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SOME PROPERTIES OF MEASURES WITH DISCRETE SUPPORT

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We give some new conditions for the support of a discrete measure on Euclidean space to be a finite union of translated lattices. In particular, we consider the case when values of masses a_λ of discrete measure satisfy the equality $G(a_\lambda, \bar{a}_\lambda) = 0$ for each analytic function $G(z, w)$.

Denote by $S(\mathbb{R}^d)$ the Schwartz space of test functions $\varphi \in C^\infty(\mathbb{R}^d)$ with finite norms

$$p_m(\varphi) = \sup_{\mathbb{R}^d}(1 + |x|)^m \max_{|k_1|+\dots+|k_d|\le m} |D^k(\varphi(x))|, \quad m = 0, 1, 2, \dots, \tag{1}$$

$k = (k_1, \dots, k_d) \in (\mathbb{N}\cup\{0\})^d$, $D^k = \partial_{x_1}^{k_1} \dots \partial_{x_d}^{k_d}$. These norms generate topology on $S(\mathbb{R}^d)$, and elements of the space $S'(\mathbb{R}^d)$ of continuous linear functionals on $S(\mathbb{R}^d)$ are called tempered distributions. For each tempered distribution f there exist $C > 0$ and $m \in \mathbb{N}\cup\{0\}$ such that for all $\varphi \in S(\mathbb{R}^d)$

$$|f(\varphi)| \le C p_m(\varphi). \tag{2}$$

Moreover, this estimate is sufficient for distribution f to be in $S'(\mathbb{R}^d)$ (see [16], Ch.3).

The Fourier transform of a tempered distribution f is defined by the equality

$$\hat{f}(\varphi) = f(\hat{\varphi}) \quad \text{for all} \quad \varphi \in S(\mathbb{R}^d), \tag{3}$$

where

$$\hat{\varphi}(y) = \int_{\mathbb{R}^d} \varphi(x) \exp\{-2\pi i\langle x, y\rangle\} dx$$

is the Fourier transform of the function φ. Note that the Fourier transform of each tempered distribution is also a tempered distribution.

In the paper we consider only the case when f is a measure μ on $\mathbb{R}^d$. We say that μ is *translation bounded*, if its variations on balls of radius 1 are uniformly bounded. If the Fourier transform $\hat{\mu}$ is an atomic measure, then *spectrum* of μ is the set $\Gamma = \{x \in \mathbb{R}^d\colon\ \hat{\mu}(x) \ne 0\}$. We denote $B(x, r) = \{y \in \mathbb{R}^d\colon\ |y - x| < r\}$, $B(r) = B(0, r)$, $A\triangle B = (A \setminus B) \cup (B \setminus A)$, and by δ_λ the unit mass at the point λ. For a measure μ denote by $|\mu|(t)$ the value of its variation on the ball $B(t)$, and by $|\mu|$ the value of its total variation, if it is finite. A measure μ is *slowly increasing*, if $|\mu|(t)$ grows at most polynomially as $t \to \infty$.

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Next, a set $E \subset \mathbb{R}^d$ is *relatively dense*, if there is $R < \infty$ such that $E \cap B(x,R) \neq \varnothing$ for all $x \in \mathbb{R}^d$. A set E is *discrete*, if $E \cap B(x,1)$ is finite for all $x \in \mathbb{R}^d$. A set E is *uniformly discrete*, if $|x - x'| \geq \varepsilon > 0$ for all $x, x' \in E$, $x \neq x'$. A measure is discrete (uniformly discrete), if its support is discrete (uniformly discrete).

Let $\mu \in S'(\mathbb{R}^d)$ be a Radon measure with discrete support Λ. Note that such measures are the main object in the theory of Fourier quasicrystals (see [1]–[12]). The following result is valid:

Theorem 1 (Y. Meyer, [11]). *Let $\mu = \sum_{\lambda\in\Lambda} a_\lambda\delta_\lambda$, $a_\lambda \in S$, be a measure on the real line $\mathbb{R}$ with discrete support Λ and some finite set $S \subset \mathbb{C} \setminus \{0\}$. If $\mu \in S'(\mathbb{R})$ and its Fourier transform $\hat\mu$ is a translation bounded measure on $\mathbb{R}$, then*

$$\Lambda = E \triangle \bigcup_{j=1}^{N} (\alpha_j\mathbb{Z} + \beta_j), \quad \alpha_j > 0,\ \beta_j \in \mathbb{R},\ E \text{ finite.} \tag{4}$$

The main tool is the following idempotent theorem by P. J. Cohen:

Theorem 2 ([2]). *Let G be a locally compact abelian group and $\hat G$ its dual group. If μ is a finite Borel measure on G such that its Fourier transform $\hat\mu(\gamma) \in \{0,1\}$ for all $\gamma \in \hat G$, then the set $\{\gamma\colon\ \hat\mu(\gamma) = 1\}$ is in the coset ring of $\hat G$.*

Recall that a *coset ring* of any topological group is the smallest collection of subsets of which is closed under finite unions, finite intersections and complements and contains all cosets of all open subgroups of G.

Note that Y. Meyer used the Cohen's theorem for measures on Bohr compactification $\mathfrak{R}$ of $\mathbb{R}$ and their Fourier transform on the dual group $\mathbb{R}_{\text{dis}}$ that is the real line in the discrete topology. Therefore the end of the proof of Meyer's theorem follows from the result of P. H. Rosenthal.

Theorem 3 ([15]). *The elements of the ring of cosets of $\mathbb{R}_{\text{dis}}$ which are discrete in the usual topology of $\mathbb{R}$ are precisely the sets of the form (4).*

To formulate the results for $\mathbb{R}^d$ with $d > 1$ we need some definitions.

A *lattice* is a discrete subgroup of R^d. If A be a lattice or a coset of some lattice in $\mathbb{R}^d$, then $\dim A$ is the dimension of the smallest translated subspace of $\mathbb{R}^d$ that contains A. Every lattice L of dimension k has the form $T\mathbb{Z}^k$, where $T\colon\ \mathbb{Z}^k \to \mathbb{Z}^d$ is a linear operator of rank k. For $k = d$ we say that L is a *full-rank* lattice.

Theorem 4 (M. Kolountzakis, [5]). *Let $\mu = \sum_{\lambda\in\Lambda} a_\lambda\delta_\lambda$, $a_\lambda \in S$, be a measure on $\mathbb{R}^d$ with discrete support Λ and some finite set $S \subset \mathbb{C} \setminus \{0\}$. If $\mu \in S'(\mathbb{R}^d)$ and its Fourier transform $\hat\mu$ is a measure with the property*

$$|\hat\mu|(t) = O(t^d) \quad \text{as} \quad t \to \infty, \tag{5}$$

then Λ is a finite union of sets of the type

$$A \setminus \Big(\bigcup_{j=1}^{N} B_j\Big), \quad A,\ B_j \text{ discrete cosets,} \quad \dim B_j < \dim A \text{ for all } j. \tag{6}$$

Note that each translation bounded measure $\hat{\mu}$ satisfies (5).

Here the following theorem was used instead of Theorem 3:

Theorem 5 ([5]). *The elements of the ring of cosets of $\mathbb{R}^d_{\mathrm{dis}}$ which are discrete in the usual topology of $\mathbb{R}^d$ are precisely finite unions of sets of the type (6).*

Note that A. Cordoba ([1]) considered a uniformly discrete measure $\mu = \sum_{\lambda\in\Lambda} a_\lambda \delta_\lambda$ with a_λ from a finite set $S \subset \mathbb{C} \setminus \{0\}$ and translation bounded measure $\hat{\mu}$ with a countable support. He proved that if this is the case, then Λ is a finite union of translates of several full-rank lattices. In our previous paper [4] we relaxed the conditions of Cordoba's theorem: we considered a uniformly discrete measure $\mu = \sum_{\lambda\in\Lambda} a_\lambda \delta_\lambda$ with $|a_\lambda|$ from a finite set S of positive numbers. We also assumed that the measure $\hat{\mu}$ had a countable support and satisfied condition (5) instead of being translation bounded.

Set for a measure μ on $\mathbb{R}^d$

$$\kappa(\mu) = \limsup_{t\to\infty} |\mu|(t)/\omega_d t^d,$$

where ω_d is the volume of the unit ball in $\mathbb{R}^d$.

The first result of the present paper is the following

Theorem 6. *Let Λ be a discrete set in $\mathbb{R}^d$, $\mu = \sum_{\lambda\in\Lambda} a_\lambda \delta_\lambda$ be a measure from $S'(\mathbb{R}^d)$, $\hat{\mu}$ be a measure such that $\kappa(\hat{\mu}) < \infty$, $G(z,w)$ be a holomorphic function on a polydisk $\{(z,w) \in \mathbb{C}^2 \colon |z| < R, |w| < R\}$ with $R > \kappa(\hat{\mu})$ and $G(0,0) = 1$. If $G(a_\lambda, \bar{a}_\lambda) = 0$ for all $\lambda \in \Lambda$, then Λ is a finite union of sets (6).*

Proof. Let $\rho(E) = \hat{\mu}(-E)$ for any Borel set $E \subset \mathbb{R}^d$. Clearly, $\hat{\rho} = \mu$. By conditions of the theorem, for each $\kappa' > \kappa(\hat{\mu})$ and sufficiently large t we have $|\rho|(t) \le \kappa' \omega_d t^d$.

Let $\varphi(|x|)$ be a nonnegative infinitely differentiable function on $\mathbb{R}^d$ such that $\varphi(|x|) = 0$ for $|x| \ge 1$ and

$$\hat{\varphi}(0) = \int_{\mathbb{R}^d} \varphi(|x|)dx = -\omega_d \int_0^1 \varphi'(t) t^d dt = 1. \tag{7}$$

Define a measure ρ_M by the equality

$$\rho_M(E) = M^{-d} \int_E \varphi(|y|/M) d\rho(y), \quad E \text{ is a Borel set in } \mathbb{R}^d.$$

Integrating by parts, we get

$$|\rho_M| \le M^{-d} \int_0^M \varphi(t/M) d|\rho|(t) \le M^{-d-1} \left(C(\kappa') - \kappa' \omega_d \int_0^M t^d \varphi'(t/M) dt \right).$$

By (7), the integral in the right-hand side equals $-M^{d+1}/\omega_d$, therefore,

$$\limsup_{M\to\infty} |\rho_M| \le \kappa(\hat{\mu}) < R. \tag{8}$$

The Fourier transform $\hat{\rho}_M$ is an infinitely differentiable (even real-analytic) function on $\mathbb{R}^d$. Let ψ be a nonnegative infinitely differentiable function on $\mathbb{R}^d$ with compact support such that $\psi(x) \equiv 1$ for $|x| \le 1$. For each point $x \in \mathbb{R}^d$ we get

$$\hat{\rho}_M(x) = (\hat{\varphi}(M\cdot) * \mu)(x) = \int \psi(x-y)\hat{\varphi}(M(x-y))d\mu(y) + \int (1-\psi(x-y))\hat{\varphi}(M(x-y))d\mu(y). \tag{9}$$

The set $\Lambda\cap\{y\colon\ \psi(x-y)\neq 0\}$ is at most finite. Since $\hat\varphi(M(x-y))\to 0$ for $x\neq y$ as $M\to\infty$, we see that the first integral tends to 0 for $x\notin\Lambda$ and tends to $a(\lambda)$ for $x=\lambda\in\Lambda$.

By (1) and (2), there is $m<\infty$ such that the second integral in (9) is bounded by the quantity

$$C\sup_{|x-y|>1}(1+|x-y|)^m\max_{|k_1|+\dots+|k_d|\le m}|D^k[(1-\psi(x-y))\hat\varphi(M(x-y))]|. \tag{10}$$

Since $\psi(x-y)\hat\varphi(M(x-y))\in S(\mathbb{R}^d)$, we see that (10) for each $N<\infty$ does not exceed

$$C'(N)M^{m-N}\sup_{|x-y|>1}|x-y|^{m-N},$$

hence it tends to 0 as $M\to\infty$.

Consider the Bohr compactification $\mathfrak{R}$ of $\mathbb{R}^d$. The dual group to $\mathfrak{R}$ is $\mathbb{R}^d_{\text{dis}}$, then $\mathbb{R}^d$ is a dense subset of $\mathfrak{R}$ with respect to the topology on $\mathfrak{R}$, and restrictions to $\mathbb{R}^d$ of continuous functions on $\mathfrak{R}$ are just almost periodic functions on $\mathbb{R}^d$, in particular, they are bounded and continuous on $\mathbb{R}^d$ (see for example [13]). By (8), variations of the measures ρ_M are uniformly bounded, the measures ρ_M act on all bounded functions on $\mathbb{R}^d$, and hence also on all functions from $C(\mathfrak{R})$. Therefore there exists a measure $\mathfrak{r}$ on $\mathfrak{R}$ with the total variation $|\mathfrak{r}|<R$, and a subsequence M' such that $\rho_{M'}\to\mathfrak{r}$ in the weak–star topology. In other words, $\langle\rho_{M'},f\rangle\to\langle\mathfrak{r},f\rangle$ as $M'\to\infty$ for all $f\in C(\mathfrak{R})$. Applying this to any character of $\mathfrak{R}$ in place of f we obtain

$$\hat{\mathfrak{r}}(x)=\lim_{M'\to\infty}\hat\rho_{M'}(x)=\begin{cases}a_\lambda, & x=\lambda\in\Lambda,\\ 0, & x\notin\Lambda.\end{cases}$$

Note that $\hat{\mathfrak{r}}(x)$ is a continuous function with respect to the discrete topology on $\mathbb{R}^d$, and $|a_\lambda|\le|\mathfrak{r}|<R$ for all a_λ.

Define a measure on $\mathfrak{R}$ by equality $\mathfrak{n}(E)=\overline{\mathfrak{r}(-E)}$. Note that $\hat{\mathfrak{n}}(x)=\bar{\hat{\mathfrak{r}}}(x)$ for all $x\in\mathbb{R}^d$ and $|\mathfrak{n}|<R$. Let $P(z,\bar z)=\sum_{1\le l+m\le r}c_{l,m}z^l\bar z^m$ be any polynomial on $\mathbb{C}$. Then the Fourier transform of the corresponding convolution polynomial $\mathfrak{p}=\sum_{1\le l+m\le r}c_{l,m}\mathfrak{r}^{*l}\mathfrak{n}^{*m}$ has the form

$$\hat{\mathfrak{p}}(x)=\begin{cases}P(a_\lambda,\bar a_\lambda), & x=\lambda\in\Lambda,\\ 0, & x\notin\Lambda.\end{cases}$$

Besides, the variation $\mathfrak{p}$ is bounded by $\sum_{1\le l+m\le p}|c_{l,m}||\mathfrak{r}|^l|\mathfrak{n}|^m$.

Furthermore, the function $1-G(z,w)$ is the absolutely convergent series $\sum_{l+m\ge 1}c_{l,m}z^lw^m$ for $|z|<R$, $|w|<R$, therefore the series $\sum_{l+m\ge1}|c_{l,m}||\mathfrak{r}|^l|\mathfrak{n}|^m$ converges, and the sums $\mathfrak{s}_r=\sum_{1\le l+m\le r}c_{l,m}\mathfrak{r}^{*l}\mathfrak{n}^{*m}$ converge in the space $C'(\mathfrak{R})$ to a measure $\mathfrak{g}$. As above we get

$$\hat{\mathfrak{g}}(x)=\begin{cases}1-G(a_\lambda,\bar a_\lambda)=1, & x=\lambda\in\Lambda,\\ 1-G(0,0)=0, & x\notin\Lambda.\end{cases}$$

Using Theorem 2 and Theorem 5, we obtain the assertion of our theorem. □

Now we consider conditions for support of a discrete measure to be a finite union of translations of a *single* lattice. We begin with the following theorem:

Theorem 7 (N. Lev, A. Olevskii, [9]). *Let $\mu = \sum_{\lambda\in\Lambda} a(\lambda)\delta_\lambda$ and $\hat{\mu} = \sum_{\gamma\in\Gamma} b(\gamma)\delta_\gamma$ be slowly increasing measures in $\mathbb{R}^d$ with countable support Λ and countable spectrum Γ. If Γ is discrete and $\Lambda - \Lambda$ is uniformly discrete, then the sets Λ is a subset of a finite union of translates of a single full-rank lattice L, and Γ is a subset of a finite union of translates of the conjugate lattice.*

Also, there is a measure μ with countable support Λ and spectrum Γ such that $\Lambda - \Lambda$ is uniformly discrete, but Λ is not contained in a finite union of translates of any lattice.

We prove the following theorem, which amplifies the previous one

Theorem 8. *Let $\mu = \sum_{\lambda\in\Lambda} a(\lambda)\delta_\lambda$ and $\hat{\mu} = \sum_{\gamma\in\Gamma} b(\gamma)\delta_\gamma$ be measures in $\mathbb{R}^d$ with countable support Λ and countable spectrum Γ, $\inf_{\lambda\in\Lambda}|a(\lambda)| > 0$, and let $\hat{\mu}$ be a slowly increasing measure. If $\Lambda - \Lambda$ is a discrete set, then Λ is a finite union of translates of a single full-rank lattice L.*

Here we need not the discreteness of spectrum Γ of the measure.

Theorem 8 is a consequence of the result on pairs of measures:

Theorem 9. *Let $\mu_j = \sum_{\lambda\in\Lambda_j} a_j(\lambda)\delta_\lambda$ be measures on $\mathbb{R}^d$ with countable Λ_j such that $\inf_{\lambda\in\Lambda_j}|a_j(\lambda)| > 0$, $\hat{\mu}_j = \sum_{\gamma\in\Gamma_j} b_j(\gamma)\delta_\gamma$ be slowly increasing measures with countable Γ_j, for $j = 1, 2$. If the set of differences $\Lambda_1 - \Lambda_2$ is discrete, then the sets Λ_1 and Λ_2 are finite unions of translates of a single full-rank lattice L.*

For $\mu_2 = \alpha\mu_1$ we get a slight strengthening of Theorem 8:

Corollary 1. *Let $\mu = \sum_{\lambda\in\Lambda} a(\lambda)\delta_\lambda$ be measures on $\mathbb{R}^d$ with countable Λ such that $\inf_{\lambda\in\Lambda}|a(\lambda)| > 0$, let $\hat{\mu} = \sum_{\gamma\in\Gamma} b(\gamma)\delta_\gamma$ be slowly increasing measures with countable Γ. If the set $\{x - \alpha x' \colon x, x' \in \Lambda\}$ for some $\alpha \in \mathbb{R}\setminus\{0\}$ is discrete, then Λ is a finite union of translates of a single full-rank lattice L.*

To prove Theorem 9 we recall some definitions connected with the notion of almost periodicity (see, for example, [10]).

A continuous function f on $\mathbb{R}^d$ is *almost periodic* if for every $\varepsilon > 0$ the set of ε-almost periods of f

$$\left\{\tau \in \mathbb{R}^d \colon \sup_{x\in\mathbb{R}^d} |f(x+\tau) - f(x)| < \varepsilon\right\}$$

is a relatively dense set in $\mathbb{R}^d$.

A (complex) measure μ on $\mathbb{R}^d$ is *almost periodic* if for every continuous function ψ on $\mathbb{R}^d$ with compact support the function $(\psi \star \mu)(t)$ is almost periodic in $t \in \mathbb{R}^d$.

A discrete set Λ is almost periodic if the measure $\sum_{\lambda\in\Lambda}\delta_\lambda$ is almost periodic.

Theorem 10 (L. Ronkin, [14]). *Every almost periodic measure is translation bounded.*

Earlier we proved an analog of Theorem 9 for almost periodic measures:

Theorem 11 ([4]). *If measures $\mu_j = \sum_{\lambda\in\Lambda_j} a_j(\lambda)\delta_\lambda$, $\inf_{\lambda\in\Lambda_j}|a_j(\lambda)| > 0$, with countable Λ_j, for $j = 1, 2$, are almost periodic, and the set of differences $\Lambda_1 - \Lambda_2$ is discrete, then the sets Λ_1 and Λ_2 are finite unions of translates of a single full-rank lattice L.*

Corollary 2 ([3]). *If Λ is an almost periodic set and $\Lambda - \Lambda$ is discrete set, then Λ is a finite union of translates of a single full-rank lattice L.*

This is a positive solution of Lagarias' (Problem 4.4, [7]).

A connection between almost periodicity of measure and properties of its Fourier transform was found by Y. Meyer.

Theorem 12 ([10])**.** *Let μ and its Fourier transform $\hat{\mu}$ be translation bounded measures. Then μ is almost periodic if and only if the spectrum of μ is countable.*

Here we need a small supplement of this result.

Theorem 13. *Let μ be a uniformly discrete measure, and let its Fourier transform $\hat{\mu}$ be a slowly increasing measure with countable support. Then μ is almost periodic.*

Proof. Let $\mu = \sum_{\lambda\in\Lambda} a_\lambda \delta_\lambda$ and $\varepsilon = \inf\{|x-x'|\colon\ x, x' \in \Lambda,\ x \neq x'\}$, let $\psi(|y|)$ be a C^∞-function such that $\operatorname{supp}\psi(|y|) \subset B(0,\varepsilon/2)$ and $\psi(0)=1$. Using (3), we have

$$\sup_{\lambda\in\Lambda}|a_\lambda| \le \sup_{x\in\mathbb{R}^d}\left|\int \psi(|x-\lambda|)d\mu(\lambda)\right| = \sup_{x\in\mathbb{R}^d}\left|\int_{\mathbb{R}^d}\hat{\psi}(y)e^{2\pi i\langle x,y\rangle}d\hat{\mu}(y)\right|. \tag{11}$$

Since $\hat{\psi}(y) \in S(R^d)$, we have $|\hat{\psi}(y)| \le c_N(1+|y|)^{-N}$ for any $N<\infty$. Therefore, the latter integral in (11) does not exceed

$$c_N\int_0^\infty \frac{d|\hat{\mu}|(t)}{(1+t)^N} \le \lim_{T\to\infty}\frac{c_N|\hat{\mu}|(T)}{(1+T)^N} + c_N N\int_0^\infty \frac{|\hat{\mu}|(t)dt}{(1+t)^{N+1}}.$$

The measure $\hat{\mu}$ is slowly increasing, hence the right-hand side is finite for appropriate N, and the numbers a_λ are uniformly bounded.

Furthermore, take any $\varphi \in S(\mathbb{R}^d)$. Since $\hat{\mu} = \sum_{\gamma\in\Gamma} b(\gamma)\delta_\gamma$ with countable Γ, we get

$$(\varphi\star\mu)(t) = \int_{\mathbb{R}^d}\varphi(t-x)d\mu(x) = \int_{\mathbb{R}^d}\hat{\varphi}(\gamma)e^{2\pi i\langle t,\gamma\rangle}d\hat{\mu}(\gamma) = \sum_{\gamma\in\Gamma} b(\gamma)\hat{\varphi}(\gamma)e^{2\pi i\langle t,\gamma\rangle}. \tag{12}$$

Note that $|\hat{\varphi}(\gamma)| \le c_N(1+|\gamma|)^{-N}$, therefore the latter sum in (12) is majorized by

$$\sum_{\gamma\in\Gamma} c_N(1+|\gamma|)^{-N}|b(\gamma)| \le c_N\int_0^\infty (1+t)^{-N}d|\hat{\mu}|(t).$$

Arguing as above, we get that the integral is finite, therefore the sum in (12) uniformly converges, and it is almost periodic in $t\in\mathbb{R}^d$.

Check that $(f\star\mu)(t)$ is almost periodic for each continuous function f with a compact support in a ball $B(R)$. Let $\varphi_n \in S(R^d)$, $\operatorname{supp}\varphi_n \subset B(R+1)$, be a sequence that uniformly converges to f. The numbers a_λ are uniformly bounded, hence the almost periodic functions $(\varphi_n\star\mu)(t)$ uniformly converge to $(f\star\mu)(t)$, and the latter function is also almost periodic. □

Combining Theorems 11 and 13 and taking into account that the discreteness of $\Lambda-\Lambda$ implies the uniformly discreteness of Λ, we obtain the proof of Theorem 9.

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