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SOME IMPROVEMENTS OF CRITERIA OF L-INDEX BOUNDEDNESS IN DIRECTION

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In this paper, we improve criteria of boundedness of L-index in direction for entire functions in $\mathbb{C}^n$. They give an estimate of maximum modulus on circles of various radius, maximum modulus by minimum modulus on circle and describe the behavior of directional logarithmic derivative and the distribution of zeros. The obtained results are also new for entire functions of bounded index in $\mathbb{C}$.

1. Introduction. This paper is an addendum to our paper [1]. We improve our solutions of some interesting problems in the theory of entire functions of bounded L-index in direction [2]. There were partially proved two conjectures about characterizations of functions from that class [1]. But we assumed that some inequalities are valid for a disk with radius in $(0,1)$. In this paper, we prove the same results without this assumption.

The derived results are new even in the one-dimensional case. As sufficient conditions of boundedness of index or l-index they improve corresponding results of G. H. Fricke (see Theorem 5 in [3] and Theorem 2 in [4]) and M. M. Sheremeta, A. D. Kuzyk (see Theorems 1 and 6 in [5]).

To state the problems we need some notation and definitions. Let $L\colon \mathbb{C}^n \to \mathbb{R}_+$ be a continuous function. An entire function $F(z)$, $z \in \mathbb{C}^n$, is called ([2],[6]–[9]) *of bounded L-index in a direction* $\mathbf{b} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$, if there exists $m_0 \in \mathbb{Z}_+$ such that for every $m \in \mathbb{Z}_+$ and every $z \in \mathbb{C}^n$

$$\frac{1}{m!L^m(z)}\left|\frac{\partial^m F(z)}{\partial \mathbf{b}^m}\right| \le \max\left\{\frac{1}{k!L^k(z)}\left|\frac{\partial^k F(z)}{\partial \mathbf{b}^k}\right| : 0 \le k \le m_0\right\}, \tag{1}$$

where $\frac{\partial^0 F(z)}{\partial \mathbf{b}^0} := F(z)$, $\frac{\partial F(z)}{\partial \mathbf{b}} := \sum\limits_{j=1}^{n} \frac{\partial F(z)}{\partial z_j} b_j = \langle \mathbf{grad} F, \overline{\mathbf{b}}\rangle$, $\frac{\partial^k F(z)}{\partial \mathbf{b}^k} := \frac{\partial}{\partial \mathbf{b}}\Big(\frac{\partial^{k-1} F(z)}{\partial \mathbf{b}^{k-1}}\Big)$, $k \ge 2$.

The least such integer $m_0 = m_0(\mathbf{b})$ is called the *L-index in the direction* $\mathbf{b} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$ *of the entire function $F(z)$ and is denoted by $N_{\mathbf{b}}(F, L) = m_0$.*

In the case $n = 1$ and $\mathbf{b} = 1$ we obtain the definition of an entire function of one variable of bounded l-index (see [5], [10]); in the case $n = 1$, $\mathbf{b} = 1$ and $L(z) \equiv 1$ it is reduced to the definition of an function of bounded index, introduced by B. Lepson ([12]).

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For $\eta > 0$, $z \in \mathbb{C}^n$, $\mathbf{b} = (b_1, \ldots, b_n) \in \mathbb{C}^n \setminus \{\mathbf{0}\}$ and a positive continuous function $L\colon \mathbb{C}^n \to \mathbb{R}_+$ we define

$$\lambda_{\mathbf{b}}(\eta) = \sup_{z\in\mathbb{C}^n} \sup \left\{ \frac{L(z+t_1\mathbf{b})}{L(z+t_2\mathbf{b})} \colon |t_1 - t_2| \le \frac{\eta}{\min\{L(z+t_1\mathbf{b}), L(z+t_2\mathbf{b})\}} \right\}.$$

By $Q^n_{\mathbf{b}}$ we denote the class of functions L which satisfy the condition $(\forall \eta \ge 0)\colon\ \lambda_{\mathbf{b}}(\eta) < +\infty$ We also use the notation $Q = Q^1_1$ for the class of positive continuous functions $l(z)$, when $z \in \mathbb{C}$, $\mathbf{b} = 1$, $n = 1$, $L \equiv l$.

2. Estimate of maximum modulus by minimum modulus. The notation $L \asymp L^*$ means that for some $\theta_1, \theta_2 \in \mathbb{R}_+$, $0 < \theta_1 \le \theta_2 < +\infty$ and for all $z \in \mathbb{C}^n$ the inequalities $\theta_1 L(z) \le L^*(z) \le \theta_2 L(z)$ hold. We need the next assertion.

Theorem 1 ([6, 9])**.** *Let $L \in Q^n_{\mathbf{b}}$, $L \asymp L^*$. An entire function $F(z)$ is of bounded L^*-index in the direction* $\mathbf{b}$ *if and only if F is of bounded L-index in the direction* $\mathbf{b}$.

There were obtained many criteria of L-index boundedness in direction [6, 9]. Later we proved that some propositions (Theorem 2 and Theorem 6 from [6]) have modified versions. They distinguish the universal quantifiers and the existential quantifiers (see Theorem 5 in [7] and Theorem 7 in [6]). In particular, the following assertion is true.

Theorem 2 ([6, 9])**.** *Let $L \in Q^n_{\mathbf{b}}$. An entire in $\mathbb{C}^n$ function F is of bounded L-index in the direction* $\mathbf{b}$ *if and only if for every r_1 and r_2 such that $0 < r_1 < r_2 < +\infty$ there exists a number $P_1 = P_1(r_1, r_2) \ge 1$ such that for each $z^0 \in \mathbb{C}^n$*

$$\max\left\{|F(z^0 + t\mathbf{b})|\colon |t| = r_2/L(z^0)\right\} \le P_1 \max\left\{|F(z^0 + t\mathbf{b})|\colon |t - t_0| = r_1/L(z_0)\right\}. \quad (2)$$

Besides, arbitrary r_1, r_2 can be replaced by certain r_1 and r_2.

Theorem 3 ([6, 9])**.** *Let $L \in Q^n_{\mathbf{b}}$. An entire in $\mathbb{C}^n$ function F is of bounded L-index in direction* $\mathbf{b}$ *if and only if there exist numbers r_1 and r_2, $0 < r_1 < 1 < r_2 < +\infty$, and $P_1 \ge 1$ such that for all $z^0 \in \mathbb{C}^n$ inequality (2) holds.*

We can relax sufficient conditions of Theorem 3 replacing the condition $0 < r_1 < 1 < r_2 < +\infty$ by $0 < r_1 < r_2 < +\infty$.

Theorem 4. *Let $L \in Q^n_{\mathbf{b}}$, F be an entire function in $\mathbb{C}^n$. If there exist numbers r_1 and r_2, $0 < r_1 < r_2 < +\infty$, and $P_1 \ge 1$ such that for all $z^0 \in \mathbb{C}^n$ inequality* (2) *holds, then the function F is of bounded L-index in the direction* $\mathbf{b}$.

Proof. Our proof is based on idea of A. D. Kuzyk and M. M. Sheremera from [11]. They proposed this method to investigate the l-index boundedness of entire solutions of linear differential equations.

From (2) with $0 < r_1 < r_2 < +\infty$ it follows that

$$\max\left\{|F(z^0 + t\mathbf{b})|\colon |t| = \frac{2r_2}{r_1 + r_2}\frac{r_1 + r_2}{2L(z^0)}\right\} \le P_1 \max\left\{|F(z^0 + t\mathbf{b})|\colon |t| = \frac{2r_1}{r_1 + r_2}\frac{r_1 + r_2}{2L(z_0)}\right\}.$$

Denoting $L^*(z) = \frac{2L(z)}{r_1 + r_2}$, we obtain

$$\max\left\{|F(z^0 + t\mathbf{b})|\colon |t| = \frac{2r_2}{(r_1 + r_2)L^*(z^0)}\right\} \le P_1 \max\left\{|F(z^0 + t\mathbf{b})|\colon |t| = \frac{2r_1}{(r_1 + r_2)L^*(z^0)}\right\},$$

where $0 < \frac{2r_1}{r_1 + r_2} < 1 < \frac{2r_2}{r_1 + r_2} < +\infty$. It means that F has bounded L^*-index in the direction $\mathbf{b}$. And by Theorem 1 the function F has bounded L-index in the direction $\mathbf{b}$. □

Theorem 5 ([6, 9])**.** *Let $L \in Q^n_{\mathbf{b}}$. An entire in $\mathbb{C}^n$ function F is of bounded L-index in the direction $\mathbf{b} \in \mathbb{C}^n$ if and only if for every $R > 0$ there exist $P_2(R) \geq 1$ and $\eta(R) \in (0, R)$ such that for all $z^0 \in \mathbb{C}^n$ and some $r = r(z^0) \in [\eta(R), R]$ the following inequality holds*

$$\max\left\{|F(z^0 + t\mathbf{b})|\colon |t| = \frac{r}{L(z^0)}\right\} \leq P_2 \min\left\{|F(z^0 + t\mathbf{b})|\colon |t| = \frac{r}{L(z^0)}\right\}. \tag{3}$$

Taking into account Theorems 2, 3 and 5 we assumed that the validity of (3) for every $R > 0$ can be replaced by the validity of (3) for some $R > 0$ (Conjecture 1 in [2]). Recently, we partially proved the conjecture by the additional restriction $R \in (0; 1)$ in [1].

In view of Theorem 4, we can relax sufficient conditions of Theorem 5 and truly proves the hypothesis.

Theorem 6. *Let $L \in Q^n_{\mathbf{b}}$, F be an entire function in $\mathbb{C}^n$. If there exist $R > 0$, $P_2 \geq 1$ and $\eta \in (0, R)$ such that for all $z^0 \in \mathbb{C}^n$ and some $r = r(z^0) \in [\eta, R]$ inequality (3) holds then the function F has bounded L-index in the direction* $\mathbf{b}$.

Proof. Proof of Theorem 6 is similar to that the corresponding theorem from [1]. But our proof also uses Theorem 4 instead of Theorem 3 as in [1].

In view of Theorem 4 it is sufficient to prove that there exists a number P_1 such that for all $z^0 \in \mathbb{C}^n$

$$\max\left\{|F(z^0 + t\mathbf{b})|\colon |t| = (R+1)/L(z^0)\right\} \leq P_1 \max\left\{|F(z^0 + t\mathbf{b})|\colon |t| = R/L(z^0)\right\}. \tag{4}$$

Let there exist $R > 0$, $P_2 \geq 1$ and $\eta \in \big(0, R\big)$ such that for all $z^0 \in \mathbb{C}^n$ and some $r = r(z^0) \in \big[\eta, R\big]$ we have

$$\max\left\{|F(z^0 + t\mathbf{b})|\colon |t| = r/L(z^0)\right\} \leq P_2 \min\left\{|F(z^0 + t\mathbf{b})|\colon |t| = r/L(z^0)\right\}.$$

We denote $L^* = \max\left\{L(z^0 + t\mathbf{b})\colon |t| \leq (2R+2)/L(z^0)\right\}$, $\rho_0 = R/L(z^0)$, $\rho_k = \rho_0 + k\eta/L^*$, $k \in \mathbb{Z}_+$. We have

$$\frac{\eta}{L^*} < \frac{R}{L^*} \leq \frac{R}{L(z^0)} < \frac{2R+2}{L(z^0)} - \frac{R+1}{L(z^0)}.$$

Then there exists $n^* \in \mathbb{N}$ independent of z^0 such that

$$\rho_{p-1} < \frac{R+1}{L(z^0)} \leq \rho_p \leq \frac{2R+2}{L(z^0)},$$

for some $p = p(z^0) \leq n^*$ because $L \in Q^n_{\mathbf{b}}$. Indeed,

$$\left(\frac{2R+2}{L(z^0)} - \rho_0\right) \Big/ \left(\frac{\eta}{L^*}\right) = \frac{(R+2)L^*}{\eta L(z^0)} =$$
$$= \frac{R+2}{\eta} \max\left\{\frac{L(z^0 + t\mathbf{b})}{L(z^0)}\colon |t| \leq \frac{2R+2}{L(z^0)}\right\} \leq \frac{R+2}{\eta}\lambda_{\mathbf{b}}(2R+2).$$

Thus, $n^* = [\frac{R+2}{\eta}\lambda_{\mathbf{b}}(2R+2)]$, where $[a]$ is the integer part of $a \in \mathbb{R}$. Let $|F(z^0 + t_k^{**}\mathbf{b})| = \max\{|F(z^0 + t\mathbf{b})|\colon t \in c_k\}$, $c_k = \{t \in \mathbb{C}\colon |t| = \rho_k\}$, and t_k^* be the intersection point of the segment $[0, t_k^{**}]$ with the circle c_{k-1}. Then for every $r > \eta$ and for each $k \leq n^*$ the relations $|t_k^{**} - t_k^*| = \frac{\eta}{L^*} \leq \frac{r}{L(z^0 + t_k^*\mathbf{b})}$ hold. Thus, for some $r = r(z^0 + t_k^*\mathbf{b}) \in [\eta, R]$ we deduce

$$|F(z^0 + t_k^{**}\mathbf{b})| \leq \max\left\{|F(z^0 + t\mathbf{b})|\colon |t - t_k^*| = r/L(z^0 + t_k^*\mathbf{b})\right\} \leq$$

$$\leq P_2 \min\left\{|F(z^0+t\mathbf{b})|\colon |t-t_k^*| = r/L(z^0+t_k^*\mathbf{b})\right\} \leq$$
$$\leq P_2 \min\left\{|F(z^0+t\mathbf{b})|\colon |t-t_k^*| = r/L(z^0+t_k^*\mathbf{b}), |t-t_0| \leq \rho_{k-1}\right\} \leq$$
$$\leq P_2 \max\{|F(z^0+t\mathbf{b})|\colon t\in c_{k-1}\}.$$

Hence,

$$\max\left\{|F(z^0+t\mathbf{b})|\colon |t| = (R+1)/L(z^0)\right\} \leq$$
$$\leq \max\{|F(z^0+t\mathbf{b})|\colon t\in c_p\} \leq P_2 \max\{|F(z^0+t\mathbf{b})|\colon t\in c_{p-1}\} \leq$$
$$\leq \ldots \leq (P_2)^p \max\{|F(z^0+t\mathbf{b})|\colon t\in c_0\} \leq$$
$$\leq (P_2)^{n^*} \max\left\{|F(z^0+t\mathbf{b})|\colon |t| = R/L(z^0)\right\}.$$

We obtain (4) with $P_1 = (P_2)^{n^*}$. Theorem 6 is proved. □

Let us to denote

$$G_r(F) := \bigcup_{z\colon F(z)=0} \{z+t\mathbf{b}\colon |t| < r/L(z)\}. \tag{5}$$

Given $z^0\in\mathbb{C}^n$, by $n_{z^0}(r,F) = n_{\mathbf{b}}\big(r,z^0,1/F\big) := \sum_{|a_k^0|\leq r} 1$ we denote the counting function of the zeros a_k^0 of the slice function $F(z^0+t\mathbf{b})$ in the disc $\{t\in\mathbb{C}\colon |t|\leq r\}$. If for a given $z^0\in\mathbb{C}^n$ and for all $t\in\mathbb{C}$ $F(z^0+t\mathbf{b})\equiv 0$, then we put $n_{z^0}(r) = -1$. Denote $n(r) = \sup_{z\in\mathbb{C}^n} n_z(r/L(z))$.

Theorem 7 ([6, 9])**.** *Let F be an entire function in $\mathbb{C}^n$, $L\in Q^n_{\mathbf{b}}$. Then the function F is of bounded L-index in the direction $\mathbf{b}\in\mathbb{C}^n\setminus\{0\}$ if and only if the following conditions hold:*

1) *for every $R>0$ there exists $P=P(R)>0$ such that for each $z\in\mathbb{C}^n\backslash G_R(F)$*

$$\left|\frac{1}{F(z)}\frac{\partial F(z)}{\partial \mathbf{b}}\right| \leq PL(z); \tag{6}$$

2) *for every $R>0$ there exists $\widetilde{n}(R)\in\mathbb{Z}_+$ such that for every $z\in\mathbb{C}^n$*

$$n_z\left(r/L(z),F\right) \leq \widetilde{n}(r). \tag{7}$$

Theorem 7 is convenient to investigate boundedness of l-index of infinite products [13, 14, 15] in one-dimensional case. In view of Theorems 2, 3, 7, we posed Conjecture 2 in [2] that inequalities (6) and (7) can be satisfied for some $R>0$. Using assumptions that (6) is valid for some $r_1>0$ and (7) is valid for some $r_2\in(0,1)$ such that $2r_1\cdot n(r_2) < r_2/\lambda_{\mathbf{b}}(r_2)$, we proved the conjecture in [1]. But in view of Theorem 6, we can remove the condition $r_2\in(0,1)$. Thus, the following statement is true.

Theorem 8. *Let $L\in Q^n_{\mathbf{b}}$, $F(z)$ be an entire function in $\mathbb{C}^n$. If the following conditions hold*

1) *there exist $r_1>0$, $P>0$ such that for each $z\in\mathbb{C}^n\backslash G_{r_1}(F)$ inequality (6) holds;*
2) *there exists $r_2>0$ such that $n(r_2)\in[-1;\infty)$ and $2r_1\cdot n(r_2) < r_2/\lambda_{\mathbf{b}}(r_2)$, where r_1 is chosen in the previous condition,*

then the function F is of bounded L-index in the direction $\mathbf{b}$.

Proof of Theorem 8 is similar to that of corresponding theorem from [1]. It also uses new Theorem 6.

Note that Conjecture 2 from [2] is partially proved under the additional restriction $2n(r_2)r_1 < r_2/\lambda_{\mathbf{b}}(r_2)$. It is currently unknown whether the condition is essential (see Problem 3 in [1]).

Note that Theorems 6 and 8 are new even for entire function in $\mathbb{C}$ (cf. [3], [4], [5]).

We remind some definitions of the theory of entire functions (more details are in [16]). The order ρ and the type σ of an entire function $f(t)$ ($t \in \mathbb{C}$) are defined as

$$\rho = \varlimsup_{r\to\infty} \frac{\ln\ln M_f(r)}{\ln r}, \ \sigma = \varlimsup_{r\to\infty} \frac{\ln M_f(r)}{r^\rho}.$$

Corollary 1. *Let $l \in Q$, f be an entire function of order $\rho \in (0,1]$ and finite type $\sigma > 0$, $t \in \mathbb{C}$,*

$$r_0 := \varliminf_{r\to\infty} \frac{1}{r^{\rho-1}\lambda_{\mathbf{b}}(r)}.$$

If $r_0 > 0$ and there exist $r_1 \in (0, \frac{r_0}{2e\rho\sigma})$, $P > 0$ such that for each $t \in \mathbb{C}\backslash G_{r_1}(f)$ the inequality $\frac{|f'(t)|}{|f(t)|} \le Pl(t)$ holds and $n(r_1) \in [-1; \infty)$ then the function f has bounded l-index.

Proof. If $f(t)$ is an entire function of finite type σ and of positive order ρ then we have ([16], p. 19, Theorem 3) $\varlimsup_{r\to\infty} \frac{n(r)}{r^\rho} \le e\rho\sigma$. Hence, we deduce

$$0 < \frac{r_0}{2e\rho\sigma} \le \varliminf_{r\to\infty} \frac{r^\rho}{2n(r)} \cdot \frac{1}{r^{\rho-1}\lambda_{\mathbf{b}}(r)} = \varliminf_{r\to\infty} \frac{r}{2n(r)\lambda_{\mathbf{b}}(r)}.$$

Thus, $r_1 < \frac{r}{2n(r)\lambda_{\mathbf{b}}(r)}$ for all $r \ge r^*$. It remains to prove $n(r) \in (0, \infty)$ for all $r \ge r^*$.

Let $t_0 \in \mathbb{C}$ be arbitrary. We consider the circle $|t - t_0| \le \frac{r}{l(t_0)}$. Since $l \in Q$, every closed disk of radius $\frac{r}{l(t_0)}$ can be covered by a finite number $m = m(r)$ of closed discs of radius $\frac{r_1}{\lambda_{\mathbf{b}}(r)l(t_0)} \le \frac{r_1}{l(t_j)}$ with centers t_j in the disk of radius $\frac{r}{l(t_0)}$. Then $n(r) \le m(r)n(r_1) < \infty$. By Theorem 8 the function f is of bounded l-index. □

Note that the similar criteria (estimates of maximum modulus, minimum modulus, logarithmic derivative and distribution of zeros for any radius) are also known for analytic in a disc and in arbitrary domain functions of one variable ([17, 18, 19]), entire functions of several variables ([8, 20, 21]) and analytic in a ball or in a polydisc functions ([22, 23]). It should be mentioned that there were only obtained estimates of maximum modulus for entire and analytic function of bounded $\mathbf{L}$-index in joint variables. Now we do not know direct analogs of Theorems 5, 7 for these classes of functions.

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