)$, we have $\Phi_1''(\varphi_1((1+o(1))x)) = (1+o(1))\Phi_1''(\varphi_1(x))$ as $x \to +\infty$. The condition (5) implies the relation $\kappa_n(\Phi_1) = \varphi_1((1+o(1))t_n)$ as $n \to \infty$. Therefore, in view of the condition $\Phi_2'(\varphi_1) \in L^0$ we have

$$\frac{\Phi_2'(\kappa_n(\Phi_1)}{\Phi_1''(\kappa_n(\Phi_1)} = \frac{\Phi_2'(\varphi_1((1+o(1))t_n))}{\Phi_1''(\varphi_1((1+o(1))t_n))} = (1+o(1))\frac{\Phi_2'(\varphi_1(t_n))}{\Phi_1''(\varphi_1(t_n))} =$$
$$= (1+o(1))\Phi_2'(\varphi_1(t_n))\varphi_1'(t_n) = (1+o(1))\xi_n, \quad n \to \infty.$$

Hence it follows that for some $\eta_n \in (0,1)$ by (3)

$$\Phi_1(\kappa_n(\Phi_1) - (1+o(1))\tau\xi_n) - \Phi_1(\kappa_n(\Phi_1)) = -\Phi_1'(\kappa_n(\Phi_1) - (1+o(1))\eta_n\tau\xi_n)(1+o(1))\tau\xi_n =$$
$$= -(1+o(1))\tau\xi_n\Phi_1'\left(\kappa_n(\Phi_1) - (1+o(1))\eta_n\tau\frac{\Phi_2'(\kappa_n(\Phi_1)}{\Phi_1''(\kappa_n(\Phi_1)}\right) = -(1+o(1))\tau\xi_n\Phi_1'(\kappa_n(\Phi_1))$$

as $n \to \infty$, and by analogy

$$\Phi_2(\kappa_n(\Phi_1) - (1+o(1))\tau\xi_n) - \Phi_2(\kappa_n(\Phi_1)) = -\Phi_2'(\kappa_n(\Phi_1) - (1+o(1))\eta_n\tau\xi_n)(1+o(1))\tau\xi_n =$$
$$= -(1+o(1))\tau\xi_n\Phi_2'\left(\kappa_n(\Phi_1) - (1+o(1))\eta_n\tau\frac{\Phi_2'(\kappa_n(\Phi_1)}{\Phi_1''(\kappa_n(\Phi_1)}\right) = -(1+o(1))\tau\xi_n\Phi_2'(\kappa_n(\Phi_1))$$

as $n \to \infty$. Thus,

$$\begin{aligned} G_2(t_n, t_{n+1}, \Phi) = G_2(t_n, t_{n+1}, \Phi_1) + \tau\Phi_2(\kappa_n(\Phi_1)) - \\ -(1+o(1))\tau\xi_n\Phi_1'(\kappa_n(\Phi_1)) - (1+o(1))\tau\xi_n\Phi_2'(\kappa_n(\Phi_1)), \quad n \to \infty. \end{aligned} \tag{22}$$

On the other hand, in view of (17) and the equality $\varphi(x) = (x\Psi(\varphi(x))'$ we have

$$G_1(t_n, t_{n+1}, \Phi) = \frac{t_n t_{n+1}}{t_{n+1} - t_n}\int_{t_n}^{t_{n+1}} \Phi(\varphi(t))d\left(-\frac{1}{t}\right) = \frac{t_n t_{n+1}}{t_{n+1} - t_n}(\Psi(\varphi(t_{n+1})) -$$
$$\Psi(\varphi(t_n)) = t_n\left(\frac{t_{n+1}\Psi(\varphi(t_{n+1})) - t_n\Psi(\varphi(t_n))}{t_{n+1} - t_n} - \Psi(\varphi(t_n))\right) =$$
$$= t_n\kappa_n(\Phi) - t_n\Psi(\varphi(t_n)) = t_n\kappa_n(\Phi_1) - (1+o(1))\tau\xi_n t_n -$$
$$-t_n\Psi_1(\varphi_1(t_n)) + (1+o(1))\tau\Phi_2(\varphi_1(t_n)) =$$
$$= G_1(t_n, t_{n+1}, \Phi_1) - (1+o(1))\tau\xi_n t_n + (1+o(1))\tau\Phi_2(\varphi_1(t_n)), \quad n \to \infty. \tag{23}$$

From the conditions $\Phi_2'(\varphi_1) \in L^0$ and $\varphi_1' \in L^0$ it follows that

$$0 < \varlimsup_{x\to+\infty} \frac{x\Phi_2'(\varphi_1(x))\varphi_1'(x)}{\Phi_2(\varphi_1(x))} \le$$
$$\le \varlimsup_{x\to+\infty} \frac{\Phi_2'(\varphi_1(x))\varphi_1'(x) + x\Phi_2''(\varphi_1(x))\varphi_1'(x)^2 + x\Phi_2'(\varphi_1(x))\varphi_1''(x)}{\Phi_2'(\varphi_1(x))\varphi_1'(x)} \le$$
$$\le 1 + \varlimsup_{x\to+\infty} \frac{x\Phi_2''(\varphi_1(x))\varphi_1'(x)}{\Phi_2'(\varphi_1(x))} + \varlimsup_{x\to+\infty} \frac{x\varphi_1''(x)}{\varphi_1''(x)} < +\infty, \tag{24}$$

i. e. $\Phi_2(\varphi_1) \in L^0$. Therefore, taking into account (5) and $\Phi_2(\kappa_n) = (1+o(1))\Phi_2(\varphi(t_n))$ as $n \to \infty$, from (22) and (23) we obtain

$$\begin{aligned} G_2(t_n, t_{n+1}, \Phi) - G_1(t_n, t_{n+1}, \Phi) = G_2(t_n, t_{n+1}, \Phi_1) - G_1(t_n, t_{n+1}, \Phi_1) - \\ -(1+o(1))\tau\xi_n\Phi_1'(\kappa_n(\Phi_1)) - (1+o(1))\tau\xi_n\Phi_2'(\kappa_n(\Phi_1)) + (1+o(1))\tau\xi_n t_n + \\ +o(\Phi_2(\varphi_1(t_n))), \quad n \to \infty. \end{aligned} \tag{25}$$

Since $\xi_n = \Phi_2'(\varphi_1(\eta_n))\varphi_1'(\eta_n)$ $(t_n < \eta_n < t_{n+1})$, $t_{n+1} = (1+o(1))t_n$ as $n \to +\infty$, $\Phi_2'(\varphi_1) \in L^0$ and $\varphi_1' \in L^0$ we have

$$\xi_n = (1+o(1))\Phi_2'(\varphi_1(t_n))\varphi_1'(t_n), \quad n \to \infty, \tag{26}$$

and in view of (24) we have

$$t_n\xi_n = O(\Phi_2(\varphi_1(t_n))), \quad n \to \infty, \tag{27}$$

whence

$$\xi_n(t_n - \Phi_1'(\kappa_n(\Phi_1))) = \xi_n(t_n - \Phi_1'(\varphi((1+o(1))t_n))) = o(\xi_n t_n) = o(\Phi_2(\varphi_1(t_n))) \tag{28}$$

as $n \to \infty$. Finally, (27) implies

$$\frac{\xi_n\Phi_2'(\kappa_n(\Phi_1))}{\Phi_2(\kappa_n(\Phi_1))} = \frac{\xi_n\Phi_2'(\varphi_1(1+o(1))t_n)}{\Phi_2(\varphi_1(1+o(1))t_n)} = o\left(\frac{\xi_n\Phi_1'(\varphi_1(1+o(1))t_n)}{\Phi_2(\varphi_1(1+o(1))t_n)}\right) =$$
$$= o\left(\frac{\xi_n t_n}{\Phi_2(\varphi_1(t_n))}\right) = o(1), \ n \to \infty. \tag{29}$$

The asymptotic equality (21) follows from (25) and (27)–(29). □

Lemma 6. *Let $\Phi_1 \in \Omega^*(A)$, $\varphi_1' \in L^0$ and Φ_2 be weakly subordinated to Φ_1. Suppose that $\Phi_2'(\sigma) = o(\sigma\Phi_1''(\sigma))$ as $\sigma \uparrow A$, the conditions (3) as $\sigma \uparrow A$ hold and*

$$\xi_n\Phi_1'(\kappa_n(\Phi_1))) = o(G_2(t_n, t_{n+1}, \Phi_1)), \quad \Phi_2(\kappa_n(\Phi_1)) = o(G_2(t_n, t_{n+1}, \Phi_1)) \tag{30}$$

as $n \to \infty$. Suppose also that

$$\Phi_*(\sigma) = \Phi_1(\sigma) + (\tau - \varepsilon)\Phi_2(\sigma), \quad \Phi^*(\sigma) = \Phi_1(\sigma) + (\tau + \varepsilon)\Phi_2(\sigma),$$

where $\varepsilon \in (0, |\tau|)$. Then the inequality $G_1(t_n, t_{n+1}, \Phi^*) \ge \Phi_*(\varkappa_n(\Phi^*))$ for some sequence $(t_n) \uparrow +\infty$ implies the inequality

$$0 < G_2(t_n, t_{n+1}, \Phi_1) - G_1(t_n, t_{n+1}, \Phi_1) = 2\varepsilon(1 + o(1)\Phi_2(\varphi_1(t_n)), \quad n \to \infty. \tag{31}$$

Proof. The inequality $G_1(t_n, t_{n+1}, \Phi^*) \ge \Phi_*(\varkappa_n(\Phi^*))$ is equivalent to the inequality

$$0 < G_2(t_n, t_{n+1}, \Phi^*) - G_1(t_n, t_{n+1}, \Phi^*) \le 2\varepsilon\Phi_2(\varkappa_n(\Phi^*)). \tag{32}$$

Using (22) with Φ^* and $\tau + \varepsilon$ instead Φ and τ we have

$$G_2(t_n, t_{n+1}, \Phi^*) = G_2(t_n, t_{n+1}, \Phi_1) + (\tau + \varepsilon)\Phi_2(\kappa_n(\Phi_1)) -$$
$$-(1 + o(1))(\tau + \varepsilon)\xi_n\Phi_1'(\kappa_n(\Phi_1)) - (1 + o(1))(\tau + \varepsilon)\xi_n\Phi_2'(\kappa_n(\Phi_1)), \quad n \to \infty.$$

On the other hand, (23) implies

$$G_1(t_n, t_{n+1}, \Phi^*) = G_1(t_n, t_{n+1}, \Phi_1) - (1+o(1))(\tau+\varepsilon)\xi_n t_n + (1+o(1))(\tau+\varepsilon)\Phi_2(\varphi_1(t_n)), \ n \to \infty.$$

Therefore, from (32) we obtain

$$0 < G_2(t_n, t_{n+1}, \Phi_1) - G_1(t_n, t_{n+1}, \Phi_1) + (\tau + \varepsilon)\Phi_2(\kappa_n(\Phi_1)) -$$
$$-(1 + o(1))(\tau + \varepsilon)\xi_n\Phi_1'(\kappa_n(\Phi_1)) - (1 + o(1))(\tau + \varepsilon)\xi_n\Phi_2'(\kappa_n(\Phi_1)) -$$
$$+(1 + o(1))(\tau + \varepsilon)\xi_n t_n - (1 + o(1))(\tau + \varepsilon)\Phi_2(\varphi_1(t_n)) \le 2\varepsilon\Phi_2(\varkappa_n(\Phi^*)). \tag{33}$$

Since $t_n = \Phi'(\varphi(t_n)) \le \Phi'(\varkappa_n(\Phi_1))$, from (30) it follows that

$$\frac{t_n\xi_n}{G_2(t_n, t_{n+1}, \Phi_1)} \to 0, \ \frac{\Phi_2(\varkappa_n(\Phi_1))}{G_2(t_n, t_{n+1}, \Phi_1)} \to 0, \quad n \to \infty.$$

Therefore, (33) implies (14) and, thus, (5). Using (5) and the equality $\Phi_2(\kappa_n(\Phi_1)) = (1 + o(1))\Phi_2(\varphi_1(t_n))$ as $n \to \infty$, as in the proof of Lemma 5, from (33) we have

$$\begin{gathered} G_2(t_n, t_{n+1}, \Phi_1) - G_1(t_n, t_{n+1}, \Phi_1) \le (1 + o(1))(\tau + \varepsilon)\xi_n\Phi_1'(\kappa_n(\Phi_1)) + \\ +(1 + o(1))(\tau + \varepsilon)\xi_n\Phi_2'(\kappa_n(\Phi_1)) - (1 + o(1))(\tau + \varepsilon)\xi_n t_n + \\ +o(\Phi_2(\varphi_1(t_n))) + 2\varepsilon\Phi_2(\varkappa_n(\Phi_1)), \quad n \to \infty. \end{gathered}$$

whence, repeating the proof of Lemma 5, we obtain (31). □

4. Main results. From Lemmas 1 and 4 the following generalization of Theorem A follows.

Theorem 1. *Let $\tau \in \mathbb{R} \setminus \{0\}$ and either $A = 0$ or $A = +\infty$. Suppose that $\Phi_1 \in \Omega(A)$, $\varphi_1' \in L^0$ and Φ_2 is weakly subordinated to Φ_1. Then the asymptotic inequality* (1) *holds with $\sigma \uparrow A$ instead $\sigma \to +\infty$ if and only if the asymptotic inequality* (2) *holds.*

Indeed, for $\Phi(\sigma) = \Phi_1(\sigma) + (\tau + \varepsilon)\Phi_2(\sigma)$ ($\sigma \in [\sigma_0(\tau + \varepsilon), A)$, where $\varepsilon \in (0, |\tau|)$ is an arbitrary number, by Lemma 4 we have

$$t\Psi(\varphi(t)) = t\Psi_1(\varphi_1(t)) - (\tau + \varepsilon)(1 + o(1))\Phi_2'(\varphi_1(t)), \quad t \to +\infty.$$

Then, Lemma 1 completes the proof of Theorem 1.

Theorem 2. *Let $\tau \in \mathbb{R} \setminus \{0\}$ and either $A = 0$ or $A = +\infty$. Suppose that $\Phi_1 \in \Omega(A)$, $\varphi_1' \in L^0$, Φ_2 is weakly subordinated to Φ_1, $\Phi_2'(\sigma) = o(\sigma\Phi_1''(\sigma))$ as $\sigma \uparrow A$ and conditions* (3) *as $\sigma \uparrow A$ hold. If the asymptotic equality* (4) *holds for some increasing to $+\infty$ sequence (t_n), satisfying* (5) *and* (6)*, then the asymptotic equality* (7) *holds as $\sigma \uparrow A$.*

Proof. By the assumptions for an arbitrary $\varepsilon \in (0, |\tau|)$ the inequality

$$P(t_n) \ge -t_n\Psi_1(\varphi_1(t_n)) + (\tau - \varepsilon)\Phi_2(\varphi_1(t_n))$$

holds for a some increasing to $+\infty$ sequence (t_n), satisfying (5) and (6). By Lemma 5 for the function $\Phi(\sigma) = \Phi_1(\sigma) + (\tau - \varepsilon)\Phi_2(\sigma)$ ($\sigma \in [\sigma_0(\tau - \varepsilon), A)$) we have (21). But in view of (16) and (3)

$$\frac{\Phi_2'(\varphi(t))}{\Phi_2(\varphi_1(t))} = \frac{1}{\Phi_2'(\varphi_1(t))}\Phi_2\left(\varphi(t) - (1 + o(1))\frac{\Phi_2'(\varphi_1(t))}{\Phi_2''(\varphi_1(t))}\right) \to 1, \quad t \to +\infty.$$

Therefore, (21) and (6) implies $G_2(t_n, t_{n+1}, \Phi) - G_1(t_n, t_{n+1}, \Phi) = o(\Phi_2(\varphi(t_n)))$, $n \to \infty$, and by Lemma 2 for all $n \ge n_0$ and all $\sigma \in [\varphi(t_n), \varphi(t_{n+1})]$

$$\begin{gathered} Q(\sigma) \ge \Phi(\sigma) + G_1(t_n, t_{n+1}, \Phi) - G_2(t_n, t_{n+1}, \Phi) = \Phi(\sigma) + o(\Phi_2(\varphi(t_n))) = \\ = \Phi(\sigma) + o(\Phi_2(\sigma)) = \Phi_1(\sigma) + (\tau - \varepsilon + o(1))\Phi_2(\sigma), \quad \sigma \uparrow A. \end{gathered}$$

In view of the arbitrariness of ε Theorem 2 is proved. □

Finally, we prove the following main theorem.

Theorem 3. *Let $\tau \in \mathbb{R} \setminus \{0\}$ and either $A = 0$ or $A = +\infty$. Suppose that $\Phi_1 \in \Omega^*(A)$, $\varphi_1' \in L^0$, Φ_2 is weakly subordinated to Φ_1, $\Phi_2'(\sigma) = o(\sigma\Phi_1''(\sigma))$ as $\sigma \uparrow A$ and the conditions (3) as $\sigma \uparrow A$ hold.*

Suppose also that either $\sigma\Phi_1'(\sigma) = O(\Phi_1(\sigma))$ as $\sigma \uparrow A$ (when $A = 0$ or $A = +\infty$), or $\Phi_2'(\sigma) = o(\Phi_1''(\sigma))$ and $\Phi_1'(\sigma) = O(\Phi_1(\sigma))$ as $\sigma \uparrow A$ when $A = +\infty$.

In order that

$$Q(\sigma) = \Phi_1(\sigma) + (1 + o(1))\tau\Phi_1(\sigma), \quad \sigma \uparrow A, \tag{34}$$

it is necessary and sufficient that for every $\varepsilon > 0$: 1) for all $t \ge t_0 = t_0(\varepsilon)$

$$P(t) \le -t\Psi_1(\varphi_1(t)) + (\tau + \varepsilon)\Phi_2(\varphi_1(t)); \tag{35}$$

2) there exists an increasing to $+\infty$ sequence (t_n) such that

$$P(t_n) \ge -t_n\Psi_1(\varphi_1(t_n)) + (\tau - \varepsilon))\Phi_2(\varphi_1(t_n)), \quad n \to +\infty, \tag{36}$$

and

$$\lim_{n\to\infty} \frac{G_2(t_n, t_{n+1}, \Phi_1) - G_1(t_n, t_{n+1}, \Phi_1)}{\Phi_2(\varphi_1(t_n))} = 0. \tag{37}$$

Proof. If (34) holds then

$$\Phi_*(\sigma) = \Phi_1(\sigma) + (\tau - \varepsilon)\Phi_2(\sigma) \le Q(\sigma) \le \Phi^*(\sigma) = \Phi_1(\sigma) + (\tau + \varepsilon)\Phi_2(\sigma)$$

for arbitrary $\varepsilon \in (0, |\tau|)$ and all $\sigma \in [\sigma(\varepsilon), A)$. Hence by Lemmas 1 and 4 we get

$$P(t) \le -t\Psi_1(\varphi_1(t)) + (\tau + \varepsilon)(1 + o(1))\Phi_2(\varphi_1(t))$$

as $t \to +\infty$, i. e. (35) holds.

By Lemma 3 there exists an increasing to $+\infty$ sequence (t_n) of positive numbers such that $P(t_n) \ge -t_n\Psi_*(\varphi_*(t_n))$ and $G_1(t_n, t_{n+1}, \Phi^*) \ge \Phi_1(\varkappa_n(\Phi^*))$. From the inequality $P(t_n) \ge -t_n\Psi_*(\varphi_*(t_n))$ in view of Lemma 4 we have (36).

Further, since the condition $\sigma\Phi_1'(\sigma) = O(\Phi_1(\sigma))$ as $\sigma \uparrow A$ (when $A = 0$ or $A = +\infty$), implies (30) (see the proof of Proposition 1), by Lemma 6 the relation (31) holds, whence in view of arbitrariness of ε we obtain (37). If $A = +\infty$ then the condition $\Phi_2'(\sigma) = o(\Phi_1''(\sigma))$ and $\Phi_1'(\sigma) = O(\Phi_1(\sigma))$ as $\sigma \to +\infty$ imply (30) (see the proof of Proposition 2) and, therefore, we obtain again (31) and, thus, (37). □

Now we will prove the sufficity. By Lemmas 1 and 4 in view of the arbitrariness of ε the condition (35) implies the inequality $Q(\sigma) \le \Phi_1(\sigma) + (\tau + \varepsilon)\Phi_2(\sigma)$. On the other hand, by Lemmas 2 and 4 from (36) we have $Q(\sigma) \ge \Phi(\sigma) + G_1(t_n, t_{n+1}, \Phi) - G_2(t_n, t_{n+1}, \Phi)$ for all $n \ge n_0$ and all $\sigma \in [\varphi(t_n), \varphi(t_{n+1})]$, where $\Phi(\sigma) = \Phi_1(\sigma) + (\tau - \varepsilon)\Phi_2(\sigma)$. Since $\Phi_1 \in \Omega^*(A)$, from (37) we obtain (5). Therefore, by Lemma 5 for all $\sigma \in [\varphi(t_n), \varphi(t_{n+1})]$ as $n \to \infty$

$$Q(\sigma) \ge \Phi_1(\sigma) + (\tau - \varepsilon)\Phi_2(\sigma) + G_1(t_n, t_{n+1}, \Phi_1) - G_2(t_n, t_{n+1}, \Phi_1) + o(\Phi_2(\varphi(t_n))),$$

whence as in the proof of Theorem 2 we obtain the inequality $Q(\sigma) \ge \Phi_1(\sigma) + (\tau - \varepsilon + o(1))\Phi_2(\sigma)$ as $\sigma \uparrow A$. In view of the arbitrariness of ε the proof of Theorem 3 is complete.

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