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TWO-MEMBER ASYMPTOTIC OF LAPLACE-STIELTJES INTEGRALS

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We study conditions, under which for the Laplace-Stieltjes integral $I(\sigma)=\int_0^{\infty}f(x)e^{x\sigma}dF(x)$ with the abscissa of the convergence $\sigma_c=A\in(-\infty,+\infty]$ the asymptotical equality $\ln I(\sigma)=\Phi_1(\sigma)+\tau(1+o(1))\Phi_2(\sigma)$ as $\sigma\uparrow A$ is true, where $\tau\in\mathbb{R}\setminus\{0\}$ and Φ_1, Φ_2 are some positive functions on $(-\infty,A)$.

1. Introduction. Let V be the class of functions F on $[0,+\infty)$ which are nonnegative nondecreasing unbounded right-continuous. We say that $F\in V(l)$ if $F\in V$ and $F(x)-F(x-0)\le l<+\infty$ for all $x\ge 0$.

For a nonnegative and measurable function f on $[0,+\infty)$ the integral

$$I(\sigma)=\int_0^{\infty}f(x)e^{x\sigma}dF(x),\quad \sigma\in\mathbb{R},\tag{1}$$

is called the *Laplace-Stietjes integral* ([1, p. 7]). It is a direct generalisation of the ordinary Laplace integral $I(\sigma)=\int_0^{\infty}f(x)e^{x\sigma}dx$ and of the Dirichlet series $D(\sigma)=\sum_{n=0}^{\infty}a_ne^{\lambda_n\sigma}$ with nonnegative coefficients a_n end exponents λ_n. It is clear that integral (1) either converges for all $\sigma\in\mathbb{R}$ or diverges for all $\sigma\in\mathbb{R}$ or there exists a number σ_c such that integral (1) converges for $\sigma<\sigma_c$ and diverges for $\sigma>\sigma_c$. In the latter case the number σ_c is called an *abscissa of the convergence of integral* (1). If integral (1) converges for all $\sigma\in\mathbb{R}$ then we put $\sigma_c=+\infty$, and if it diverges for all $\sigma\in\mathbb{R}$ then we put $\sigma_c=-\infty$. Let

$$\mu(\sigma)=\mu(\sigma,I)=\sup\{f(x)e^{x\sigma}\colon\ x\ge 0\},\quad \sigma\in\mathbb{R},$$

be the maximum of the integrand. Then either $\mu(\sigma)<+\infty$ for all $\sigma\in\mathbb{R}$ or $\mu(\sigma)=+\infty$ for all $\sigma\in\mathbb{R}$ or there exists a number σ_μ such that $\mu(\sigma)<+\infty$ for all $\sigma<\sigma_\mu$ and $\mu(\sigma)=+\infty$ for for all $\sigma>\sigma_\mu$. By analogy the number σ_μ is called the abscissa of the maximum of the integrand.

Lemma 1 ([1], p. 13). *If $F\in V$ and $\ln F(x)=o(x)$ as $x\to+\infty$ then $\sigma_c\ge\sigma_\mu$.*

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In the general case the equality $\sigma_c = \sigma_\mu$ can be not held. We will say ([1, p. 21]) that a nonnegative function f has regular variation in regard to F if there exist $a \geq 0$, $b \geq 0$ and $h > 0$ such that for all $x \geq a$

$$\int\limits_{x-a}^{x+b} f(t)dF(t) \geq hf(x).$$

Lemma 2 ([1], p. 21). *If $F \in V$ and f has regular variation in regard to F then $\sigma_c \leq \sigma_\mu$.*

Thus, if $F \in V$, $\ln F(x) = o(x)$ as $x \to +\infty$ and f has regular variation in regard to F then $\sigma_c = \sigma_\mu$.

Let L^0 be the class of positive continuously differentiable on $(0, +\infty)$ functions l such that $xl'(x) = O(l(x))$ as $x \to +\infty$. We remark that if $l \in L^0$ then $l((1+o(1))x) = (1+o(1))l(x)$ as $x \to +\infty$.

By $\Omega(A)$ we denote the class of positive unbounded on $(-\infty, A)$ functions Φ such that the derivative Φ' is positive continuously differentiable and increasing to $+\infty$ on $(-\infty, A)$. From now on, we denote by φ the inverse function to Φ', and let $\Psi(x) = x - \Phi(x)/\Phi'(x)$ be the function associated with Φ in the sense of Newton. It is clear that the function φ is continuously differentiable and increasing to A on $(0, +\infty)$. The function Ψ is continuously differentiable and increasing to A on $(-\infty, A)$ [1, p. 30; 2–3].

Let $A \in (-\infty, +\infty]$ and $\Phi_1 \in \Omega(A)$. As in [4] we will say that a positive twice continuously differentiable increasing to $+\infty$ on $(-\infty, +\infty)$ function Φ_2 is *subordinated to* $\Phi_1 \in \Omega(A)$ if $\Phi_2''(\sigma) = o(\Phi_1''(\sigma))$, $\Phi_2'(\sigma) = o(\sigma\Phi_1''(\sigma))$ as $\sigma \uparrow A$ and $\Phi_2'(\varphi_1) \in L^0$. We remark that $\Phi_2'(\varphi_1) \in L^0$ iff $\Phi_2''(\sigma)/\Phi_2'(\sigma) = O(\Phi_1''(\sigma)/\Phi_1'(\sigma))$ as $\sigma \uparrow A$. Moreover, we will say that Φ_2 is *strongly subordinated to* Φ_1 if Φ_2 is subordinated to Φ_1 and

$$\Phi_j'(\sigma + O(\Phi_2'(\sigma)/\Phi_1''(\sigma))) = (1+o(1))\Phi_j'(\sigma) \quad (\sigma \to +\infty), \quad j \in \{1, 2\}.$$

Finally, by $\Omega^*(A)$ we denote the class of functions $\Phi \in \Omega(A)$ such that for every increasing to $+\infty$ sequence (t_n) of positive numbers from the relation

$$G_2(t_n, t_{n+1}, \Phi_1) = (1+o(1))G_1(t_n, t_{n+1}, \Phi), \quad n \to \infty,$$

it follows that $t_{n+1} = (1+o(1))t_n$ as $n \to \infty$, where ([1, p.34; 5])

$$G_1(a, b, \Phi) := \frac{ab}{b-a}\int_a^b \frac{\Phi(\varphi(t))}{t^2}\,dt < G_2(a, b, \Phi) := \Phi\left(\frac{1}{b-a}\int_a^b \varphi(t)\,dt\right)$$

for $0 < a < b < +\infty$.

Let $\tau \in \mathbb{R} \setminus \{0\}$. In this paper we will explore conditions, under which

$$\ln I(\sigma) = \Phi_1(\sigma) + \tau(1+o(1))\Phi_2(\sigma), \quad \sigma \uparrow A. \tag{2}$$

Replacing $\sigma - A$ by σ, the general case $A \in (-\infty, +\infty)$ can be reduced to the case $A = 0$. Thus, we will consider that either $A = 0$ or $A = +\infty$. By $LS_A(F)$ we denote the class of integrals (1) with a given function F such that $\sigma_\mu = A$.

2. Two-member asymptotic of functions from $LS_{+\infty}(F)$. We use some results from [6] and [7]. If we choose $Q(\sigma) = \ln \mu(\sigma, I)$ and $P(x) = \ln f(x)$ then Theorem 3 from [7] implies the following statement.

Lemma 3. *Let $\tau \in \mathbb{R} \setminus \{0\}$ and either $A = 0$ or $A = +\infty$. Suppose that $\Phi_1 \in \Omega^*(A)$, $\varphi_1' \in L^0$ and Φ_2 is strongly subordinated to Φ_1. Suppose also that either $\sigma\Phi_1'(\sigma) = O(\Phi_1(\sigma))$ as $\sigma \uparrow A \in \{0, +\infty\}$ or $\Phi_2'(\sigma) = o(\Phi_1''(\sigma))$ and $\Phi_1'(\sigma) = O(\Phi_1(\sigma))$ as $\sigma \uparrow A$ in the case $A = +\infty$.*

In order that $\ln \mu(\sigma, I) = \Phi_1(\sigma) + (1+o(1))\tau\Phi_2(\sigma)$, as $\sigma \uparrow A$, it is necessary and sufficient that for every $\varepsilon > 0$:

1) *$\ln f(t) \le -t\Psi_1(\varphi_1(t)) + (\tau + \varepsilon)\Phi_2(\varphi_1(t))$ for all $t \ge t_0 = t_0(\varepsilon)$;*
2) *there exists an increasing to $+\infty$ sequence (t_n) such that $\ln f(t_n) \ge -t_n\Psi_1(\varphi_1(t_n)) + (\tau - \varepsilon))\Phi_2(\varphi_1(t_n))$, $n \to +\infty$, and $\lim\limits_{n\to\infty} \frac{G_2(t_n,t_{n+1},\Phi_1) - G_1(t_n,t_{n+1},\Phi_1)}{\Phi_2(\varphi_1(t_n))} = 0$.*

On the other hand, the following lemma is proved in [6].

Lemma 4. *Let $F \in V$, $\Phi \in \Omega(+\infty)$ and $\gamma\colon [0, +\infty) \to [0, +\infty)$ be a continuous function such that $\gamma(t) \uparrow +\infty$ as $t \to +\infty$.*

If $\Phi'(\sigma) = O(\Phi'(\Psi(\sigma))$ as $\sigma \to +\infty$, the function $\gamma(t)/t$ is nonincreasing on $[t_0, +\infty)$ and $\gamma(t) = O(\Phi(\Psi(\varphi(t)))$ as $t \to +\infty$ then the condition

$$\overline{\lim_{x\to+\infty}} \frac{\ln F(x)}{\gamma(x)} = 0 \tag{3}$$

is sufficient, and if $F \in V(l)$ and the function γ is continuously differentiable on $[0, +\infty)$, $\ln (1/\gamma'(t)) = o(\gamma(t))$ and $\ln \varphi(t) = o(\gamma(t))$ as $t \to +\infty$ then the condition (3) is necessary in order that for every integral $I \in LS_{+\infty}(F)$ the inequality

$$\ln \mu(\sigma, I) \le \Phi(\sigma), \quad \sigma \ge \sigma_0, \tag{4}$$

imply the estimate

$$\ln I(\sigma) \le \Phi(\sigma) + o(\gamma(\Phi'(\sigma)), \quad \sigma \to +\infty. \tag{5}$$

On the other hand, if a function f has regular variation in regard to F then the inequality

$$\ln I(\sigma) \le \Phi(\sigma), \quad \sigma \ge \sigma_0, \tag{6}$$

implies the estimate

$$\ln \mu(\sigma, I) \le \Phi(\sigma) + O(\sigma), \quad \sigma \to +\infty. \tag{7}$$

Remark 1. In the proof of Theorem 1 from [6] it is established that

$$\ln I(\sigma) \le \ln \mu(\sigma, I) + o(\gamma(\Phi'(\sigma)), \quad \sigma \uparrow A, \tag{8}$$

and if $A = +\infty$ and a function f has regular variation in regard to F then

$$\ln \mu(\sigma, I) \le I(\sigma) + O(\sigma), \quad \sigma \to +\infty. \tag{9}$$

Firstly we prove the following theorem.

Theorem 1. *Let $A = +\infty$, $\tau \in \mathbb{R} \setminus \{0\}$, $F \in V(l)$ and the function $\Phi_1 \in \Omega^*(+\infty)$ be such that $\varphi_1' \in L^0$, $\Phi_1(\sigma + o(1)) = O(\Phi_1(\sigma))$ and $\Phi_1'(\sigma) = O(\Phi_1'(\sigma - (1 + o(1))\Phi_1(\sigma)/\Phi_1'(\sigma))$ as $\sigma \to +\infty$. Suppose that a function Φ_2 is strongly subordinated to Φ_1 and satisfies the conditions $\Phi_2(\sigma) = O(\Phi_1(\Psi_1(\sigma)))$, $\ln \Phi_1''(\sigma) = o(\Phi_2(\sigma))$, $\sigma = o(\Phi_2(\sigma))$ as $\sigma \to +\infty$ and*

$\Phi_2(\sigma)/\Phi_1'(\sigma) \searrow 0$ as $\sigma_0 \le \sigma \to +\infty$. Suppose also that either $\sigma\Phi_1'(\sigma) = O(\Phi_1(\sigma))$ or $\Phi_2'(\sigma) = o(\Phi_1''(\sigma))$ and $\Phi_1'(\sigma) = O(\Phi_1(\sigma))$ as $\sigma \to +\infty$, a function f has regular variation in regard to F and

$$\varlimsup_{x\to+\infty} \frac{\ln F(x)}{\Phi_2(\varphi(x))} = 0. \tag{10}$$

Then in order that for every integral $I \in LS_{+\infty}(F)$ equality (2) hold it is necessary and sufficient that for every $\varepsilon > 0$ conditions 1) and 2) of Lemma 3 hold.

Proof. Since Φ_2 is subordinated to Φ_1, that is $\Phi_2''(\sigma) = o(\Phi_1(\sigma))$, as $\sigma \to +\infty$, there exists a function $\Phi \in \Omega(+\infty)$ such that

$$\Phi(\sigma) = \Phi_1(\sigma) + \tau\Phi_2(\sigma), \quad \sigma \ge \sigma_0(\tau). \tag{11}$$

The condition $\Phi'(\sigma) = O(\Phi'(\Psi(\sigma))$ as $\sigma \to +\infty$ for the function (11) holds if $\Phi_1'(\sigma) = = O\left(\Phi_1'(\sigma - (1+o(1))\Phi_1(\sigma)/\Phi_1'(\sigma))\right)$ as $\sigma \to +\infty$. We choose $\gamma(t) = \Phi_2(\varphi_1(t))$. Clearly, if $\Phi_2(\sigma)/\Phi_1'(\sigma) \searrow 0$ as $\sigma_0 \le \sigma \to +\infty$ then $\gamma(t)/t$ is nonincreasing on $[t_0, +\infty)$. It is proved [7], that for function (11)

$$\varphi(t) = \varphi_1(t) - (1+o(1))\tau\Phi_2'(\varphi_1(t))\varphi_1'(t) = (1+o(1))\varphi_1(t), \quad t \to +\infty, \tag{12}$$

and

$$t\Psi(\varphi(t)) = t\Psi_1(\varphi_1(t)) - (1+o(1))\tau\Phi_2(\varphi_1(t)), \quad t \to +\infty. \tag{13}$$

Since $\Phi_2(\varphi_1(t))/t \searrow 0$ as $t \to +\infty$, by the condition $\Phi_1(\sigma + o(1)) = O(\Phi_1(\sigma))$ as $\sigma_0 \le \sigma \to +\infty$ from (13) we obtain

$$\Phi(\Psi(\varphi(t))) = (1+o(1))\Phi_1(\Psi(\varphi(t))) = (1+o(1))\Phi_1(\Psi_1(\varphi_1(t) + o(1)) = O(\Phi_1(\Psi_1(\varphi_1(t)))$$

as $t \to +\infty$. Therefore, if $\Phi_2(\sigma) = O(\Phi_1(\Psi_1(\sigma)))$ as $\sigma \to +\infty$ then $\gamma(t) = O(\Phi(\Psi(\varphi(t)))$ as $t \to +\infty$.

We remark also that if $\ln \Phi_1''(\sigma) = o(\Phi_2(\sigma))$ as $\sigma \to +\infty$ then $\ln \frac{\Phi_1''(\varphi_1(t))}{\Phi_2(\varphi_1(t))} = = o(\Phi_2(\varphi_1(t)))$ as $t \to +\infty$ and, thus, $\ln(1/\gamma'(t)) = o(\gamma(t))$ as $t \to +\infty$. Finally, if $A = +\infty$ and $\sigma = o(\Phi_2(\sigma))$ as $\sigma \to +\infty$ then $\ln \sigma = o(\Phi_2(\sigma))$ as $\sigma \to +\infty$ and in view of (12) $\ln \varphi(t) = \ln \varphi_1(t) + o(1) = o(\gamma(t))$ as $t \to +\infty$. Therefore, by Lemma 4 and condition (10) is necessary and sufficient in order that for every integral $I \in LS_{+\infty}(F)$ inequalities

$$\ln \mu(\sigma, I) \le \Phi_1(\sigma) + \tau(1+o(1))\Phi_2(\sigma), \quad \sigma \to +\infty, \tag{14}$$
$$\ln I(\sigma) \le \Phi_1(\sigma) + \tau(1+o(1))\Phi_2(\sigma), \quad \sigma \to +\infty, \tag{15}$$

are equivalent. Moreover, condition (10) is sufficient for the equivalence of the equalities

$$\ln \mu(\sigma, I) = \Phi_1(\sigma) + \tau(1+o(1))\Phi_2(\sigma), \quad \sigma \to +\infty, \tag{16}$$
$$\ln I(\sigma) = \Phi_1(\sigma) + \tau(1+o(1))\Phi_2(\sigma), \quad \sigma \to +\infty. \tag{17}$$

Further, since $\Phi_2(\varphi_1) \in L^0$ and $\gamma(t) = \Phi_2(\varphi_1(t))$, then for function (11) we have $\gamma(\Phi'(\sigma)) = \Phi_2(\varphi_1((1+o(1))\Phi_1'(\sigma))) = (1+o(1))\Phi_2(\sigma)$ as $\sigma \to +\infty$ and in view of (8) $\ln I(\sigma) \le \ln \mu(\sigma, I) + o(\Phi_2(\sigma))$ as $\sigma \uparrow A$. On the other hand, by condition $\sigma = o(\Phi_2(\sigma))$ as $\sigma \to +\infty$ from (9) we obtain $\ln \mu(\sigma, I) \le I(\sigma) + o(\Phi_2(\sigma))$ as $\sigma \to +\infty$. Thus, $\ln \mu(\sigma, I) + o(\Phi_2(\sigma)) \le \ln I(\sigma) \le \ln \mu(\sigma, I) + o(\Phi_2(\sigma))$ as $\sigma \to +\infty$, whence the equivalence of (14) and (15) follows. If view of Lemma 3 the proof of Theorem 1 is complete. □

We remark that in Lemma 4 the condition of the nonicreasing of $\gamma(x)/x$ can be replaced by the condition $\gamma(2x) = O(\gamma(x))$ as $x \to +\infty$, when Φ has power growth. The following lemma is proved in [6] .

Lemma 5. *Let $F \in V$, $\Phi \in \Omega(+\infty)$ and $\gamma\colon [0, +\infty) \to [0, +\infty)$ be a continuous function such that $\gamma(t) \uparrow +\infty$ as $t \to +\infty$.*

If $\sigma\Phi'(\sigma)/\Phi(\sigma) \geq h > 1$ and $\sigma\Phi''(\sigma)/\Phi'(\sigma) \leq H < +\infty$ for $\sigma \geq \sigma_0$, $\gamma(2t) = O(\gamma(t))$ and $\gamma(t) = O(t\Psi(\varphi(t)))$ as $x \to +\infty$ then condition (3) is sufficient, and if $F \in V(l)$ and the function γ is continuously differentiable on $[0, +\infty)$, $\ln \gamma'(t) = o(\gamma(t))$ and $\ln \varphi(t) = o(\gamma(t))$ as $t \to +\infty$ then condition (3) is necessary in order that for every integral $I \in LS_{+\infty}(F)$ inequality (4) imply inequality (5).

If a function f has regular variation in regard to F then inequality (6) implies estimate (7).

Using Lemmas 3 and now we prove the following theorem.

Theorem 2. *Let $A = +\infty$, $\tau \in \mathbb{R} \setminus \{0\}$, $F \in V(l)$ and a function $\Phi_1 \in \Omega^*(+\infty)$ be such that $\varphi_1' \in L^0$, $\sigma\Phi_1'(\sigma)/\Phi_1(\sigma) \geq h_1 > 1$ and $\sigma\Phi_1''(\sigma)/\Phi_1'(\sigma) \leq H_1 < +\infty$ for $\sigma \geq \sigma_0^*$. Suppose that a function Φ_2 is strongly subordinated to Φ_1 and satisfies the conditions $\ln \Phi_1''(\sigma) = o(\Phi_2(\sigma))$ and $\sigma = o(\Phi_2(\sigma))$ as $\sigma \to +\infty$. Suppose also that either $\sigma\Phi_1'(\sigma) = O(\Phi_1(\sigma))$ or $\Phi_2'(\sigma) = o(\Phi_1''(\sigma))$ and $\Phi_1'(\sigma) = O(\Phi_1(\sigma))$ as $\sigma \to +\infty$, a function f has regular variation in regard to F and condition (10) holds. Then in order that for every integral $I \in LS_{+\infty}(F)$ equalities (2) holds it is necessary and sufficient that for every $\varepsilon > 0$ conditions 1) and 2) of Lemma 3 hold.*

Proof. For function (11) the conditions $\sigma\Phi'(\sigma)/\Phi(\sigma) \geq h > 1$ and $\sigma\Phi''(\sigma)/\Phi'(\sigma) \leq$ $\leq H < +\infty$ hold for $\sigma \geq \sigma_0$ if $\sigma\Phi_1'(\sigma)/\Phi_1(\sigma) \geq h_1 > 1$ and $\sigma\Phi_1''(\sigma)/\Phi_1'(\sigma) \leq H_1 < +\infty$ for $\sigma \geq \sigma_0^*$. If we choose $\gamma(t) = \Phi_2(\varphi_1(t))$ then $\gamma \in L^0$ and, therefore [8], $\gamma(2t) = O(\gamma(t))$ as $t \to +\infty$. The condition $\gamma(t) = O(t\Psi(\varphi(t)))$ as $t \to +\infty$ in view of (13) holds if $\Phi_2(\varphi_1(t)) = O(t\Psi_1(\varphi_1(t)))$ as $t \to +\infty$. The last condition follows from the condition $\sigma\Phi_1'(\sigma)/\Phi_1(\sigma) \geq h_1 > 1$. Finally, as above, the conditions $\ln \Phi_1''(\sigma) = o(\Phi_2(\sigma))$ and $\sigma = o(\Phi_2(\sigma))$ as $\sigma \to +\infty$ imply the conditions $\ln \gamma'(t) = o(\gamma(t))$ and $\ln \varphi(t) = o(\gamma(t))$ as $t \to +\infty$. Therefore, by Lemma 5, if the function f has regular variation in regard to F then condition (10) is necessary and sufficient in order that for every integral $I \in LS_{+\infty}(F)$ inequalities (14) and (15) are equivalent. Moreover, condition (10) is sufficient for the equivalence of equalities (16) and (17). Hence and from Lemma 3, as above, we obtain the conclusion of Theorem 2. □

3. Two-member asymptotic of functions from $LS_0(F)$. The following lemma is proved in [7].

Lemma 6. *Let $F \in V$, $\Phi \in \Omega(0)$ and $\gamma\colon [0, +\infty) \to [0, +\infty)$ be a continuous function such that $\gamma(x) \uparrow +\infty$ as $x \to +\infty$.*

If $\Phi'(\sigma) = O(\Phi'(\Psi(\sigma))$ as $\sigma \uparrow 0$, the function $\gamma(x)/x$ is nonincreasing on $[x_0, +\infty)$ and $\gamma(x) = O(\Phi(\Psi(\varphi(x)))$ as $x \to +\infty$ then condition (3) is sufficient, and if $F \in V(l)$, the function γ is continuously differentiable on $[0, +\infty)$ and $\ln \gamma'(x) = o(\gamma(x))$ as $x \to +\infty$ then condition (3) is necessary in order that for every integral $I \in LS_0(F)$ the inequality

$$\ln \mu(\sigma, I) \leq \Phi(\sigma), \quad \sigma \in [\sigma_0, 0), \tag{18}$$

imply the estimate

$$\ln I(\sigma) \le \Phi(\sigma) + o(\gamma(\Phi'(\sigma)), \quad \sigma \uparrow 0. \tag{19}$$

On the other hand, if the function f has regular variation in regard to F then the inequality

$$\ln I(s) \le \Phi(\sigma), \quad \sigma \in [\sigma_0, 0), \tag{20}$$

implies the estimate

$$\ln \mu(\sigma, I) \le \Phi(\sigma) + O(\sigma), \quad \sigma \uparrow 0. \tag{21}$$

Using Lemmas 3 and 6 we can prove the following theorem.

Theorem 3. *Let $A = 0$, $\tau \in \mathbb{R} \setminus \{0\}$, $F \in V(l)$ and a function $\Phi_1 \in \Omega^*(0)$ be such that $\varphi_1' \in L^0$,*

$$\sigma\Phi_1'(\sigma) = O(\Phi_1(\sigma)), \quad \Phi_1'(\sigma) = O(\Phi_1'(\sigma - (1 + o(1))\Phi_1(\sigma)/\Phi_1'(\sigma))), \quad \sigma \uparrow 0.$$

Suppose that a function Φ_2 is strongly subordinated to Φ_1 and satisfies the conditions

$$\Phi_2(\sigma) = O(\Phi_1(\sigma - (1 + o(1))\Phi_1(\sigma)/\Phi_1'(\sigma))), \ \ln \Phi_1''(\sigma) = o(\Phi_2(\sigma)), \quad \sigma \uparrow 0.$$

Suppose also that a function f has regular variation in regard to F and condition (10) holds. Then in order that for every integral $I \in LS_0(F)$ equality (2) as $\sigma \uparrow 0$ hold it is necessary and sufficient that for every $\varepsilon > 0$ conditions 1) and 2) of Lemma 3 hold.

Finally, we consider the case, when a function $\Phi \in \Omega(0)$ has slow growth. A function $\Phi\colon (-\infty, 0) \to [0, +\infty)$ is called *slowly increasing* if $\Phi(\sigma) \uparrow +\infty$ and $|\sigma|\Phi'(\sigma)/\Phi(\sigma) \to 0$ as $\sigma \uparrow 0$. By L_{si} we denote the class of such function.

The following lemma is proved in [6].

Lemma 7. *Let $F \in V$, $\Phi \in \Omega(0)$ and $\gamma\colon [0, +\infty) \to [0, +\infty)$ be a continuous function such that $\gamma(x) \uparrow +\infty$ as $x \to +\infty$. If*

$$\gamma(\Phi'(\sigma)) = O(\gamma(\Phi'(2\sigma))), \ \gamma(\Phi'(\sigma)) = O(|\sigma|\Phi'(\Psi^{-1}(\sigma))), \ \gamma(\Phi'(\sigma)) = O(\gamma(\Phi'(\Psi(\sigma)))), \quad \sigma \uparrow 0$$

then condition (3) is sufficient, and if $F \in V(l)$, a function γ is continuously differentiable on $[0, +\infty)$ and $\ln \gamma'(x) = o(\gamma(x))$ as $x \to +\infty$ then condition (3) is necessary in order that for every integral $I \in LS_0(F)$ inequality (18) imply estimate (19). On the other hand, if a function f has regular variation in regard to F then inequality (20) implies estimate (21).

From Lemmas 3 and 7 we obtain the following theorem.

Theorem 4. *Let $A = 0$, $\tau \in \mathbb{R} \setminus \{0\}$, $F \in V(l)$, $\Phi_1 \in \Omega^*(0) \cap L_{si}$, $\sigma\Phi_1'(\sigma) = O(\Phi_1(\sigma))$ as $\sigma \uparrow 0$ and $\varphi_1' \in L^0$. Suppose that a function $\Phi_2 \in L_{si}$ is strongly subordinated to Φ_1 and satisfies the conditions $\Phi_2(-\Phi_1(\sigma)/\Phi_1'(\sigma)) = O(\Phi_1(\sigma))$, $\Phi_2(\sigma) = O(\Phi_2(-\Phi_1(\sigma)/\Phi_1'(\sigma)))$ and $\ln \Phi_1''(\sigma) = o(\Phi_2(\sigma))$ as $\sigma \uparrow 0$. Suppose also that a function f has regular variation in regard to F and condition (10) holds.*

Then in order that for every integral $I \in LS_0(F)$ equality (2) hold it is necessary and sufficient that for every $\varepsilon > 0$ conditions 1) and 2) of Lemma 3 hold.

REFERENCES

1. Sheremeta M.M. Asymptotical behaviour of Laplace-Stiltjes integrals, Lviv: VNTL Publishers, 2010. – 211 p.
2. Sheremeta M.M., Fedynyak S.I. *On the derivative of a Dirichlet series*// Sibirsk. Mat. Zh. – 1998. – V.39, №1. – P. 206–223, (in Russian); Engl. Transl.: Siberian Math. J. – 1998. – V.39, №1. – P. 181–197.
3. Sheremeta M.M., Sumyk O.M. *A connection detween the growth of Young conjugated function*// Mat. Stud. – 1999. – V.11, №11. – P. 41–47. (in Ukrainian)
4. Sheremeta M.M. *Two-member asymptotics of Young conjugated functions and problems of behaviour of positive sequences*// Mat. Stud. – 2000. – V.14, №2. – P. 217–220.
5. Zabolotskyi M.V., Sheremeta M.M. *Generalization of the Lindelöf theorem*// Ukr. Mat. J. – 1998. – V.50, №9. – P. 1177–1192, (in Ukrainian); Engl. Transl.: Ukr. Math. Zh. – 1998. – V.50, №9. – P. 1346–1364.
6. Dobushovskyy M.S., Sheremeta M.M. *Estimates of Laplace-Stieltjes integrals*// Ukr. Mat. J. – 2016. – V.68, №11. – P. 1467–1482, (in Ukrainian); Engl. Transl.: Ukr. Math. Zh. – 2017. – V.68, №11. – P. 1694–1714.
7. Sheremeta M.M., Dobushovskyy M.S. *On the two-member asymptotics of Young conjugated functions*// Mat. Stud. – 2016. – V.46, №2. – P. 178–188.
8. Sheremeta M.M. *On two classes of positive functions and the belonging to them main characteristics of entire functions*// Mat. Stud. – 2003. – V.19, №1. – P. 73–82.

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