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REMOVABILITY RESULTS FOR SUBHARMONIC FUNCTIONS, FOR HARMONIC FUNCTIONS AND FOR HOLOMORPHIC FUNCTIONS

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We begin with an improvement to Blanchet's extension result for subharmonic functions. With the aid of this improvement we then give extension results both for harmonic and for holomorphic functions. Our results for holomorphic functions are related to Besicovitch's and Shiffman's extension results, at least in some sense.

1. Introduction. 1.1. We will consider extension problems for subharmonic, harmonic and holomorphic functions. Our results are based on an extension result for subharmonic functions, see Theorem 1 below. The starting point for this result is a result of Blanchet. As a matter of fact, Blanchet has shown that hypersurfaces of class $\mathcal{C}^1$ are removable singularities for subharmonic functions, provided the considered subharmonic functions satisfy certain assumptions. We have showed that, in certain cases, it is sufficient that the exceptional sets are of finite (n-1)-dimensional Hausdorff measure, (see [21], Theorem, p. 568).

We will then apply our subharmonic function result to get extension results both for harmonic and for holomorphic functions, see Sections 3 and 4 below.

1.2. Notation. Our notation is more or less standard (see [19, 20]). However, for the convenience of the reader we recall here the following. We use the common convention $0\cdot\pm\infty=0$. For each $n\geq 1$ we identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$. In integrals we will write dx for the Lebesgue measure in $\mathbb{R}^n$, $n\in\mathbb{N}$. Let $0\leq\alpha\leq n$ and $A\subset\mathbb{R}^n$, $n\geq 1$. Then we write $\mathcal{H}^\alpha(A)$ for the α-dimensional Hausdorff (outer) measure of A. Recall that $\mathcal{H}^0(A)$ is the number of points of A. If $x=(x_1,\dots,x_n)\in\mathbb{R}^n$, $n\geq 2$, and $j\in\mathbb{N}$, $1\leq j\leq n$, then we write $x=(x_j,X_j)$, where $X_j=(x_1,\dots,x_{j-1},x_{j+1},\dots,x_n)$. Moreover, if $A\subset\mathbb{R}^n$, $1\leq j\leq n$, and $x_j^0\in\mathbb{R}$, $X_j^0\in\mathbb{R}^{n-1}$, we write

$$A(x_j^0)=\{\,X_j\in\mathbb{R}^{n-1}\colon x=(x_j^0,X_j)\in A\,\},\ \ A(X_j^0)=\{\,x_j\in\mathbb{R}\colon x=(x_j,X_j^0)\in A\,\}.$$

If $\Omega\subset\mathbb{R}^n$ and $p>0$, then $\mathcal{L}^p_{\rm loc}(\Omega)$, $p>0$, is the space of functions u in Ω for which $|u|^p$ is locally integrable on Ω.

For the definition and properties of harmonic and subharmonic functions, see e.g. [7, 8, 9, 14, 16], for the definition of holomorphic functions see e.g. [3, 10, 11].

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2. Lemmas.

2.1. A result of Federer. The following important result of Federer on geometric measure theory will be used repeatedly.

Lemma. ([4], Theorem 2.10.25, p. 188, and [24], Corollary 4, Lemma 2, p. 114) *Suppose that* $E \subset \mathbb{R}^n$, $n \geq 2$.

1. *If* $\mathcal{H}^{n-1}(E) = 0$, *then for all* j, $1 \leq j \leq n$, *and for* $\mathcal{H}^{n-1}$*-almost all* $X_j \in \mathbb{R}^{n-1}$ *the set* $E(X_j)$ *is empty.*
2. *If* $\mathcal{H}^{n-1}(E) < +\infty$, *then for all* j, $1 \leq j \leq n$, *and for* $\mathcal{H}^{n-1}$*-almost all* $X_j \in \mathbb{R}^{n-1}$ *the set* $E(X_j)$ *is finite.*

2.2. A result of Blanchet and our improvement.

2.2.1. For the sake of orientation we begin by recalling the result of Blanchet [2], Theorems 3.1, 3.2 and 3.3, pp. 312-313:

Blanchet's theorem. *Let* Ω *be a domain in* $\mathbb{R}^n$, $n \geq 2$, *and let* S *be a hypersurface of class* $\mathcal{C}^1$ *which divides* Ω *into two subdomains* Ω_1 *and* Ω_2. *Let* $u \in \mathcal{C}^0(\Omega) \cap \mathcal{C}^2(\Omega_1 \cup \Omega_2)$ *be subharmonic (respectively convex (or respectively plurisubharmonic provided* Ω *is then a domain in* $\mathbb{C}^n$, $n \geq 1$*)) in* Ω_1 *and* Ω_2. *If* $u_i = u|\Omega_i \in \mathcal{C}^1(\Omega_i \cup S)$, $i = 1, 2$, *and*

$$\frac{\partial u_i}{\partial \overline{n}^k} \geq \frac{\partial u_k}{\partial \overline{n}^k} \tag{1}$$

on S *with* $i, k = 1, 2$, *then* u *is subharmonic (respectively convex (or respectively plurisubharmonic)) in* Ω.

Above $\overline{n}^k = (\overline{n}_1^k, \overline{n}_2^k, \ldots, \overline{n}_n^k)$ is the unit normal exterior to Ω_k, and $u_k \in \mathcal{C}^1(\Omega_k \cup S)$, $k = 1, 2$, means that there exist n functions v_k^j, $j = 1, 2, \ldots, n$, continuous on $\Omega_k \cup S$, such that

$$v_k^j(x) = \frac{\partial u_k}{\partial x_j}(x)$$

for all $x \in \Omega_k$, $k = 1, 2$ and $j = 1, 2, \ldots, n$.

The following example shows that one cannot drop the above condition (1) in Blanchet's theorem.

Example. The function $u\colon \mathbb{R}^2 \to \mathbb{R}$,

$$u(z) = u(x + iy) = u(x, y) := \begin{cases} 1 + x, & \text{when } x < 0; \\ 1 - x, & \text{when } x \geq 0, \end{cases}$$

is continuous in $\mathbb{R}^2$ and subharmonic, even harmonic in $\mathbb{R}^2 \setminus (\{0\} \times \mathbb{R})$. It is easy to see that u does not satisfy the condition (1) on $S = \{0\} \times \mathbb{R}$ and that u is not subharmonic in $\mathbb{R}^2$.

Remark 1. For related results, previous and later, see [12], Lemma 2.2, p. 201, Fundamental Theorem 2.1, p. 200-201, and [13], Lemma 4.1, p. 503, Theorem 2.1, p. 498, Theorems 3.1 and 3.2, p. 500-501. In this connection, see also [7], 1.4.3, p. 21-22.

2.2.2. An improvement to the result of Blanchet. Already in [19], Theorem 4, p. 181-182, we have given partial improvements to the cited subharmonic removability results of Blanchet. Below we recall, however, our more recent improvement to Blanchet's result, see

Theorem 1 below. Instead of hypersurfaces of class $\mathcal{C}^1$, now arbitrary sets of finite $(n-1)$-dimensional Hausdorff measure are allowed as exceptional sets. Then, however, the condition (1) is replaced by another, related condition, the condition (iv) below. Moreover, an additional integrability condition on the second partial derivatives $\frac{\partial^2 u}{\partial x_j^2}$, $j = 1, 2, \ldots, n$, must be imposed, see (iii) below.

Theorem 1 ([21], Theorem, p. 568)**.** *Suppose that Ω is a domain in $\mathbb{R}^n$, $n \geq 2$. Let $E \subset \Omega$ be closed in Ω and let $\mathcal{H}^{n-1}(E) < +\infty$. Let $u\colon \Omega \to [-\infty, +\infty]$ be such that the following conditions are satisfied:*

(i) $u \in \mathcal{L}^1_{\text{loc}}(\Omega)$.

(ii) $u \in \mathcal{C}^2(\Omega \setminus E)$.

(iii) For each j, $1 \leq j \leq n$, $\frac{\partial^2 u}{\partial x_j^2} \in \mathcal{L}^1_{\text{loc}}(\Omega)$.

(iv) For each j, $1 \leq j \leq n$, and for $\mathcal{H}^{n-1}$-almost all $X_j \in \mathbb{R}^{n-1}$ such that $E(X_j)$ is finite, the following condition holds: For each $x_j^0 \in E(X_j)$ there exist sequences $x_{j,l}^{0,1}, x_{j,l}^{0,2} \in (\Omega \setminus E)(X_j)$, $l = 1, 2, \ldots$, such that $x_{j,l}^{0,1} \nearrow x_j^0$, $x_{j,l}^{0,2} \searrow x_j^0$, and that

(iv(a)) $\lim_{l\to+\infty} u(x_{j,l}^{0,1}, X_j) = \lim_{l\to+\infty} u(x_{j,l}^{0,2}, X_j) \in \mathbb{R}$,

(iv(b)) $-\infty < \lim_{l\to+\infty} \frac{\partial u}{\partial x_j}(x_{j,l}^{0,1}, X_j) \leq \lim_{l\to+\infty} \frac{\partial u}{\partial x_j}(x_{j,l}^{0,2}, X_j) < +\infty$.

(v) u is subharmonic in $\Omega \setminus E$.

Then $u|\Omega \setminus E$ has a subharmonic extension to Ω.

Corollary 1 ([19], Theorem 4, p. 181–182)**.** *Suppose that Ω is a domain in $\mathbb{R}^n$, $n \geq 2$. Let $E \subset \Omega$ be closed in Ω and let $\mathcal{H}^{n-1}(E) < +\infty$. Let $u\colon \Omega \to \mathbb{R}$ be such that*

(i) $u \in \mathcal{C}^0(\Omega)$.

(ii) $u \in \mathcal{C}^2(\Omega \setminus E)$.

(iii) For each j, $1 \leq j \leq n$. $\dfrac{\partial^2 u}{\partial x_j^2} \in \mathcal{L}^1_{\text{loc}}(\Omega)$.

(iv) For each j, $1 \leq j \leq n$, and for $\mathcal{H}^{n-1}$-almost all $X_j \in \mathbb{R}^{n-1}$ such that $E(X_j)$ is finite, one has

$$-\infty < \liminf_{\epsilon\to 0+0} \frac{\partial u}{\partial x_j}(x_j^0 - \epsilon, X_j) \leq \limsup_{\epsilon\to 0+0} \frac{\partial u}{\partial x_j}(x_j^0 + \epsilon, X_j) < +\infty$$

for each $x_j^0 \in E(X_j)$.

(v) u is subharmonic in $\Omega \setminus E$.

Then u is subharmonic.

Corollary 2 ([19], Corollary 4A, p. 185–186)**.** *Suppose that Ω is a domain in $\mathbb{R}^n$, $n \geq 2$. Let $E \subset \Omega$ be closed in Ω and let $\mathcal{H}^{n-1}(E) < +\infty$. Let $u\colon \Omega \to \mathbb{R}$ be such that*

(i) $u \in \mathcal{C}^1(\Omega)$,

(ii) $u \in \mathcal{C}^2(\Omega \setminus E)$,

(iii) for each j, $1 \leq j \leq n$, $\dfrac{\partial^2 u}{\partial x_j^2} \in \mathcal{L}^1_{\text{loc}}(\Omega)$,

(iv) *u is subharmonic in $\Omega \setminus E$.*

Then u is subharmonic.

3. Removability results for harmonic functions.

3.1. Removability results for harmonic functions are stated, among others, in [5, 6, 15, 25].

Now, using our Theorem 1, we give the following extension result for harmonic functions:

Theorem 2. *Suppose that Ω is a domain in $\mathbb{R}^n$, $n \geq 2$. Let $E \subset \Omega$ be closed in Ω and let $\mathcal{H}^{n-1}(E) < +\infty$. Let $u \colon \Omega \to [-\infty, +\infty]$ be such that the following conditions are satisfied:*

(i) $u \in \mathcal{L}^1_{\rm loc}(\Omega)$.

(ii) *For each j, $1 \leq j \leq n$,* $\frac{\partial^2 u}{\partial x_j^2} \in \mathcal{L}^1_{\rm loc}(\Omega)$.

(iii) *For each j, $1 \leq j \leq n$, and for $\mathcal{H}^{n-1}$-almost all $X_j \in \mathbb{R}^{n-1}$ such that $E(X_j)$ is finite, the following condition holds: For each $x_j^0 \in E(X_j)$ there exist sequences $x_{j,l}^{0,1}, x_{j,l}^{0,2} \in (\Omega \setminus E)(X_j)$, $l = 1, 2, \ldots,$ such that $x_{j,l}^{0,1} \nearrow x_j^0$, $x_{j,l}^{0,2} \searrow x_j^0$, and that*

(iii(a)) $\lim_{l \to +\infty} u(x_{j,l}^{0,1}, X_j) = \lim_{l \to +\infty} u(x_{j,l}^{0,2}, X_j) \in \mathbb{R}$,

(iii(b)) $\lim_{l \to +\infty} \frac{\partial u}{\partial x_j}(x_{j,l}^{0,1}, X_j) = \lim_{l \to +\infty} \frac{\partial u}{\partial x_j}(x_{j,l}^{0,2}, X_j) \in \mathbb{R}$.

(iv) *u is harmonic in $\Omega \setminus E$.*

Then $u|\Omega \setminus E$ has a harmonic extension to Ω.

Proof. Since the assumptions of Theorem 1 do hold for the subharmonic function u, u has a subharmonic extension u^* to Ω. On the other hand, the assumptions of Theorem 1 hold also for the subharmonic function $v = -u$. Thus $v = -u$ has a subharmonic extension $v^* = (-u)^*$ to Ω. Since $-v^* = u^*$, the extension u^* of u is both subharmonic and superharmonic, thus harmonic and the claim follows. □

3.2. Then a concise special case to our above Theorem 2:

Corollary 3. *Suppose that Ω is a domain in $\mathbb{R}^n$, $n \geq 2$. Let $E \subset \Omega$ be closed in Ω and let $\mathcal{H}^{n-1}(E) = 0$. Let $u \colon \Omega \to [-\infty, +\infty]$ be such that the following conditions are satisfied:*

(i) $u \in \mathcal{L}^1_{\rm loc}(\Omega)$,

(ii) *for each j, $1 \leq j \leq n$,* $\frac{\partial^2 u}{\partial x_j^2} \in \mathcal{L}^1_{\rm loc}(\Omega)$,

(iii) *u is harmonic in $\Omega \setminus E$.*

Then $u|\Omega \setminus E$ has a harmonic extension to Ω.

Proof. With the aid of the above lemma one sees easily that the assumptions of Theorem 2 are satisfied. □

4. Removability results for holomorphic functions.

4.1. Below we will give certain counterparts to some of Shiffman's well-known extension results for holomorphic functions. For Shiffman's results, see, among others, [24, 5, 6, 15].

4.2. We consider first a counterpart to the following result:

Shiffman's theorem. ([24], Lemma 3, p. 115, and [6], Theorem 1.1 (b), p. 703) *Let Ω be a domain in $\mathbb{C}^n$, $n \geq 1$. Let $E \subset \Omega$ be closed in Ω and let $\mathcal{H}^{2n-1}(E) < +\infty$. If $f \colon \Omega \to \mathbb{C}$ is continuous and $f|\Omega \setminus E$ is holomorphic, then f has a unique holomorphic extension to Ω.*

Shiffman's proof was based on coordinate rotation, on use of the Cauchy integral formula and on the cited result of Federer, the lemma above.

For slightly more general versions of Shiffman's result with different proofs, see [17], Theorem 3.1, p. 49, Corollary 3.2, p. 52, and [18], Theorem 3.1, p. 333, Corollary 3.3, p. 336.

4.3. Using also here our above Theorem 1, or more directly Theorem 2, we get the following counterpart to Shiffman's above result:

Theorem 3. *Suppose that Ω is a domain in $\mathbb{C}^n$, $n \geq 1$. Let $E \subset \Omega$ be closed in Ω and let $\mathcal{H}^{2n-1}(E) < +\infty$. Let $f\colon \Omega \setminus E \to \mathbb{C}$ be such that the following conditions are satisfied:*

(i) $f \in \mathcal{L}^1_{\mathrm{loc}}(\Omega)$.

(ii) For each j, $1 \leq j \leq 2n$, $\frac{\partial^2 f}{\partial x_j^2} \in \mathcal{L}^1_{\mathrm{loc}}(\Omega)$.

(iii) For each j, $1 \leq j \leq 2n$, and for $\mathcal{H}^{2n-1}$-almost all $X_j \in \mathbb{R}^{2n-1}$ such that $E(X_j)$ is finite, the following condition holds: For each $x_j^0 \in E(X_j)$ there exist sequences $x_{j,l}^{0,1}, x_{j,l}^{0,2} \in (\Omega \setminus E)(X_j)$, $l = 1, 2, \ldots$, such that $x_{j,l}^{0,1} \nearrow x_j^0$, $x_{j,l}^{0,2} \searrow x_j^0$, and that

(iii(a)) $\lim_{l \to +\infty} f(x_{j,l}^{0,1}, X_j) = \lim_{l \to +\infty} f(x_{j,l}^{0,2}, X_j) \in \mathbb{C}$,

(iii(b)) $\lim_{l \to +\infty} \frac{\partial f}{\partial x_j}(x_{j,l}^{0,1}, X_j) = \lim_{l \to +\infty} \frac{\partial f}{\partial x_j}(x_{j,l}^{0,2}, X_j) \in \mathbb{C}$.

(iv) f is holomorphic in $\Omega \setminus E$.

Then $f|\Omega \setminus E$ has a holomorphic extension to Ω.

Proof. Write $f = u + iv$. It is sufficient to show that u and v have harmonic extensions u^* and v^* to Ω. As a matter of fact, then $f^* = u^* + iv^*\colon \Omega \to \mathbb{C}$ is $\mathcal{C}^\infty$ and thus a continuous function. Therefore the claim follows from Shiffman's theorem or from [17, 18].

Another possibility for the proof is just to observe that the in $\Omega \setminus E$ harmonic functions u and v have by Theorem 2 harmonic extensions u^* and v^* to Ω. Since u^* and v^* are thus $\mathcal{C}^\infty$ functions, the holomorphy of the extension $f^* = u^* + iv^*$ in Ω follows easily.

Therefore it remains only to check that both u and v satisfy the assumptions of our above Theorem 2. But this is seen at once! □

4.4. Then a counterpart to another result of Shiffman.

The following result of Besicovitch is well-known.

Besicovitch's theorem. ([1], Theorem 1, p. 2) *Let D be a domain in $\mathbb{C}$. Let $E \subset D$ be closed in D and let $\mathcal{H}^1(E) = 0$. If $f\colon D \setminus E \to \mathbb{C}$ is holomorphic and bounded, then f has a unique holomorphic extension to D.*

Much later Shiffman gave the following general version:

Another theorem of Shiffman. ([24], Lemma 3, p. 115, and [6], Theorem 1.1 (c), p. 703)) *Let Ω be a domain in $\mathbb{C}^n$, $n \geq 1$. Let $E \subset \Omega$ be closed in Ω and let $\mathcal{H}^{2n-1}(E) = 0$. If $f\colon \Omega \setminus E \to \mathbb{C}$ is holomorphic and bounded, then f has a unique holomorphic extension to Ω.*

Shiffman's proof was based on Besicovitch's result, on coordinate rotation, on the use of Cauchy integral formula and on the already stated important result of Federer, Lemma 2.

For slightly more general versions of Shiffman's result with different proofs, see [17], Theorem 3.1, p. 49, Corollary 3.2, p. 52, and [18], Theorem 3.1, p. 333, Corollary 3.3, p. 336.

4.5. Now we give the following counterpart to Shiffman's above result. The proof will be based, in addition to Federer's cited Lemma above, again on our Theorem 1, or more directly on our above Corollary 3.

Theorem 4. *Suppose that Ω is a domain in $\mathbb{C}^n$, $n \geq 1$. Let $E \subset \Omega$ be closed in Ω and let $\mathcal{H}^{2n-1}(E) = 0$. Let $f \colon \Omega \setminus E \to \mathbb{C}$ be holomorphic and such that the following conditions are satisfied:*

(i) $f \in \mathcal{L}^1_{\mathrm{loc}}(\Omega)$,

(ii) for each j, $1 \leq j \leq 2n$, $\frac{\partial^2 f}{\partial x_j^2} \in \mathcal{L}^1_{\mathrm{loc}}(\Omega)$.

Then f has a holomorphic extension to Ω.

Proof. Write $f = u + iv$. It is sufficient to show that u and v have subharmonic extensions to Ω. As a matter of fact, then f will be locally bounded in Ω, and thus the claim will follow from Shiffman's theorem or also from the already cited slightly more general results from [17, 18].

And to see that u and v have indeed subharmonic extensions to Ω, we use our Theorem 1 as follows.

It is sufficient to show that the assumption (iv) of Theorem 1 is satisfied. For that purpose take j, $1 \leq j \leq 2n$, arbitrarily. By Federer's result, lemma above, we know that for $\mathcal{H}^{2n-1}$ almost all $X_j \in \mathbb{R}^{2n-1}$ the set $E(X_j)$ is empty. Thus for $\mathcal{H}^{2n-1}$ almost all $X_j \in \mathbb{R}^{2n-1}$ the functions $u(\cdot, X_j) \colon \Omega(X_j) \to \mathbb{R}$ and $v(\cdot, X_j) \colon \Omega(X_j) \to \mathbb{R}$ are $\mathcal{C}^\infty$ functions. Therefore, the assumption (iv) is satisfied both for u and for v, concluding the proof.

Another, and perhaps more direct, possibility for the proof is just to observe that the in $\Omega \setminus E$ harmonic functions u and v have by Corollary 3 harmonic extensions u^* and v^* to Ω. Since u^* and v^* are $\mathcal{C}^\infty$ functions, the holomorphy of the extension $f^* = u^* + iv^*$ in Ω follows easily. □

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