УДК 517.55, 517.57

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ON ASYMPTOTIC BEHAVIOR OF THE pTH MEANS OF THE GREEN POTENTIAL FOR $0 < p \le 1$

I. E. Chyzhykov, M. A. Voitovych. *On asymptotic behavior of the pth means of the Green potential for* $0 < p \le 1$, Mat. Stud. **46** (2016), 159–170.

For $0 < p \le 1$ we prove sharp estimates of pth means of the invariant Green potentials in the unit ball in $\mathbb{C}^n$ in terms of smoothness properties of a measure.

1. Introduction and main result. For $n \in \mathbb{N}$, let $\mathbb{C}^n$ denote the n-dimensional complex space with the inner product

$$\langle z, w\rangle = \sum_{j=1}^{n} z_j \overline{w}_j, \ \ z, w \in \mathbb{C}^n.$$

Let B denote the unit ball $\{z \in \mathbb{C}^n \colon |z| < 1\}$ with the boundary $S = \{z \in \mathbb{C}^n \colon |z| = 1\}$, where $|z| = \sqrt{\langle z, z\rangle}$.

For $z, w \in B$, define the *involutive automorphism* φ_w of the unit ball B given by

$$\varphi_w(z) = \frac{w - P_w z - (1 - |w|^2)^{1/2} Q_w z}{1 - \langle z, w\rangle},$$

where $P_0 z = 0,\ P_w z = \frac{\langle z,w\rangle}{|w|^2} w,\ w \ne 0$, is the orthogonal projection of $\mathbb{C}^n$ onto the subspace generated by w and $Q_w = I - P_w$. We note that ([10, p.11])

$$1 - |\varphi_w(z)|^2 = \frac{(1 - |w|^2)(1 - |z|^2)}{|1 - \langle z, w\rangle|^2}. \tag{1}$$

The *invariant Laplacian* $\tilde{\Delta}$ on B is defined by

$$\tilde{\Delta} f(a) = \Delta(f \circ \varphi_a)(0),$$

where $f \in C^2(B)$, $\Delta = 4\sum_{i=1}^{n}(\partial^2/\partial z_i \partial \bar{z}_i)$ is the ordinary Laplacian. The operator $\tilde{\Delta}$ is invariant with respect to any holomorphic automorphism of B, i.e., $\tilde{\Delta}(f \circ \psi) = (\tilde{\Delta} f) \circ \psi$ for all $\psi \in \mathcal{M}$, the group of holomorphic automorphisms of B ([8, Chap.4], [10]).

The *Green's function* for the invariant Laplacian is defined by $G(z, w) = g(\varphi_w(z))$, where $g(z) = \frac{n+1}{2n} \int_{|z|}^{1} (1 - t^2)^{n-1} t^{-2n+1} dt$ ([10, Chap.6.2]).

2010 *Mathematics Subject Classification:* 31B25; 31C05.
Keywords: Green potential; unit ball; invariant Laplacian; $\mathcal{M}$-subharmonic function; Riesz measure.
doi:10.15330/ms.46.2.159-170

If μ is a nonnegative Borel measure on B, the function G_μ defined by

$$G_\mu(z) = \int_B G(z,w)d\mu(w)$$

is called the *(invariant) Green potential* of μ, provided $G_\mu \not\equiv +\infty$. It is known that ([10, Chap. 6.4]) the condition $G_\mu \not\equiv +\infty$ is equivalent to

$$\int_B (1-|w|^2)^n d\mu(w) < \infty. \tag{2}$$

The Green potential is closely connected to the notion of an $\mathcal{M}$-subharmonic function ([10, Chap. 3]). A function u on B is called *$\mathcal{M}$-harmonic* if $u \in C^2(B)$ and $\tilde\Delta u = 0$. A function u on B is called *$\mathcal{M}$-subharmonic* if it is upper semicontinuos and $\tilde\Delta u \geq 0$ in the sense of distributions. In particular, $-G_\mu$ is $\mathcal{M}$-subharmonic. Note that in the case $n=1$ the classes of $\mathcal{M}$-subharmonic functions and subharmonic functions coincide.

Let u be a measurable function locally integrable on B. For $0<p<\infty$ we define

$$m_p(r,u) = \left(\int_S |u(r\xi)|^p \, d\sigma(\xi) \right)^{\frac{1}{p}},$$

where $d\sigma$ is the Lebesgue measure on S normalized so that $\sigma(S)=1$.

The following Riesz Decomposition Theorem holds.

Theorem A ([11]). *Suppose that u is $\mathcal{M}$-subharmonic in B and*

$$\sup_{1/2\leq r<1} m_1(r,u) < \infty.$$

Let μ be the Riesz measure of u in B with '$d\mu(z) = \tilde\Delta u(z)(1-|z|^2)^{-n-1}dV(z)$' where V is the Lebesgue measure on B. Then there exists a signed Borel measure ν on S such that for all $z \in B$

$$u(z) = P[\nu](z) - G_\mu(z) \tag{3}$$

where

$$P[\nu](z) = \int_S \frac{(1-|z|^2)^n}{|1-\langle z,\zeta\rangle|^{2n}} d\nu(\zeta)$$

is the Poisson-Stieltjes integral.

Growth of the integral $P[\nu](z)$ in the uniform metric is described in terms of smoothness properties of the measure ν in [1] for $n=1$, and in [4] for arbitrary $n \in \mathbb{N}$. Growth of $m_p(r, P[\nu])$ for $n=1$ and $p \geq 1$ is described in [15].

In the case $n>1$, sharp estimates of the growth rate of $m_p(r,G_\mu)$ for the whole class of Borel measures satisfying (2) are proved by M. Stoll in [9]. The case $n=1$ is studied much more deeper, see e.g. [12, 13, 14]).

Theorem B ([9]). *Let G_μ be the Green potential on B.*

(1) If $1 \leq p < \frac{2n-1}{2(n-1)}$, then

$$\lim_{r\to 1-} (1-r^2)^{n(1-1/p)} m_p(r,G_\mu) = 0. \tag{4}$$

(2) If $n \geq 2$ and $\frac{2n-1}{2(n-1)} \leq p < \frac{2n-1}{2n-3}$, then

$$\liminf_{r\to 1-} (1-r^2)^{n(1-1/p)} m_p(r,G_\mu) = 0. \tag{5}$$

Theorem B gives the maximal growth rate of the pth mean of the Green potentials, but does not take into account particular properties of a measure μ. It appears that smoothness properties of the so called complete measure (in the sense of Grishin [7, 2, 3]) or the *related measure* (see [6]) of a subharmonic function allow us to describe its growth. Here we just note that in the case when $n = 1$ and $u = -G_\mu$, the *complete measure* $\lambda = \lambda_u$ of u is the weighted Riesz measure $d\lambda(z) = (1 - |z|)d\mu(z)$.

Define for $a, b \in \bar{B}$ the *nonisotropic metric* on S by $d(a, b) = |1 - \langle a, b\rangle|^{1/2}$ ([8, Chap.5.1]).

For $\xi \in S$ and $\delta > 0$ we set

$$C(\xi, \delta) = \{z \in B\colon d(z, \xi) < \delta^{1/2}\}, \quad D(\xi, \delta) = \{z \in B\colon d(z, \xi) < \delta\}, \quad d\lambda(z) = (1 - |z|)^n d\mu(z).$$

The growth of $m_p(r, G_\mu)$ in terms of properties of the measure μ are described in [5] for $n > 1$. One dimensional analogue has been established earlier in [3] for all $p > 1$.

Theorem C ([5])**.** *Let $n \in \mathbb{N}$, $1 < p < \frac{2n-1}{2(n-1)}$, $0 \le \gamma < 2n$, and let μ be a Borel measure satisfying (2). Then*

$$m_p(r, G_\mu) = O\left((1 - r)^{\gamma - n}\right), \ r \uparrow 1 \tag{6}$$

holds if and only if

$$\left(\int_S \lambda^p (C(\xi, \delta))\, d\sigma(\xi)\right)^{\frac{1}{p}} = O(\delta^\gamma), \ 0 < \delta < 1. \tag{7}$$

In this paper we would like to consider the case $0 < p \le 1$. For this interval one can obtain an analogue of necessity part of Theorem C.

Theorem 1. *Let $n > 1$, $0 < p \le 1$, $0 \le \gamma < 2n$, and let μ be a Borel measure satisfying* (2) *and*

$$m_p(r, G_\mu) = O\left((1 - r)^{\gamma - n}\right), \ r \uparrow 1 \tag{8}$$

hold. Then

$$\left(\int_S \lambda^p (C(\xi, \delta))\, d\sigma(\xi)\right)^{\frac{1}{p}} = O(\delta^\gamma), \ 0 < \delta < 1. \tag{9}$$

Proof. The proof repeats that of necessity in Theorem C. □

The following theorem is the main result of the paper.

Theorem 2. *Let $n > 1$, $0 < p \le 1$, $0 \le \gamma < 2n$, and let μ be a Borel measure satisfying* (2) *and*

$$\int_S \lambda (C(\xi, \delta))\, d\sigma(\xi) = O(\delta^\gamma), \ 0 < \delta < 1, \tag{10}$$

hold. Then

$$m_p(r, G_\mu) = O\left((1 - r)^{\gamma - n}\right), \ r \uparrow 1. \tag{11}$$

Remark 1. An example in Section 4 shows that estimate (11) is sharp for all $p \in (0, 1]$. As a corollary we obtain a criterion of the growth of $m_p(r, G_\mu)$ in terms of properties of the measure μ in the case $p = 1$.

Corollary 1. *Let $n > 1$, $0 \le \gamma < 2n$, and let μ be a Borel measure satisfying* (2)*. Then*

$$\int_S \lambda\left(C(\xi,\delta)\right) d\sigma(\xi) = O\left(\delta^\gamma\right), \ 0 < \delta < 1, \tag{12}$$

holds if and only if

$$m_1\left(r, G_\mu\right) = O\left((1-r)^{\gamma-n}\right), \ r \uparrow 1. \tag{13}$$

Remark 2. Due to Proposition 1.10 ([5]) we always have

$$\int_S \lambda\left(C(\xi,\delta)\right) d\sigma(\xi) = o(\delta^n), \ \delta \downarrow 0.$$

This agrees with the relation $m_1(r, G_\mu) = o(1)$, $r \uparrow 1$ as it was shown by Ulrich ([11], see also [10]).

2. Some properties of the Green's function. The following lemma gives some basic properties of g which will be needed later.

Lemma A ([10])**.** *Let $0 < \delta < \frac{1}{2}$ be fixed. Then g satisfies the following two inequalities:*

$$g(z) \ge \frac{n+1}{4n^2}(1-|z|^2)^n, \ z \in B,$$
$$g(z) \le c(\delta)(1-|z|^2)^n, \ z \in B, |z| \ge \delta, \tag{14}$$

where $c(\delta)$ is a positive constant. Furthermore, if $n > 1$ then

$$g(z) \asymp |z|^{-2n+2}, \quad |z| \le \delta. \tag{15}$$

We need an estimate of p-means of the Green's function for $0 < p \le 1$. Analogues estimates for $p > 1$ are established by Stoll ([9, Lemma 5]). His proof does not work for $p \le 1$, though we use some ideas and notation from [9].

For fixed $\delta, 0 < \delta < 1/2$, denote $B^*(z,\delta) = \{w \in B \colon |\varphi_w(z)| < \delta\}$ and for $0 < r < 1$ denote

$$E(r) = \bigcup_{t \in S} B^*(rt, \delta).$$

Lemma 1. *Let $0 < p \le 1$, $n \in \mathbb{N}$. Then there exists $r_0 \in (0,1)$ such that for all $r \in (r_0, 1)$ and $w \in E(r)$*

$$m_p(G(\cdot,w),r) \asymp (1-r^2)^{n/p}, \quad \text{if } \ p \in (0,1] \setminus \left\{\frac{1}{2(n-1)}\right\},$$
$$m_p(G(\cdot,w),r) = O\left((1-r^2)^{n/p}\left(\ln\frac{1}{1-r}\right)^{1/p}\right), \ \text{if}\ p = \frac{1}{2(n-1)}, \ n > 1.$$

Proof. Let $w \in E(r), |w| = \rho$. Since σ is invariant under the group of unitary transformations of $\mathbb{C}^n$,

$$\int_S g(\varphi_w(rt))^p d\sigma(t) = \int_S g(\varphi_{\rho e}(rt))^p d\sigma(t) = \int_S g(\varphi_{re}(\rho t))^p d\sigma(t),$$

where $e = (1, 0, \ldots, 0) \in \mathbb{C}^n$.

For $0 < r, \rho < 1$, and fixed $\delta \in (0, \frac{1}{2}]$, let $N_r^\rho = \{t \in S \colon \rho t \in B^*(re,\delta)\}$.

For $t \in S \setminus N_r^\rho$, we have ([9, p. 491])

$$\int_S g(\varphi_{re}(\rho t))^p d\sigma(t) \le c(1-\rho^2)^{pn}(1-r^2)^{-n(p-1)} \le c(1-r^2)^n. \tag{16}$$

Also, for $c > 0$, let $\Omega_r^c = \{se^{i\theta} \colon 0 < 1-s < c(1-r^2), |\theta| < c(1-r^2)\}$ and $Q_r^c = \{t = (t_1, \dots, t_n) \in S \colon t_1 \in \Omega_r^c\}$.

By the definition of N_r^ρ, one has $|\varphi_{re}(\rho t)| < \delta$ for $t \in N_r^\rho$. Hence by (15) and (1)

$$g(\varphi_{re}(\rho t)) \asymp |\varphi_{re}(\rho t)|^{-2(n-1)} = c_1 \frac{|1 - r\rho t_1|^{2(n-1)}}{(|1-r\rho t_1|^2 - (1-r^2)(1-\rho^2))^{n-1}}, \tag{17}$$

where $c_1 = c_1(n)$.

It is known that ([9, Lemma 3]) there exist $c_2 = c_2(\delta)$ and $r(\delta)$ such that $N_r^\rho \subset Q_r^{c_2}$ for all ρ with $\rho e \in B^*(re, \delta)$, and all $r > r(\delta)$. Moreover, one can choose $r_0 \in (0,1)$ such that the inclusion holds for all $r \in (r_0, 1)$ and $0 < \delta \le \frac{1}{2}$.

By (1), $\rho t \in B^*(re, \delta)$ if and only if $(1-r^2)(1-\rho^2) > (1-\delta^2)|1-r\rho t_1|^2$, i.e.

$$|1-r\rho t_1|^2 \le \frac{1}{1-\delta^2}(1-r^2)(1-\rho^2) \le \frac{4}{3}(1-r^2)(1-\rho^2).$$

Since $t \in N_r^\rho$, we can apply the previous inequality to deduce

$$\int_{N_r^\rho} g(\varphi_{re}(\rho t))^p d\sigma(t) \le c_3 (1-r^2)^{p(n-1)}(1-\rho^2)^{p(n-1)} \times$$
$$\times \int_{Q_r^{c_2}} \big(|1-r\rho t_1|^2 - (1-r^2)(1-\rho^2)\big)^{-p(n-1)} d\sigma(t) =: c_3(1-r^2)^{p(n-1)}(1-\rho^2)^{p(n-1)} I_r. \tag{18}$$

Since ([9, p. 488])

$$|1-r\rho s e^{i\theta}|^2 - (1-r^2)(1-\rho^2) = (\rho - r)^2 + 2\rho r(1-s) - r^2\rho^2(1-s^2) + 4r\rho s \sin^2\frac{\theta}{2} \ge$$
$$\ge (r-\rho)^2 + (1-s)(1-r) + \frac{\theta^2}{\pi^2}, \quad \min\{\rho r, s\} \ge \frac{1}{2}, \tag{19}$$

by formula 1.4.5(2) in [8],

$$I_r = c_4(n) \iint_{\Omega_r^{c_2}} (1-s^2)^{n-2} \big(|1-r\rho s e^{i\theta}|^2 - (1-r^2)(1-\rho^2)\big)^{-p(n-1)} s\,ds\,d\theta \le$$
$$\le c_5 \int\limits_{1-c_2(1-r^2)}^{1} \left[\int\limits_0^{c_2(1-r^2)} (1-s)^{n-2} \Big[(r-\rho)^2 + (1-s)(1-r) + \frac{\theta^2}{\pi^2}\Big]^{-p(n-1)} d\theta \right] ds.$$

So

$$I_r \le c_5 \int\limits_{1-c_2(1-r^2)}^{1} (1-s)^{n-2} \Bigg[\int\limits_0^{\pi\sqrt{(1-s)(1-r)}} \left((1-s)(1-r) + \frac{\theta^2}{\pi^2}\right)^{-p(n-1)} d\theta +$$

$$+\left|\int\limits_{\pi\sqrt{(1-s)(1-r)}}^{c_2(1-r^2)}\left((1-s)(1-r)+\frac{\theta^2}{\pi^2}\right)^{-p(n-1)}d\theta\right|\Bigg]ds\le$$

$$\le c_5\int\limits_{1-c_2(1-r^2)}^{1}(1-s)^{n-2}\Bigg[\int\limits_{0}^{\pi\sqrt{(1-s)(1-r)}}((1-s)(1-r))^{-p(n-1)}\,d\theta+$$

$$+\left|\int\limits_{\pi\sqrt{(1-s)(1-r)}}^{c(1-r^2)}\left(\frac{\theta}{\pi}\right)^{-2p(n-1)}d\theta\right|\Bigg]ds.$$

Direct calculation shows that for $0\le 1-s\le c_2(1-r^2)$

$$\left|\int\limits_{\pi\sqrt{(1-s)(1-r)}}^{c_2(1-r^2)}\theta^{-2p(n-1)}d\theta\right|\le\begin{cases}c_6(1-r)^{1-2p(n-1)}, & p\in(0,1]\setminus\left\{\frac{1}{2(n-1)}\right\};\\ c_6\ln\frac{1}{1-r}, & p=\frac{1}{2(n-1)}.\end{cases}$$

Let us consider three cases. Firstly, let $0<p<\frac{1}{2(n-1)}$. Since $0<1-s<2c_2(1-r)$, we get

$$I_r\le c_7\int\limits_{1-c_2(1-r^2)}^{1}(1-s)^{n-2}(1-r)^{1-2p(n-1)}ds\le c_8(1-r^2)^{n-2p(n-1)}.$$

Now let $1\ge p>\frac{1}{2(n-1)}$. Then

$$I_r\le c_9\int\limits_{1-c_2(1-r^2)}^{1}\left((1-s)^{n-\frac32-p(n-1)}(1-r)^{\frac12-p(n-1)}+(1-s)^{n-2}(1-r)^{1-2p(n-1)}\right)ds\le$$

$$\le c_{10}(1-r^2)^{n-2p(n-1)}.$$

Finally, if $p=\frac{1}{2(n-1)}$, $n>1$, then

$$I_r\le c_9\int\limits_{1-c_2(1-r^2)}^{1}(1-s)^{n-2}\left(1+\ln\frac{1}{1-r}\right)ds\le c_{11}(1-r^2)^{n-1}\ln\frac{1}{1-r}.$$

Therefore from the latter inequalities, (16) and (18) we get

$$m_p(G(\cdot,w),r)\le c_{11}[(1-r^2)^{p(n-1)}(1-\rho^2)^{p(n-1)}(1-r^2)^{n-2p(n-1)}]^{1/p}=$$
$$=c_{11}\frac{(1-\rho^2)^{n-1}}{(1-r^2)^{n-1-n/p}}\le c(n,p)(1-r^2)^{n/p},\quad p\ne\frac{1}{2(n-1)},$$
$$m_p(G(\cdot,w),r)\le c_{12}\left(((1-r^2)(1-\rho^2))^{\frac12}(1-r^2)^{n-1}\ln\frac{1}{1-r}\right)^{1/p}\le$$
$$\le c(n)(1-r^2)^{n/p}\ln^{1/p}\frac{1}{1-r},\quad p=\frac{1}{2(n-1)}.$$

The upper estimates are proved. Let us prove the lower estimate. By (17) we have

$$\int_S g(\varphi_{re}(\rho t))^p d\sigma(t) \geq \tilde{c}_1 \int_{Q_r^c} |\varphi_{re}(\rho t)|^{-2p(n-1)} d\sigma(t) =$$
$$= \tilde{c}_1 \int_{Q_r^c} \frac{|1-r\rho t_1|^{2p(n-1)}}{\left(|1-r\rho t_1|^2-(1-r^2)(1-\rho^2)\right)^{p(n-1)}} d\sigma(t) \geq$$
$$\geq \tilde{c}_1 \int_{Q_r^c} \frac{(1-r\rho)^{2p(n-1)}}{\left(|1-r\rho t_1|^2-(1-r^2)(1-\rho^2)\right)^{p(n-1)}} d\sigma(t).$$

Equality (19) implies

$$|1-r\rho se^{i\theta}|^2-(1-r^2)(1-\rho^2) \leq (r-\rho)^2+2(1-s)(1-r\rho s)+\theta^2 \leq \tilde{c}_2(1-r)^2, \quad se^{i\theta} \in Q_r^c.$$

Then

$$\int_S g(\varphi_{re}(\rho t))^p d\sigma(t) \geq \tilde{c}_3 |1-r\rho|^{2p(n-1)} \times$$
$$\times \int\limits_{1-c(1-r^2)}^{1} \left[\int\limits_{0}^{c(1-r^2)} (1-s^2)^{n-2}\left(|1-r\rho se^{i\theta}|^2-(1-r^2)(1-\rho^2)\right)^{-p(n-1)} sds\right] d\theta \geq$$
$$\geq \tilde{c}_4(1-r)^{2p(n-1)} \int\limits_{1-c(1-r^2)}^{1} \left[\int\limits_{0}^{c(1-r^2)} (1-s^2)^{n-2}(1-r)^{-2p(n-1)} sds\right] d\theta = \tilde{c}_5(1-r^2)^n.$$

So, $m_p(G(\cdot,w),r) \geq \tilde{c}_6(1-r^2)^{n/p}$, $p \in (0;1]\backslash\{\frac{1}{2(n-1)}\}$. □

3. Proof of Theorem 2. Since, by the convexity, $m_p(r, G_\mu) \leq m_1(r, G_\mu)$, $0<p\leq 1$, it is enough to prove (11) for $p=1$. We follow the scheme from [5].

Let us estimate the absolute values of

$$u_1(z) := \int\limits_{B^*\left(z,\frac{1}{4}\right)} G(z,w)d\mu(w) \quad\text{and}\quad u_2(z) := \int\limits_{B\backslash B^*\left(z,\frac{1}{4}\right)} G(z,w)d\mu(w).$$

We start with u_1. By definition

$$0 \leq u_1(z) = \int_{B^*\left(z,\frac{1}{4}\right)} G(z,w)d\mu(w) = \int_{B^*\left(z,\frac{1}{4}\right)} g(\varphi_w(z))d\mu(w).$$

By (15) we have $g(z) \leq c|z|^{-2n+2}$ for $|z| \leq \frac{1}{4}$ and some positive constant c. Thus,

$$|u_1(z)| \leq c\int_{B^*\left(z,\frac{1}{4}\right)} |\varphi_w(z)|^{-2n+2}\, d\mu(w).$$

Denote $z = r\xi$, where $r=|z|$, $\frac{1}{2}<r<1$ and $w=|w|\eta$, $\xi,\eta \in S$. Let

$$K(z,\sigma_1,\sigma_2) = \{w \in B \colon |r-|w|| \leq \sigma_1, d(\xi,\eta) \leq \sigma_2\}.$$

In [5] it is proved that

$$B^*\left(z,\frac{1}{4}\right)\subset K(z,c_{13}(1-r),c_{14}(1-r)^{\frac{1}{2}}) \tag{20}$$

where $c_{13}=\frac{2}{3}$ and $c_{14}=4\sqrt{2}$. We denote

$$K(z):=K\left(z,\frac{2}{3}(1-r),4\sqrt{2}(1-r)^{\frac{1}{2}}\right),\ \tilde{K}(z):=K\left(z,\frac{2}{3}(1-r),8\sqrt{2}(1-r)^{\frac{1}{2}}\right).$$

The inclusion (20) implies

$$I_1:=\int_S|u_1(r\xi)|d\sigma(\xi)\le c_{15}\int_S\int_{B^*\left(r\xi,\frac{1}{4}\right)}|\varphi_w(r\xi)|^{-(2n-2)}d\mu(w)d\sigma(\xi)\le$$
$$\le c_{15}\int\limits_S\int\limits_{K(r\xi)}\frac{d\mu(w)}{|\varphi_w(r\xi)|^{2n-2}}d\sigma(\xi)$$

where $c_{15}=c_{15}(p)$. Then, by Fubini's theorem we deduce ($z=r\xi$, $w=|w|\eta$)

$$I_1\le c_{16}(n,p)\int\limits_{\substack{\eta\in S\\ ||w|-r|<\frac{2}{3}(1-r)}}\int\limits_{d(\xi,\eta)<4\sqrt{2}(1-r)^{1/2}}\frac{d\sigma(\xi)}{|\varphi_w(r\xi)|^{2n-2}}\,d\mu(|w|\eta)\le$$
$$\le c_{16}(p,n)\int\limits_{||w|-r|<\frac{2}{3}(1-r)}\int\limits_S\frac{d\sigma(\xi)}{|\varphi_w(r\xi)|^{2n-2}}d\mu(w). \tag{21}$$

Applying to (21) subsequently (1), (14) and Lemma 1, we obtain that for $0<p\le1$

$$\int_S\frac{d\sigma(\xi)}{|\varphi_w(r\xi)|^{2n-2}}=\int_S\frac{d\sigma(\xi)}{|\varphi_{r\xi}(w)|^{2n-2}}\le\int_S g(\varphi_{r\xi}(w))d\sigma(\xi)\le c_{17}(1-r^2)^n,\quad \frac{1}{2}<r<1.$$

Substituting the estimate of the inner integral into (21) we get

$$I_1\le c_{18}(1-r)^n\int\limits_{||w|-r|<\frac{2}{3}(1-r)}d\mu(|w|\eta). \tag{22}$$

We need the following lemma that plays a key role in the proof of Theorem C.

Lemma B ([5])**.** *Let ν be a finite positive Borel measure on S, $0<\delta<\frac{1}{2}$, and $p\ge1$. Then*

$$\int_S\nu^{p-1}(D(\xi,\delta))d\nu(\xi)\le\frac{N^p}{\delta^{2n}}\int_S\nu^p(D(\xi,\delta))\,d\sigma(\xi),$$

where N is a positive constant independent of p and δ.

To obtain the final estimate of I_1, for a fixed $r\in(\frac{1}{2},1)$, we define the measure ν_1 on the balls $\{D(\eta,t)\colon \eta\in S, t>0\}$ by

$$\nu_1(D(\eta,t))=\lambda\Big(\Big\{\rho\zeta\in B\colon |\rho-r|<\frac{2}{3}(1-r),d(\zeta,\eta)<t\Big\}\Big).$$

It can be expanded to the family of all Borel sets on B in the standard way. It is clear that

$$\nu_1(D(\eta,t)) \asymp (1-r)^n \mu\Big(\Big\{\rho\zeta \in B\colon |\rho - r| < \frac{2}{3}(1-r), d(\zeta,\eta) < t\Big\}\Big).$$

By using of (22) and Lemma B we get

$$I_1 \le c_{19} \int\limits_{||w|-r|<\frac{2}{3}(1-r)} d\lambda(|w|\eta) = c_{19}\int_S d\nu_1(\eta) \le$$

$$\le \frac{c_{19}N}{(1-r)^n}\int_S \nu_1\left(D(\eta, 8\sqrt{2}(1-r)^{\frac{1}{2}})\right) d\sigma(\eta) = \frac{c_{20}(n,p)}{(1-r)^n}\int_S \lambda\left(\tilde{K}(r\eta)\right) d\sigma(\eta).$$

Note that if $\rho\zeta \in \tilde{K}(r\eta)$ then

$$|1 - \langle \rho\zeta, \eta\rangle| \le |1 - \langle \zeta,\eta\rangle| + (1-\rho)\,|\langle\zeta,\eta\rangle| \le (4c_{14}^2 + c_{13} + 1)(1-r) = c_{21}(1-r). \quad (23)$$

Hence,

$$I_1 \le \frac{c_{20}}{(1-r)^n}\int_S \lambda\left(C(\eta, c_{21}(1-r))\right) d\sigma(\eta). \quad (24)$$

By the assumption of the theorem we deduce

$$I_1 = O((1-r)^{\gamma - n}),\ r \uparrow 1. \quad (25)$$

Let us estimate

$$u_2(z) = \int_B G(z,w)(1-|w|)^{-n} d\tilde{\lambda}(w)$$

where $d\tilde{\lambda}(w) = (1-|w|)^n \chi_{B\setminus B^*\left(z,\frac{1}{4}\right)}(w)d\mu(w)$, χ_E is the characteristic function of a set E. We may assume that $|z| \ge \frac{1}{2}$.

We denote

$$E_k = E_k(z) = \left\{w \in B\colon \left|1 - \left\langle \frac{z}{|z|}, w\right\rangle\right| < 2^{k+1}(1-|z|)\right\}, \quad k \in \mathbb{Z}_+.$$

Since $|1-\langle z,w\rangle| \ge \frac{1}{2}\left|1 - \left\langle \frac{z}{|z|}, w\right\rangle\right|$, one has that for $w \in E_{k+1}\setminus E_k$, $|1-\langle z,w\rangle| \ge 2^{k-1}(1-|z|)$. Combining Lemma A with the equality in (1) for $z \in B$ such that $|z| \ge \frac{1}{2}$ we get that $0 \le G(z,w) \le c_{22}\left(\frac{(1-|w|^2)(1-|z|^2)}{|1-\langle z,w\rangle|^2}\right)^n$ holds. So

$$|u_2(z)| \le c_{22}\int_B \left(\frac{(1+|w|)(1-|z|^2)}{|1-\langle z,w\rangle|^2}\right)^n d\tilde{\lambda}(w) \le$$

$$\le \sum_{k=1}^{[\log_2 \frac{1}{1-r}]} c_{22}\int_{E_{k+1}\setminus E_k}\left(\frac{(1+|w|)(1-|z|^2)}{2^{2(k-1)}(1-|z|)^2}\right)^n d\tilde{\lambda}(w) + c_{22}\int_{E_1}\left(\frac{(1+|w|)(1-|z|^2)}{(1-|z|)^2}\right)^n d\tilde{\lambda}(w) \le$$

$$\le \sum_{k=1}^{\infty}\int_{E_{k+1}\setminus E_k}\frac{4^n c_{22}}{\left(2^{2(k-1)}(1-|z|)\right)^n} d\tilde{\lambda}(w) + \int_{E_1}\frac{4^n c_{22}}{(1-|z|)^n} d\tilde{\lambda}(w) \le$$

$$\leq \frac{4^n c_{22}}{(1-|z|)^n}\left(\sum_{k=1}^{\infty}\frac{\tilde{\lambda}(E_{k+1})}{2^{2n(k-1)}}+\tilde{\lambda}(E_1)\right)\leq \frac{4^n c_{22}}{(1-|z|)^n}\sum_{k=1}^{\infty}\frac{\tilde{\lambda}(E_k)}{2^{2n(k-2)}}.$$

Therefore

$$\int_S |u_2(r\xi)|d\sigma(\xi) \leq \frac{c_{23}}{(1-r)^n}\sum_{k=1}^{\infty}\int_S \frac{\tilde{\lambda}(E_k(r\xi))}{2^{2n(k-2)}}d\sigma(\xi) =$$

$$= \frac{c_{23}}{(1-r)^n}\sum_{k=1}^{\infty}\frac{1}{2^{2n(k-2)}}\int_S \tilde{\lambda}\left(C\left(\xi, 2^{k+1}(1-r)\right)\right)d\sigma(\xi) \leq \frac{c_{24}}{(1-r)^n}\sum_{k=1}^{\infty}\frac{2^{\gamma(k+1)}(1-r)^{\gamma}}{2^{2n(k-2)}} =$$

$$= \frac{c_{25}}{(1-r)^n}\sum_{k=1}^{\infty}2^{k(\gamma-2n)} = \frac{c_{25}}{(1-r)^{n-\gamma}}\frac{2^{\gamma-2np}}{1-2^{\gamma-2n}} = \frac{c_{26}(n,\gamma)}{(1-r)^{n-\gamma}}.$$

Hence

$$m_p(r,G_\mu) \leq m_1(r,G_\mu) \leq \int_S |u_1(r\xi)|d\sigma(\xi) + \int_S |u_2(r\xi)|d\sigma(\xi) \leq \frac{c(n,\gamma)}{(1-r)^{n-\gamma}}.$$

4. An example.

Proposition 1. *For $n>1$, $0<p\leq 1$, $n<\gamma<2n$, there exists a Borel measure μ on B sutisfying (2) and such that*

$$G_\mu(z) = O\left((1-|z|)^{\gamma-n}\right),\ |z|\uparrow 1 \tag{26}$$

and for some $C>0$

$$\lambda\left(C(\xi,\delta)\right) \geq C\delta^{\gamma},\ 0<\delta<1. \tag{27}$$

Proof. We define $d\mu(z) = \frac{dV(z)}{(1-|z|)^{2n+1-\gamma}}$, where V is the Lebesgue measure on B.

We write

$$G_\mu(z) = \int_B G(z,w)d\mu(w) = \int_{B^*\left(z,\frac14\right)} G(z,w)d\mu(w) + \int_{B\setminus B^*\left(z,\frac14\right)} G(z,w)d\mu(w) =: J_1 + J_2.$$

Since, by (20) $1-|w| \asymp 1-|z|$ holds for $w\in B^*\left(z,\frac14\right)$, we get

$$J_1 \leq c_{27}\int_{B^*\left(z,\frac14\right)}\frac{G(z,w)dV(w)}{(1-|z|)^{2n+1-\gamma}} \leq \frac{c_{27}}{(1-|z|)^{2n+1-\gamma}}\int_{r-c_1(1-r)}^{r+c_1(1-r)}\int_S G(z,\rho\eta)d\sigma(\eta)\rho^{2n-1}d\rho.$$

Using Lemma 1 for $p=1$, we obtain

$$J_1 \leq \frac{c_{28}}{(1-|z|)^{2n+1-\gamma}}\int_{r-c_1(1-r)}^{r+c_1(1-r)}(1-\rho)^n\rho^{2n-1}d\rho \leq \frac{c_{29}}{(1-r)^{n-\gamma}}.$$

For $w\in B\setminus B^*\left(z,\frac14\right)$ we have (see (1))

$$0 \leq G(z,w) \leq c\left(\frac{(1-|w|^2)(1-|z|^2)}{|1-\langle z,w\rangle|^2}\right)^n.$$

Then by the above inequality and [8, Chap.1.4.10] it follows that

$$J_2 \le c(1-|z|)^n \int_B \frac{(1-|w|^2)^{-n-1+\gamma}}{|1-\langle z,w\rangle|^{2n}} dV(w) \le c_{30}(1-|z|)^n(1-|z|)^{-2n+\gamma} = c_{30}(1-|z|)^{\gamma-n}.$$

Thus $m_1(r,G_\mu) = O((1-r)^{\gamma-n}),\ r\uparrow 1$.

Let us prove (27). We have $d\lambda(w) = \frac{dV(w)}{(1-|w|)^{n+1-\gamma}}$. Then

$$\lambda(C(\xi,\delta)) \ge \int\limits_{C(\xi,\delta)\cap\left\{1-\frac{\delta}{2}\le|w|\le 1-\frac{\delta}{4}\right\}} \frac{dV(w)}{(1-|w|)^{n+1-\gamma}} \ge$$

$$\ge \delta^{\gamma-n-1} \int\limits_{C(\xi,\delta)\cap\left\{1-\frac{\delta}{2}\le|w|\le 1-\frac{\delta}{4}\right\}} dV(w) \ge c\delta^{\gamma-n-1}\delta^{n+1} = \delta^\gamma.$$

The latter estimates follow from the inclusion

$$C(\xi,\delta)\cap\left\{1-\frac{\delta}{2}\le|w|\le 1-\frac{\delta}{4}\right\} \supset \left\{|w|\eta\colon \frac{\delta}{4}\le 1-|w|\le\frac{\delta}{2}, d(\xi,\eta)\le\sqrt{\frac{\delta}{2}}\right\}.$$

Let us prove this. We denote $v = (1-\frac{\delta}{2})\zeta \in \partial C(\xi,\delta)$, $\zeta\in S$. Since $\min\{\delta(\xi,\eta)\colon |w|\eta \in C(\xi,\delta)\cap\left\{1-\frac{\delta}{2}\le|w|\le 1-\frac{\delta}{4}\right\}\}$ is attained at v, it is enough to estimate $d(\xi,\zeta)$ from below.

$$d(\xi,\zeta) = \sqrt{|1-\langle\xi,\zeta\rangle|} = \sqrt{|1-\langle\xi,\zeta\rangle - \langle\xi,|v|\zeta\rangle + \langle\xi,|v|\zeta\rangle|} \ge$$

$$\ge \sqrt{|1-\langle\xi,|v|\zeta\rangle| - |\langle\xi,\zeta\rangle - \langle\xi,|v|\zeta\rangle|} = \sqrt{\delta - \frac{\delta}{2}|\langle\xi,\zeta\rangle|} \ge \sqrt{\frac{\delta}{2}}.$$

The estimate (27) is proved. □

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Received 2.07.2016