In memory of
ANDRIY ANDRIYOVYCH KONDRATYUK
(March 5, 1941 – April 22, 2016)

Andriy Andriyovych Kondratyuk (1941 – 2016), Mat. Stud. **47** (2017), 100–112.

The famous Ukrainian mathematician **Andriy Kondratyuk** was born on March 5, 1941 in the village *Hremyache*. It is situated near the city of Ostrog, Rivne Region. The glorious Volyn land and the close relation to an ancient center of Ukrainian Science Ostrog Academy, an ancient center of the Ukrainian sciense, promoted in young Andriy two main features: patriotism and love for science.

In 1958 he graduated from the Dubno Pedagogical College and started to work as a teacher of mathematics in Voronkivska seven-year school in Volodymyrets, Rivne region.

In 1960 A. Kondratyuk entered the Lviv University and his life was attached to it forever on. He graduated with distinction from the Faculty of Mechanics and Mathematics and entered the PhD courses in 1965. However, for the next year, Andriy Kondratyuk was doing his military service in the army. After demobilization he returned to the postgraduate study. Also he was accepted as an Assistant Professor of Department of Function Theory and Probability.

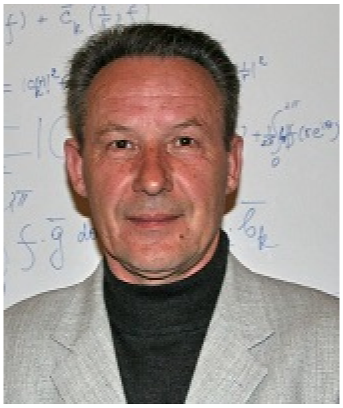
The first scientific paper was published in the journal “Lithuanian mathematical collection” (“Lietuvos matematikos rinkinys”) in 1967.

Andriy Kondratyuk defended his PhD thesis *“Estimates of indicators of entire functions”* in 1969.

He continued with Lviv University as an Assistant Proffesor of the Department of Function Theory and Probability. In 1971 he became an Associate Professor (Docent) of the same department.

During 1972–1975 he works as a lecturer of mathematics in Algeria and Cuba. After his arrival in Lviv Andriy Kondratyuk returned to the work at the Lviv University and began to conduct his scientific research very actively.

He published a lot of scientific works concerning meromorphic functions of completely regular growth and subharmonic functions during the next ten years. As a result of such productive work Kondratyuk defended the DSci thesis *“Fourier and Fourier-Laplace series method for meromorphic and subharmonic functions of completely regular growth”* in 1989.



Andriy Kondratyuk was a Professor since 1990 and the Head of the Department of Mathematical and Functional Analysis of Lviv University since 1992. He conducted active scientific work. Many PhD theses and DSci thesis were defended under his guidance and supervision. In 2010 he was given the honorary title of Distinguished Professor of the Ivan Franko Lviv National University.

Professor Andriy Kondratyuk died on April 22, 2016. A lot of scientific ideas, thoughts, plans left unfinished.

Andriy Kondratyuk always had an active public position. He took an active part in democratic processes of the reformation and collapse of the Soviet Union and the establishment of the independent Ukraine. He was a deputy of district council for a period, and one of the founders of the Statute of the Lviv University. Several times he presided at the first democratic conferences of the university stuff. Professor Kondratyuk always had his own independent view, often ahead of time.

Andriy Kondratyuk was a real intellectual. He loved and respected nature of his native land. He was a great fisherman and a mushroom gather. He had many favorite places in many parts of Ukraine. He mastered skiing in adulthood. His active way of life, optimism, cheerfulness, benevolence had always been and remain an example for his students, colleagues and friends.

Andriy Kondratyuk had two sons Taras and Yarema and five grandchildren.

The most essential scientific achievements of A. A. Kondratyuk

Ph.D. thesis

Bounds for Indicators of Entire Functions (1969)

To describe the asymptotic behavior of an entire function f of order $\rho > 0$ and of normal type along the rays $\{z : \arg z = \theta\}$, the function $h(\theta, f) = \limsup_{r\to+\infty} r^{-\rho} \ln |f(re^{i\theta})|$, $0 \le \theta \le 2\pi$ is used. It is called the indicator and was introduced by Fragmen and Lindelöf in 1908. Also there is the notion of lower indicator $\underline{h}(\theta, f)$, defined as $\underline{h}(\theta, f) = \liminf_{r\to+\infty} r^{-\rho} \ln |f(re^{i\theta})|$, $0 < \theta < 2\pi$ if f is an entire function with positive zeros and $\underline{h}(\theta, f) = \liminf^*_{r\to+\infty} r^{-\rho} \ln |f(re^{i\theta})|$, $0 \le \theta \le 2\pi$ in the case of an arbitrary entire function f of order $\rho > 0$, where $\liminf^*$ means that before passing to the limit C_0-sets of discs of zero linear density are removed from the plane $\mathbb{C}$ and then supremum in all C_0-sets is taken.

Let us start with classes of entire functions f with positive zeros. It is said that the zeros of f have density δ if the counting function $n(r)$ of the zeros of f possesses the property $n(r) \sim \delta r^\rho, r \to +\infty$. It was Lindelöf who had shown in 1902, even before the introduction of the notion of the indicator, that in this case the indicator coincides with the lower indicator (if ρ is an integer then the indicators are taken with respect to $r^\rho \ln r$) and can be expressed in terms of simple functions of δ, ρ and θ. Later G.Valiron (1914) and E.Titchmarsh (1927) showed that the existence of the density of the zeros is necessary for the coincidence of the indicator and lower indicator in the case of noninteger order.

The following problem arises when the density of the zeros does not exist: to give precise lower and upper estimates for the indicator and lower indicator. First step in this direction was done by B. Ya. Levin, who got sharp upper bound for the indicator of an entire function of order $\rho = \frac{1}{2}$ with positive zeros and prescribed upper density. Further results in this direction belong to A. F. Leont'ev (1956), M. I. Andrashko (1960), Y. V. Ostrovskyi (1961), N. V. Govorov (1966). The problem of finding sharp estimates of the indicator and lower indicator for the class of entire functions with positive zeros and prescribed conditions on the lower and upper densities was completely solved by A. A. Gold'berg (1962-1966).

In A. A. Kondratyuk's thesis precise estimates of growth of entire functions f of finite order ρ along the rays depending on the distribution of their zeros were established.

In particular, he got sharp estimates of $h(\theta, f)$, $\underline{h}(\theta, f)$ in the class of entire functions with positive zeros under some conditions on the maximal and minimal densities of zeros introduced by D. Polya (1929). Moreover, the results of Polya were substantially generalized.

Note that in the general case the upper (lower) angular density of zeros is not an additive measure of an angle. In 1962-1965 A. A. Goldberg developed the theory of the integral on nonadditive measure and used it to get precise estimates of the indicators with some conditions on the upper and lower angular densities of zeros. A. A. Kondratyuk introduced a notion of the maximal angular density of zeros of an entire function. An advantage of his definition is the fact that the maximal and minimal densities of zeros are additive measures of an angle. Based on this notion he completely solved the problem of finding precise estimates for the indicator and lower indicator of entire functions having finite maximal density. The finiteness of the maximal density means that the sequence of zeros does not have large accumulations. Such class of functions is distinguished by the fact that, firstly, the lower indicators of these functions unlike the general case are bounded and, secondly, the estimates in this class have very simple form.

Doctoral Thesis

Fourier and Fourier-Laplace series methods
for meromorphic and subharmonic functions of completely regular growth (1989)

In 1936-42 B. Levin and A. Pflüger developed the theory of entire functions of completely regular growth (c.r.g for short). This theory has wide applications in a number of sections of modern analysis such as analytic theory of differential equations, Riemann boundary problem theory, theory of almost periodic functions, interpolation of entire functions and their representation by ordinary and generalized Laurent series, solvability of the equations of convolution type in various classes of analytic functions, theory of characteristic functions of probabilistic laws. Also, directly or indirectly, in a number of theoretical physics, radiophysics, electrical and radio engineering, computer tomography problems. Therefore the problem of contensive generalization of this theory on meromorphic functions, and even for a wider scale than in the case of the ordinary or generalized order naturally arises. This problem turned out to be extremely difficult, because of a number of nontrivial circumstances. In particular, it was unclear how to define an indicator of a meromorphic function reasonably. Furthermore, unlike the case of entire functions, good asymptotics of the logarithm of the modulus of a meromorphic function does not imply regular (in usual interpretation) distribution of its zeros and poles. Therefore to solve this problem new research tools have to be used.

Using Fourier series method developed by L. A. Rubel and B. A. Taylor and some results of A. A. Gol'dberg and V. S. Azarin on regular behavior of the logarithm of the modulus of an entire function of a proximate order and their Fourier coefficients, A. A. Kondratyuk in series of his publications, starting in 1978, constructed complete and contensive theory of meromorphic functions of completely regular growth in a quite general comparison functions (growth functions) scale. The theory of meromorphic functions of c.r.g. in sense of A. A. Kondratyuk covers even wider class of entire functions than the classical Levin-Pflüger theory does. The monograph [26] of A. A. Kondratyuk contains comprehensive statement of this theory.

Subharmonic functions were introduced by F. Hartogs and F. Riesz though the concept was laid earlier in the works of H. Poincaré. They are in some sense generalizations of convex functions for the case of several variables. The theory of harmonic and subharmonic functions plays a significant role in analytic function theory. This is in particular due to the fact that the real part $\operatorname{Re} f$ and the imaginary part $\operatorname{Im} f$ of an analytic function f in some domain are harmonic functions and the functions $\log|f|$ and $|f|^p$, $p > 0$, are subharmonic in this domain. Thus theories of harmonic and subharmonic functions offer an efficient and powerful tool investigation of properties of analytic functions. However the utmost importance these theories have because of direct connections between subharmonic functions and the potential theory, since the fundamental F. Riesz representation theorem states that any subharmonic function is a locally sum of some harmonic function and a potential. Hence the studying of subharmonic and harmonic functions is one of the most important aspects of potential theory which plays a crucial role in investigation of problems of mathematical physics and field theory, including the newest ones.

Generalization of Rubel-Taylor theory on subharmonic functions in the plane did not cause essential complications and was done by P. Noverazz and Ya. V. Vasyl'kiv. Concerning multidimensional space the problem of the development of Fourier-Laplace series method, i.e. spherical harmonic series, for subharmonic functions was posed by L. A. Rubel in 1973. However it could not be solved for a long time. The main task in this problem was to describe Riesz measures of subharmonic functions of possibly most general type of growth. Also the problem of the application of results to studying subharmonic functions of c.r.g. in higher dimensions was of considerable interest.

Using the Poisson transformation of a generalized function on the unit sphere A. A. Kondratyuk introduced and developed spherical harmonic method (Fourier-Laplace series method) for subharmonic functions in the space. Thus the problem of L. A. Rubel was completely solved. The main result out of others obtained by A. A. Kondratyuk is a criterion on the description of Riesz measures of subharmonic functions with arbitrary growth restrictions. As a consequence analogs of classical Lindelöf, Borel, Weierstraß theorems were obtained. These and other results were applied to the investigation of differences of subharmonic functions, subharmonic functions of c.r.g. in the space and subharmonic functions of infinite order with radially distributed masses. The most important new scientific problems solved in this thesis are the following: 1) the problem of the construction of complete and contensive theory of meromorphic functions of c.r.g.; 2) the problem of the development of spherical harmonics method for subharmonic in the space functions and its application to the investigation of subharmonic functions of c.r.g. To solve these problems wide enough class of comparising functions was used as well as Fourier series method for meromorphic functions was further developed.

Further results

General Paley problem

In 1932, Paley conjectured that the following inequality is true for an arbitrary entire function f of order $p \geq 0$:

$$\liminf_{r\to+\infty} \frac{B(r,\ln|f|)}{T(r,\ln|f|)} \leq \begin{cases} \pi\rho, & \rho \geq 1/2; \\ \frac{\pi\rho}{\sin\pi\rho}, & 0 \leq \rho < 1/2. \end{cases}$$

where $B(r,\ln|f|)$, $T(r,\ln|f|)$ are commonly known growth characteristics for subharmonic function $u = \log|f|$. The equality holds, in particular, for Mittag-Leffler's entire functions. G. Valiron (1930) and A. Valund (1929) were the first who proved it for the case $0 \leq \rho < 1/2$. N. V. Govorov, using methods developed by him in order to solve the Riemann boundary problem with infinite index, completely proved Paley's hypothesis. It follows from the results of V. P. Petrenko that Paley's relation remains true if the lower order λ is taken instead of the order ρ, and f can be assumed meromorphic. In the case of entire functions and $\lambda < 1/2$ the possibility of replacement ρ by λ in Paley's relation was proved by A. A. Gol'dberg and Y. V. Ostrovskiy.

The problem of finding an exact upper bound for

$$\liminf_{r\to+\infty} \frac{m_q(r,u^+)}{T(r,u)}, \quad 1 < q \leq +\infty,$$

where u is a subharmonic in $\mathbb{R}^{p+2}$, $p \in \mathbb{N}$ function of finite lower order, is called the generalized Paley problem. It was solved by M. L. Sodin in 1983 for subharmonic functions in $\mathbb{R}^2$. A. A. Kondratyuk together with S. I. Tarasyuk and Ya. V. Vasylkiv using the Phragmén-Lindelöf principle for δ-subharmonic functions and properties of A. Bernstein star-function u^* completely solved the above mentioned generalized Paley problem. They proved that

$$\liminf_{r\to+\infty} \frac{m_q(r,u^+)}{T(r,u)} \leq M_q(\lambda,p),$$

$$M_q(\lambda,p) = m_q(Q_\lambda^{p/2}),\ Q_\lambda^{p/2}(\theta) = \begin{cases} A(\lambda,p)C_\lambda^{p/2}(\cos\theta), & 0 \leq \theta \leq \alpha_\lambda; \\ 0, & \alpha_\lambda \leq \theta \leq \pi, \end{cases}$$

$$\alpha_\lambda = \min\{\theta \in (0,\pi) \colon C_\lambda^{p/2}(\cos\theta) = 0\},\ A(\lambda,p) = \Big(\frac{\sigma_p}{\sigma_{p+1}} \int_0^{\alpha_\lambda} C_\lambda^{p/2}(\cos\theta)\sin^p\theta d\theta\Big)^{-1},$$

where σ_p being the measure of the unit sphere in $\mathbb{R}^p$, $C_\lambda^{p/2}$ is the Gegenbauer polynomial.

Functions conjugate to subharmonic functions

The following two fundamental facts are well-known in the modern complex and real analyses:

1. Real and imaginary parts of holomorphic in simply-connected domain function are harmonically conjugated functions. This is caused by the very nature of a derivative in complex analysis sense and is expressed in Cauchy-Riemann conditions.
2. Harmonically conjugated in the unit disk $\mathbb{D}$ functions are connected with each other through harmonic conjugation operator (Hilbert operator for a circle). This operator

permits to combine methods of the theories of functions of complex and real variables with methods of commutative harmonic analysis, in particular, with methods of the metric theory of conjugated functions.

Since on the one hand $\log B$ of Blaschke product B is a holomorphic function in the simply-connected domain $\mathbb{D}^*$, the unit disk $\mathbb{D}$ with radial slits from its zeros to the boundary $\mathbb{S}$, the real and imaginary parts of $\log B$ should also be expressed by a harmonic conjugation operator. On the other hand, the function $u = \log |B|$ is not harmonic in $\mathbb{D}$ (it is subharmonic) and this special fact should be properly reckoned with. A connection between logarithm of the modulus and the argument of Blaschke product was firstly established by A. A. Kondratyuk and Ya. V. Vasylkiv in 1996. It has the following form

$$\arg B(re^{i\theta}) = \widetilde{u}(re^{i\theta}) - \widetilde{p}(re^{i\theta})$$

for almost all $\theta \in [0; 2\pi]$ and any fixed $r \in (0, 1)$, where by $\sim$ the harmonic conjugation operator is denoted. A compensating function $p(z)$, called the zero distribution function of a product $B(z)$, has the following form

$$p(re^{i\theta}) = \int\limits_0^r \sum_{|a_j| \le t} P\left(r, te^{i(\theta - a_j)}\right) t^{-1}\, dt, \quad 0 < r < 1, \quad \theta \in [0, 2\pi],$$

where $P(r, w) = \mathrm{Re}\left[(r + w)(r - w)^{-1}\right], |w| < r$, is the Poisson kernel.

The introduction of the notion of the function conjugated to a subharmonic function in a domain starred with respect to the origin and studying its properties inspired new research direction.

In 2000, A. A. Kondratyuk jointly with Ya. V. Vasyl'kiv laid the foundations of the theory of functions conjugated to subharmonic functions. Namely, they introduced the notion of a function $\breve{u}$ conjugated to a subharmonic function u in a domain starred with respect to the origin; established a series of representations of such functions, in particular, they showed that if $u = \log |f|$ and f is holomorphic then the conjugated function $\breve{u}$ to the function u is a branch of $\mathrm{Arg}\, f$. They also obtained initial relations for the Fourier coefficients of a pair of functions $F = u + i\breve{u}$.

The main result is the following. Let $u(z)$ be a subharmonic in the disk $\{z \in \mathbb{C} : |z| < R\}$ function, harmonic in some neighborhood of the origin, $u(0) = 0$, let $\mu[u]$ be the Riesz measure of u. Then for any $r \in (0, R)$ $\breve{u}(re^{i\theta}) = \widetilde{u}(re^{i\theta}) - \widetilde{p}(re^{i\theta})$ holds for almost all $\theta \in [0; 2\pi]$. Here

$$p(z) = \int\limits_0^{|z|} \frac{dt}{t} \int\limits_{|a| \le t} P\left(|z|, t\frac{\bar{a}z}{|az|}\right) d\mu_a[u] = \int\limits_0^{|z|} \frac{dt}{t} \int\limits_0^{2\pi} P\left(|z|, t\frac{z}{|z|}e^{-i\varphi}\right) ds(t, \varphi), \quad |z| = r < R,$$

where $s(t, \varphi) = \mu[u]\left(\{z = |z|e^{i\theta} : |z| \le t, 0 \le \theta \le \pi\}\right)$.

It should be emphasized that since the function $\breve{u}$ conjugated to a subharmonic function u is expressed in terms of Hilbert operator for a circle, this crucial to the metric theory of conjugated functions (which by-turn belongs to the commutative harmonic analysis, one of the classical directions of mathematics) fact allows to combine effectively methods of this theory with methods of the theory of functions of real and complex variables by means of the potential theory and Nevanlinna theory methods. Note that this representation allowed to solve completely generalized A.Zigmund's problem, i.e. to describe the behavior of the Lebesgue integral p-means ($1 \le p < +\infty$) of $\log B$ and $\log |B|$ of Blaschke product B in terms of its zero distribution.

Distribution of zeros of the Riemann ζ-function

The Riemann ζ-function is defined as follows,

$$\zeta(s) = \sum_{n\in\mathbb{N}} \frac{1}{n^s} \quad \text{or} \quad \zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1},$$

where $\operatorname{Re} s > 1$, and the product is taken over all prime numbers. This function was first considered by L. Euler in 1737 for real s. In particular, he proposed its representation as the infinite product over all prime numbers p. In 1859, G. F. B. Riemann showed that $\zeta(s)$ has a meromorphic continuation to the whole complex s-plane $\mathbb{C}$ with a simple pole at $s = 1$. He put forward a hypothesis that all non-trivial (non-real) zeros of the function $\zeta(s)$ lie on *"the critical line"* $\operatorname{Re} s = \frac{1}{2}$. This hypothesis is considered to be one of the greatest unsolved problems in mathematics and is called *the Riemann hypothesis.*

In 1924 J. Littlewood established an analog of Jensen theorem for a rectangle and derived from it the following relation

$$\int_s^1 N(\eta, T)d\eta = O\left(T \log \frac{1}{\sigma - \frac{1}{2}}\right), \quad \sigma > \frac{1}{2}, \quad T \to +\infty,$$

where $N(\sigma, T)$ is the number of zeros of the Riemann ζ-function with real parts in $[0, T]$ and imaginary parts greater than σ.

A. A. Kondratyuk jointly with A. M. Brydun generalized the Jensen-Littlewood theorem for a rectangle and studied the properties of the Fourier coefficients of the logarithm of the Riemann ζ-function. In particular, they showed that the Fourier coefficients $l_k(\sigma, T)$ of the function $\log \zeta(s)$ are continuous functions with respect to σ for any fixed $T > 0$, and bounded when $\sigma \ge \sigma_0 > 1/2$, $T \ge 1$ by a constant depending only on σ_0. The coefficient $l_0(\sigma, T)$ is bounded when $\sigma \ge 1/2$, $T \ge 1$. Taking this into account they showed that the Riemann hypothesis is equivalent to the following statement: for any fixed σ such that $\frac{1}{2} < \sigma < 1$, and for any fixed $T > 0$ there exists a constant $C(\sigma, T)$ such that the inequality

$$\left(\frac{1}{T} \int_0^T |\log |\zeta(\sigma + it)||^q dt\right)^{\frac{1}{q}} \le C(\sigma, T)$$

holds for all $q \ge 1$.

Another important results obtained by A. A. Kondratyuk in this direction include the following.

a) Summation of the logarithm of the modulus of the Riemann zeta-function on the critical line with the kernel $1/|s|^4$. Jointly with P. A. Yatsulka the following new formulation of the Riemann hypothesis is obtained. Let $\{\rho_j\}$ be the sequence of nontrivial zeros of $\zeta(s)$. Then

$$\frac{1}{2\pi} \int_{\operatorname{Re} s = \frac{1}{2}} \frac{\log |\zeta(s)|}{|s|^4} |ds| = 1 - \gamma + 2 \sum_{\operatorname{Re} \rho_j > \frac{1}{2}} \left|\log \frac{\rho_j}{1 - \rho_j}\right| + \sum_{\operatorname{Re} \rho_j > \frac{1}{2}} \frac{(|\rho_j|^2 - \operatorname{Re} \rho_j)(2 \operatorname{Re} \rho_j - 1)}{|\rho_j(1 - \rho_j)|^2},$$

where γ is the Euler constant. The Riemann hypothesis is true if and only if

$$\frac{1}{2\pi}\int\limits_{\mathrm{Re}\, s=\frac{1}{2}}\frac{\log|\zeta(s)|}{|s|^4}|ds| = 1-\gamma.$$

b) An analog of the Carleman-Nevanlinna formula for meromorphic in a rectangle functions is established and applied to the summation of the logarithm of the modulus of the Riemann zeta-function on the critical and another vertical lines. In particular, let

$$I(\varepsilon) = \int\limits_0^{+\infty} e^{-\varepsilon t}\log\Big|\zeta\Big(\frac{1}{2}+it\Big)\Big|dt, \quad \varepsilon > 0,$$

and let $\{\rho_j\}$ be nontrivial zeros of $\zeta(s)$. Then

$$\frac{\pi}{2}\sum_j\Big|\mathrm{Re}\,\rho_j - \frac{1}{2}\Big| = I(+0) + \frac{\pi}{2}$$

where $I(+0) = \lim\limits_{\varepsilon\to 0} I(\varepsilon)$. Hence, the Riemann hypothesis is true if and only if $I(+0) = -\frac{\pi}{2}$.

Theory of meromorphic and subharmonic functions in multiply connected domains

Wide range of known and actual complex analysis problems leads to the study of meromorphic functions in multiply connected domains. In particular, considering the composition $f\circ\mathcal{R}$ of a transcendental meromorphic in $\mathbb{C}$ function f and a rational function $\mathcal{R}$ with $n-1$ distinct poles in $\mathbb{C}$, we obtain a meromorphic function in an n-connected domain. Nonetheless a profound and elegant theory operating with convenient notions and notation had not been constructed yet.

The case of a doubly connected domain is an important special case. By the Doubly Connected Mapping Theorem any such domain is conformally equivalent to an annulus or to the punctured plane which can be considered as a generalized annulus. The functions of the variable z holomorphic on annuli centered at the origin possess some remarkable properties. They admit the Laurent expansion, the Hadamard Three Circle Theorem holds, the integral means of their moduli as well as their logarithms are convex with respect to $\log|z|$. Nevertheless, there is a fundamental difference in a topological sense between simply connected and doubly connected domains which has its reflection in the theory of meromorphic functions. The fundamental (Poincaré) group of a simply connected domain is trivial, while for a doubly connected domain we have a group isomorphic to the additive group $\mathbb{Z}$.

The geometrical approach to the study of meromorphic functions in multiply connected domains was first proposed in 1940 by G. Hällström, the student of Nevanlinna and Lindelöf. He used Green functions for multiply connected domains which do not have explicit representation. Many other authors had studied later meromorphic functions in multiply connected domains and generalized the Nevanlinna theory for such classes of functions. Among them we mention N. Oğuztöreli, J. A. Jenkins, H. P. Kunzi, H. Wittich, V. A. Zmorovich, G. U. Mathevossian, R. Korhonen.

A. A. Kondratyuk proposed to consider the model case of a doubly connected domain, the annulus $A = \{z\colon 1/r_0 < |z| < R_0\}$, $R_0 \in (1, +\infty)$. Thus, if $f(z)$ is a holomorphic function on A then $f\left(\frac{1}{z}\right)$ and $z^m f(z)$, where $m \in \mathbb{Z}$, are holomorphic on A as well.

For meromorphic in finitely connected domains A. A. Kondratyuk also proposed two approaches.

The first approach uses the fact that any given finitely connected domain is conformally equivalent to some circular domain. The whole complex plane as well as a single point supposed to be circular domains of radius $+\infty$ and 0, respectively. This approach based on the generalization of the Jensen theorem and the decomposition lemma for circular domains. By this lemma, any given meromorphic in a circular domain function can be presented as a product of a meromorphic function and functions meromorphic in the complement of some internal discs.

The second approach is based on the fact that every finitely connected domain can be presented as the intersection of a simply connected domain with compact exteriors of which are components of the complement to the considered domain. Mapping conformally the exteriors of these compacts onto the unit disc some generalization of the decomposition lemma is obtained. The Riemann conformal mappings theorem guarantees the existence of such conformal mapping if a component of the boundary does not degenerate to a point.

The main aspects of this theory can be found in the monographs [55], [59].

Using these new approaches A. A. Kondratyuk jointly with his students A. Ya. Khrystiyanyn, I. P. Kshanovskyy, M. O. Hanyak laid the foundations of the Nevanlinna value distribution theory of the functions meromorphic in multiply connected domains and extended the Fourier series method to this class of functions. Moreover, jointly with O. P. Gnatiuk and O. V. Stashyshyn he also extended and generalized the main aspects of the this theory to the case of subharmonic functions. In particular, the one- as well as two-parametric approaches for the investigation of functions subharmonic in annuli invariant with respect to the inversion and in ball layers were proposed. They also obtained explicit expression for the Green function and the Poisson-Jensen formula as well as an analog of the Jensen theorem and established relations between various growth functions. These results were applied to some electrostatics problems.

Theses of A. A. Kondratyuk and his students

1. A.A. Kondratyuk, *Bounds for indicators of entire functions* (1969)
2. A.A. Kondratyuk, *Fourier and Fourier-Laplace series methods for meromorphic and subharmonic functions of completely regular growth* (1989)
3. Ya.V. Vasylkiv, *Investigation of asymptotic properties of entire and subharmonic functions by the Fourier series method* (1986)
4. O.V. Veselovska, *Investigation of properties of harmonic and δ-subharmonic functions by the method spherical harmonics* (1991)
5. S.I. Tarasyuk, *Integral means of δ-subharmonic in $\mathbb{R}^n$ functions and classes of completely regular growth* (1992)
6. A.Ya. Khrystiyanyn, *Properties of analytic in a half-plane and meromorphic in annuli functions* (2006)
7. O.Ya. Brodyak, *Growth and zero distribution of entire functions of finite λ-type* (2007)
8. A.M. Brydun, *Fourier series method for meromorphic functions in a half-strip* (2008)

9. Ya.V. Vasylkiv, *Development of the harmonical analysis methods for the investigation of asymptotic properties of meromorphic and subharmonic functions* (2008)

10. I.P. Kshanovskyy, *Properties of meromorphic functions in double-connected domains* (2009)

List of monographs and papers of A. A. Kondratyuk

1. *Extremal indicator of entire functions with positive zeros I,* Litovsk. Mat. Sb. VII, **1** (1967), 79–117. (in Russian)
2. *Extremal indicator of entire functions with positive zeros II,* Litovsk. Mat. Sb. VIII, **1** (1967), 65–85. (in Russian)
3. *Entire functions with positive zeros having the finite maximal density,* Func. Theory, Func. Anal. and Appl., **7** (1968), 37–52. (in Russian)
4. *Bounds for indicators of entire functions,* Abstr. of Ph.D. Thesis, Lviv, 1969. (in Russian)
5. *Entire functions with finite maximal density of zeros I,* Func. Theory, Functional Anal. and their Appl., **10** (1970), 57–70. (in Russian)
6. *Entire functions with finite maximal density of zeros II,* Func. Theory, Functional Anal. and their Appl., **11** (1970), 35–40. (in Russian)
7. *On the extremal indicator of entire functions with positive zeros,* Sib. Mat. Zh. XI, **5** (1970), 1084–1092. (in Russian)
8. (joint with Fridman A.N.) *On the lower indicator of entire function of zero genus with positive zeros,* Ukr. Mat. Zh., **24** (1972), №1, 106–109. (in Ukrainian)
9. (joint with Fridman A.N.) *Limit value of the lower indicator of entire functions with positive zeros,* Ukr. Mat. Zh., **24** (1972), №4, 488–494. (in Ukrainian)
10. (joint with Sheremeta M.M.) *Asymptotic properties of one class of entire functions with transfinitely measurable density of zeros,* Func. Theory, Func. Anal. and Appl., **30** (1978), 67–71. (in Russian)
11. *The Fourier series method for entire and meromorphic functions of completely regular growth I,* Mat. Sb., **106(148)** (1978), 386–408. (in Russian): English transl. in Math. USSR Sb., **35** (1979), №1.
12. *On the Fourier series method for entire and meromorphic functions of completely regular growth,* Dokl. Akad. Nauk Ukraine SSR., Ser. A, **6** (1978), 436–438. (in Ukrainian)
13. *The Fourier series method for entire and meromorphic functions of completely regular growth II,* Mat. Sb., **113(155)** (1980), 118–132. (in Russian): English transl. in Math. USSR Sb., **41** (1982).
14. *Asymptotic behavior and the number of deficient values of entire functions of completely regular growth,* Dokl. Akad. Nauk Ukr. SSR, Ser. A, **5** (1981), 11–13. (in Russian)
15. *Two properties of meromorphic functions of completely regular growth,* Visn. Lviv Univ. Ser. Mech. Mat., **18** (1981), 81–85. (in Ukrainian)
16. *On the generalization of classes of entire functions of completely regular growth,* Visn. Lviv Univ. Ser. Mech. Mat., **18** (1981), 85–89. (in Ukrainian)
17. *On the method of spherical harmonics for subharmonic functions,* Mat. Sb., **116(158)** (1981), №2, 147–165. (in Russian): English transl. in Math. USSR Sb., **44** (1983).
18. (joint with Vasyl'kiv Ya.V.) *Asymptotic behavior of L_q-norms of* $\log|f|$ *in one class of entire functions,* Ukr. Mat. Zh., **34** (1982), №1, 9–14. (in Ukrainian)

19. *Application of one theorem of compactness of subharmonic functions families to meromorphic functions,* Usp. Mat. Nauk, **37** (1982), №4, 107. (in Russian)
20. *The Fourier series method for entire and meromorphic functions of completely regular growth III,* Mat. Sb., **120(162)** (1983), 331–343. (in Russian)
21. *Spherical harmonics and subharmonics functions,* Dokl. Akad. Nauk USSR 268, (1983), №3, 541–544. (in Russian): English transl. in Soviet. Math. Dokl. **27** (1983), №1.
22. *Asymptotic behavior of meromorphic functions of completely regular growth,* Ukr. Mat. Zh., **35** (1983), №3, 315–320. (in Ukrainian)
23. *Linear combinations of meromorphic functions of completely regular growth,* Visn. Lviv. Univ., **13** (1983), 21–25. (in Ukrainian)
24. *On the deficient value of meromorphic functions of completely regular growth,* Visn. Lviv Univ., **35** (1983), №3, 321–325. (in Ukrainian)
25. *Spherical harmonics and subharmonic functions,* Mat. Sb., **125(167)** (1984), 147–166. (in Russian)
26. *Fourier series and meromorphic functions,* Lviv: Vyshcha Shkola, 1988, 198 pp. (in Russian)
27. *One application of spherical star-function,* In "Funct. Theory and Approx.", Saratov, 1988, 99–100. (in Russian)
28. *On the growth of integral q-means and the deficient value of subharmonic in $\mathbb{R}^m$ functions,* Ukr. Mat. Zh., **141** (1989), №6, 750–757.
29. (joint with Tarasyuk S.I.) *Comparison of Lebesgue means and Nevanlinna characteristic of subharmonic functions,* Mat. Stud., **1** (1991), 74–80. (in Ukrainian)
30. (joint with Tarasyuk S.I.) *Comparison of Lebesgue means and Nevanlinna characteristic of subharmonic functions,* Ukr. Dep., Kyiv, 1991, V.6. (in Russian)
31. (joint with Tarasyuk S.I.) *Comparison of Lebesgue means and Nevanlinna characteristic of subharmonic functions.* (manuscript of Dep. Ukr. NDINTI 7.07.92, L91080 Uk. 92), V.8, 1992.
32. (joint with Korobeynik Yu.F.) *Zeros separability of entire functions of exponential type and character completeness,* Mat. Zametki, **52** (1992), №8, 83–91. (in Russian)
33. (joint with Hirnyk M.O.) *Blaschke products of given quantity index,* Mat. Stud., **2** (1993), 49–52.
34. (joint with Tarasyuk S.I. and Vasyl'kiv Ya.V.) *General Paley problem,* Ukr. Mat. Zh., **48** (1996), №1, 25–34. (in Ukrainian)
35. *Some problems concerning admissible sequences in the theory of entire and meromorphic functions,* Mat. Stud., **7** (1997), №2, 212.
36. *Growth of integral logarithmic means for Blaschke products,* Mat. Zametki, **64** (1998), 199–206.
37. (joint with Kalynets R.) *On the growth regularity of the module and argument of entire function in $L_p[0, 2\pi]$-metric,* Ukr. Mat. Zh., **50** (1998), №7, 889–896. (in Ukrainian)
38. (joint with Kalynets R.) *Growth regularity of module and argument of an entire function in $L^p[0, 2\pi]$-metric,* Bul. de la Soc. des Sci. et des Lett. de Łodz, Série : Rech. sur les Déform., **XXV** (1998), 51–56.
39. (joint with Vasyl'kiv Ya.V.) *Integral logarithmic means of Blaschke products,* In Contemp. Probl. of Math. 1, Chernivtsi–Kyiv, 1998, 96–98. (in Ukrainian)
40. (joint with Vasyl'kiv Ya.V.) *Comparison of the growth characteristics of positive functions,* Mat. Stud., **10** (1998), №1, 23–32; *Erratum,* Mat. Stud., **12** (1999), №2, 221. (in Ukrainian)

41. (joint with Vasyl'kiv Ya.V.) *On the boundedness of the Blaschke product logarithm square means,* Ukr. Mat. Zh., **51** (1999), №2, 225–259. (in Ukrainian)
42. (joint with Vasyl'kiv Ya.V.) *Integral means of the Blaschke product logarithm,* Visn. Lviv. Univ., **53** (1999), 52–61. (in Ukrainian)
43. (joint with Vasyl'kiv Ya.V.) *Conjugate of subharmonic function,* Mat. Stud., **13** (2000), №2, 173–180.
44. (joint with Tarasyuk S.I.) *Generalized Lindelöf–Brelot conditions of λ-type finiteness of subharmonic functions,* In "Actual problems of mathematical analysis", Rostov–Don., 2000, 81–85. (in Russian)
45. (joint with Vasyl'kiv Ya.V.) *Generalized Lindelöf conditions of λ-type finiteness of subharmonic function,* Ukr. Mat. Zh., **54** (2002), №2, 276–279. (in Ukrainian)
46. (joint with Vasyl'kiv Ya.V.) *Minimal growth majorants of entire functions and quotient representations of meromorphic functions,* Dop. Nat. Sci. Akad. Ukraine, **44** (2002), №4.
47. (joint with Vasyl'kiv Ya.V.) *Growth majorants and quotient representations of meromorphic functions,* Comp. Meth. Func. Theory, **1** (2001), №2, 555–606.
48. (joint with Vasyl'kiv Ya.V., Tarasyuk S.I.) *On boundedness of integral means of Blaschke product logarithms,* Math. Modelling and Analysis, **8** (2003), №3, 265–274.
49. (joint with Kshanovskyy I.P.) *On the logariyhmmic derivative of a meromorphic function,* Mat. Stud., **21** (2004), №1, 98–100.
50. (joint with Brydun A.M.) *On the Fourier series of the zeta-function logarithm on the vertical lines,* Mat. Stud., **22** (2004), №1, 97–104.
51. *A Carleman-Nevanlinna theorem and summation of the Riemann zeta-function logarithm,* Comp. Meth. Func. Theory, **4** (2004), 391–403.
52. (joint with Yatsulka P.A.) *Entire functions connected with the sum of divisors function $\sigma(n)$*, Voronoy's impact on modern science, Book 3: Proceedings of the 3-rd Voronoï Conference on Analitic Number Theory and Spatial Tessellations, Kyiv, 2005, 33–36.
53. (joint with Khrystiyanyn A.Ya.) *On the Nevanlinna theory for meromorphic functions on annuli I,* Mat. Stud., **23** (2005), №1, 19–30.
54. (joint with Khrystiyanyn A.Ya.) *On the Nevanlinna theory for meromorphic functions on annuli II*, Mat. Stud., **24** (2005), №1, 58–68.
55. (joint with Laine I.) Meromorphic functions in multiply connected domains, Joensuu – Lviv (2006), 116 p.
56. (joint with Hanyak M.O.) *Meromorphic functions in m-punctured complex planes,* Mat. Stud., **27** (2007), №1, 53–69.
57. (joint with Yatsulka P.A.) *Summation of the Riemann zeta-function logarithm on the critical line,* Voronoj's impact on modern science, Book 4: Proceedings of the 4-rd Voronoj Conference on Analitic Number Theory and Spatial Tessellations, Kyiv, 2008, 59–62.
58. *Meromorphic functions with several essential singularities. I*, Mat. Stud., **30** (2008), №2, 125–131.
59. Meromorphic functions in multiply connected domains, Lviv: LNU, 2009, 168 p.
60. (joint with Stashyshyn O.V.) *Subharmonic functions on annuli*, Mat. Stud., **33** (2010), №1, 92–96.
61. (joint with Gnatiuk O.P.) *Subharmonic functions and electric fields in ball layers. I*, Mat. Stud., **34** (2010), №2, 180–192.

62. (joint with Stashyshyn O.) *A two-parameter appoach,* Mat. Visn. NTSh, **7** (2010), 368–381.
63. (joint with Gnatiuk O.P.) *Subharmonic functions and electric fields in ball layers. II*, Mat. Stud., **35** (2011), №1, 50–59.
64. (joint with Gnatiuk O.P., Kudyavina Yu.V.) *Classification of isolated singularities of subharmonic functions,* Visnyk Lviv. Univ. Ser. Mech. Math., **74** (2011), 52–60. (in Ukrainian)
65. (joint with Khrystiyanyn A.Ya., Khylynska O.B.) *Nevanlinna characteristics of sequences of meromorphic functions and Julia's exceptional functions,* Mat. Stud., **36** (2011), №1, 65–72.
66. (joint with Khrystiyanyn A.Ya., Sokulska N.B.) *Growth characteristics of loxodromic and elliptic functions,* Mat. Stud., **37** (2012), №1, 52–57.
67. (joint with Khrystiyanyn A.Ya.) *Meromorphic mappings of torus onto the Riemann sphere,* Carpathian Math. Publ., **4** (2012), №1, 155–159.
68. (joint with Brydun A.M., Khrystiyanyn A.Ya.) *On simultaneous regular growth of the modulus and argument of an entire function*, Bul. de la Soc. des Sci. et des Lett. de Łodz, Série: Rech. sur les Déform., **LXII** (2012), 107–110.
69. *Loxodromic meromorphic and δ-subharmonic functions,* Proceedings of the Workshop on Complex Analysis and its Applications to Differential and Functional Equations. Reports and Studies in Forestry and Natural Sciences, №14 (2014), 89–100.
70. (joint with Khoroshchak V.S.) *The Riesz measures and a representation of multiplicatively periodic δ-subharmonic functions in a punctured Euclidean space,* Mat. Stud., **43** (2015), №1, 61–65.
71. (joint with Khoroshchak V.S.) *Stationary harmonic functions on homogenous space,* Ufa Math. J., **7** (2015), №4, 149–153.
72. (joint with Khrystiyanyn A.Ya.) *Modulo-loxodromic meromorphic functionsin* $\mathbb{C}\setminus\{0\}$, Ufa Math. J., **8** (2016), №4, 156–162.

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