UDC 519.21+62

A. V. Kinash, postgraduate student,
U. T. Khimka, Ph. D.
National University Lviv Polytechnic, Lviv

ASYMPTOTIC DISSIPATIVITY OF LORENZ MODEL WITH MARKOV SWITCHING FOR THE CITY DEVELOPMENT

In the work activity of the city system, described by generalized Lorentz model, is considered. It is assumed the existence of external influence on the land rate change in the city that is described by diffusive term and ergodic Markov processes. With the made assumptions, asymptotic dissipativity conditions of the initial system are set. Stationary points of the model of city economical activity are studied based on the results obtained for asymptotic dissipativity of generalized Lorentz model.

Key words: *Lorenz model, city system, asymptotic dissipativity, stationary point.*

Introduction. Lorentz system was first formulated in 1963 by E. Lorenz in considering the convective fluid motion heated from the bottom. Lorentz system is a system of three first order differential equations, which has the form

$$\begin{cases} \dfrac{dX}{dt} = \sigma Y - \sigma X, \\ \dfrac{dY}{dt} = rX - Y - XZ, \\ \dfrac{dZ}{dt} = XY - bZ, \end{cases}$$

where $X(t)$ describes the intensity of convective motion, $Y(t)$ — temperature difference between ascending and descending liquid streams, and $Z(t)$ — vertical change of temperature. Lorentz model is widely used in meteorology, economics, physics and others.

In particular, in economic theory Lorenz system can be used to study the distribution function of demand for products [1], and economic dynamics of the city system. In this paper the generalized system of Lorentz that describes the economic system of the city in the presence of external disturbances, which have diffusive nature. Namely, asymptotic dissipativity of such a model is studied based on the results obtained in [2]. Particularly, the presence of Lyapunov function that satisfies sufficient conditions of asymptotic dissipativity of the initial system. In this article, such Lyapunov function is constructed, that allows us to state the dissipativity of the considered Lorentz model.

Statement of the problem. Considering the generalized Lorenz system, which describes economical model of the city system. Economic activity of the city is assumed to be very small compared with the outside world [3].

$$\begin{cases} \dfrac{dX}{dt} = a_1(a_2Y - a_3X), \\ \dfrac{dY}{dt} = b_1(b_2X - b_3Y) - b_4XZ, \\ \dfrac{dZ}{dt} = c_1XY - c_2Z, \end{cases} \tag{1}$$

where X — volume of products manufactured by the city system; Y — the number of indigenous people; Z — land rent.

Constants are defined as follows a_1 — rate of formation speed; a_2 — parameter, which represents the magnitude of demand for the products of the city, normalized per person; a_3 — level of supply of products within the city; b_2 — the demand for labor on the part of firms for the production of a unit of output; b_3 — defined as a part of residents who choose work in the city; c_1 та c_2 — determine the level of production and population of the city positive impact, as well as the negative impact of rent growth on its current value, respectively.

System (1) can not fully describe the system of the city, because it does not include the impact of the world on the economic situation in the city. Thus, it is worthwhile to consider perturbed system. In particular, we introduce a diffusional perturbed term $\sigma(X,Y,Z)dw(t)$ in the third equation, which describes the influence of the world on the rate of land rent change in the city[4, p. 101].

$$\begin{cases} \dfrac{dX}{dt} = a_1(a_2Y - a_3X), \\ \dfrac{dY}{dt} = b_1(b_2X - b_3Y) - b_4XZ, \\ \dfrac{dZ}{dt} = c_1XY - c_2Z + \sigma(X,Y,Z)dw(t), \end{cases} \tag{2}$$

Let us also consider the impact of external factors on the coefficient c_2, which defines the change of the city land rent. This influence is given as ergodic Markov process $x(t)$ in phase space of states $\{-0.2, 0.2\}$ with stationary distribution $\{0.5, 0.5\}$, namely

$$x = \begin{cases} -0.2, p = 0.5, \\ 0.2, p = 0.5, \end{cases}$$

due to random rent changes.

$$\begin{cases} \dfrac{dX}{dt} = a_1(a_2Y - a_3X), \\ \dfrac{dY}{dt} = b_1(b_2X - b_3Y) - b_4XZ, \\ \dfrac{dZ}{dt} = c_1XY - (c_2 + x(t)c_2)Z + \sigma(X, Y, Z)dw(t), \end{cases} \quad (3)$$

According to the theorem on asymptotic dissipativity [2] in order that the process was asymptotically dissipative, dissipative should be its limited process. Specifically, conditions must be satisfied

$$C(u)V'(u) < -A_1V(u), \quad (4)$$

$$\sup_{u \in R^d} \| \sigma(u) \| < A_2, \quad (5)$$

where $A_1 > 0, A_2 > 0$ i $V(X,Y,Z) \in C^3(R^d)$ Lyapunov function

$$\frac{du}{dt} = C(u).$$

Example. Let us consider specific model of the city

$$\begin{cases} \dfrac{dX}{dt} = Y - 6X, \\ \dfrac{dY}{dt} = 3X - 3Y - 7XZ, \\ \dfrac{dZ}{dt} = 6XY - 4(1 + x(t))Z + dw(t). \end{cases}$$

The average system has the form

$$\begin{cases} \dfrac{dX}{dt} = Y - 6X, \\ \dfrac{dY}{dt} = 3X - 3Y - 7XZ, \\ \dfrac{dZ}{dt} = 6XY - 4Z + dw(t). \end{cases} \quad (6)$$

In this case $C(X,Y,Z)$ has the presentation

$$C(X,Y,Z) = \begin{pmatrix} Y - 6X \\ 3X - 3Y - 7XZ \\ 6XY - 4Z \end{pmatrix}^T.$$

We verify the conditions of the theorem for the system (6). Considering Lyapunov function in the form

$$V(X,Y,Z) = X^2 + Y^2 + Z^2. \tag{7}$$

Hence

$$V'(X,Y,Z) = (2X, 2Y, 2Z)^T.$$

Particularly for condition (4), have

$$C(X,Y,Z)V'(X,Y,Z) = \begin{pmatrix} Y-6X \\ 3X-3Y-7XZ \\ 6XY-4Z \end{pmatrix}^T \begin{pmatrix} 2X \\ 2Y \\ 2Z \end{pmatrix} =$$

$$= (Y-6X, 3X-3Y-7XZ, 6XY-4Z)\begin{pmatrix} 2X \\ 2Y \\ 2Z \end{pmatrix} =$$

$$= 2XY - 12X^2 + 6XY - 6Y^2 - 14XYZ + 12XYZ - 8Z^2 =$$

$$= -12X^2 + 8XY - 6Y^2 - 8Z^2 - 2XYZ.$$

Under $A_1 = 1$ condition (4) takes the form

$$-12X^2 + 8XY - 6Y^2 - 8Z^2 - 2XYZ < -X^2 - Y^2 - Z^2,$$

$$-12X^2 + 8XY - 6Y^2 - 8Z^2 - 2XYZ + X^2 + Y^2 + Z^2 < 0,$$

$$-11X^2 + 8XY - 5Y^2 - 7Z^2 - 2XYZ < 0,$$

$$-\left(11X^2 - 8XY + \left(\frac{16}{11}\right)Y^2 - \left(\frac{16}{11}\right)Y^2\right) - 5Y^2 - 7Z^2 - 2XYZ < 0,$$

$$-\left(\sqrt{11}X - \frac{4}{\sqrt{11}}Y\right)^2 + \left(\frac{16}{11}\right)Y^2 - 5Y^2 - 7Z^2 - 2XYZ < 0,$$

$$-\left(\sqrt{11}X - \frac{4}{\sqrt{11}}Y\right)^2 - \frac{39}{11}Y^2 - 7Z^2 - 2XYZ < 0.$$

And, as, $\sigma(X,Y,Z) = 1$, that condition (5) is satisfied.

Thus, system (6) is dissipative. Therefore, initial system is asymptotically dissipative.

The asymptotic dissipativity of initial system. Set the conditions of asymptotic dissipativity for the problem (3). Namely, for (4) have

$$C(X,Y,Z) = \begin{pmatrix} a_1(a_2Y - a_3X) \\ b_1(b_2X - b_3Y) - b_4XZ \\ c_1XY - c_2Z \end{pmatrix}^T = \begin{pmatrix} a_1a_2Y - a_1a_3X \\ b_1b_2X - b_1b_3Y - b_4XZ \\ c_1XY - c_2Z \end{pmatrix}^T.$$

So, for the Lyapunov function (7) get

$$C(X,Y,Z)V'(X,Y,Z)=\begin{pmatrix} a_1a_2Y-a_1a_3X \\ b_1b_2X-b_1b_3Y-b_4XZ \\ c_1XY-c_2Z \end{pmatrix}^T \begin{pmatrix} 2X \\ 2Y \\ 2Z \end{pmatrix}=$$

$$=(a_1a_2Y-a_1a_3X, b_1b_2X-b_1b_3Y-b_4XZ, c_1XY-c_2Z)\begin{pmatrix} 2X \\ 2Y \\ 2Z \end{pmatrix}=$$

$$=-2a_1a_3X^2+2(a_1a_2+b_1b_2)XY-2b_1b_3Y^2-2(b_4-c_1)XYZ-2c_2Z^2.$$

Substituting the obtained result in (4)

$$-2a_1a_3X^2+2(a_1a_2+b_1b_2)XY-2b_1b_3Y^2-2(b_4-c_1)XYZ-2c_2Z^2<$$
$$<-A_1X^2-A_1Y^2-A_1Z^2,$$

$$-2a_1a_3X^2+2(a_1a_2+b_1b_2)XY-2b_1b_3Y^2-2(b_4-c_1)XYZ-2c_2Z^2+$$
$$+A_1X^2+A_1Y^2+A_1Z^2<0,$$

$$-\left(\sqrt{2a_1a_3-A_1}\,X-\frac{a_1a_2+b_1b_2}{\sqrt{2a_1a_3-A_1}}Y\right)^2-2(b_4-c_1)XYZ-$$
$$-\left(2b_1b_3-\frac{(a_1a_2+b_1b_2)^2}{2a_1a_3-A_1}-A_1\right)Y^2-(2c_2-A_1)Z^2<0.$$

Therefore, the following conditions must be satisfied

$$\begin{cases} 2a_1a_3-A_1>0, \\ 2b_1b_3-\dfrac{(a_1a_2+b_1b_2)^2}{2a_1a_3-A_1}-A_1\ge 0, \\ b_4-c_1\ge 0, \\ 2c_2-A_1\ge 0. \end{cases}$$

$$\begin{cases} A_1<2a_1a_3, \\ 2b_1b_3-\dfrac{(a_1a_2+b_1b_2)^2}{2a_1a_3-A_1}-A_1\ge 0, \\ b_4\ge c_1, \\ A_1\le 2c_2. \end{cases} \qquad (8)$$

Consider separately the second inequality

$$2b_1b_3-\frac{(a_1a_2+b_1b_2)^2}{2a_1a_3-A_1}-A_1\ge 0,$$

$$\frac{2b_1b_3(2a_1a_3 - A_1) - (a_1a_2 + b_1b_2)^2 - A_1(2a_1a_3 - A_1)}{2a_1a_3 - A_1} \geq 0.$$

Since, $A_1 < 2a_1a_3$, then the numerator of left side of inequality should be nonnegative

$$2b_1b_3(2a_1a_3 - A_1) - (a_1a_2 + b_1b_2)^2 - A_1(2a_1a_3 - A_1) \geq 0.$$

Solve the inequality relative to A_1

$$A_1^2 - (2a_1a_3 + 2b_1b_3)A_1 + (4b_1b_3a_1a_3 - a_1^2a_2^2 - 2a_1a_2b_1b_2 - b_1^2b_2^2) \geq 0,$$

$$D = (2a_1a_3 - 2b_1b_3)^2 + (2a_1a_2 + 2b_1b_2)^2,$$

$$A_1^{1,2} = \frac{1}{2}\left(2a_1a_3 + 2b_1b_3 \pm \sqrt{(2a_1a_3 - 2b_1b_3)^2 + (2a_1a_2 + 2b_1b_2)^2}\right).$$

Therefore, system (8) has repredentation

$$\left[\begin{array}{l} \begin{cases} A_1 < 2a_1a_3, \\ A_1 \leq a_1a_3 + b_1b_3 - \sqrt{(a_1a_3 - b_1b_3)^2 + (a_1a_2 + b_1b_2)^2}, \\ b_4 \geq c_1, \\ A_1 \leq 2c_2. \end{cases} \\ \begin{cases} A_1 < 2a_1a_3, \\ A_1 \geq a_1a_3 + b_1b_3 + \sqrt{(a_1a_3 - b_1b_3)^2 + (a_1a_2 + b_1b_2)^2}, \\ b_4 \geq c_1, \\ A_1 \leq 2c_2. \end{cases} \end{array}\right. \tag{9}$$

Consider separately second system in the (9). It is obvious that in order to have a solution of the system the inequality should be fulfilled

$$a_1a_3 + b_1b_3 + \sqrt{(a_1a_3 - b_1b_3)^2 + (a_1a_2 + b_1b_2)^2} < 2a_1a_3.$$

Herefrom

$$\sqrt{(a_1a_3 - b_1b_3)^2 + (a_1a_2 + b_1b_2)^2} < a_1a_3 - b_1b_3.$$

Since all coefficients are positive, then bring to square both sides of inequality, we get

$$(a_1a_2 + b_1b_2)^2 < 0.$$

Then, system (9) have a form

$$\begin{cases} A_1 < 2a_1a_3, \\ A_1 \leq a_1a_3 + b_1b_3 - \sqrt{(a_1a_3 - b_1b_3)^2 + (a_1a_2 + b_1b_2)^2}, \\ b_4 \geq c_1, \\ A_1 \leq 2c_2. \end{cases} \tag{10}$$

According to fulfillment of these conditions and the condition (5) system (2) is dissipative, and therefore (3) asymptotically dissipative.

Hence, for the system (6) we obtain the following interval $A_1 \in (0;4]$.

Stationary points. Let us find the stationary points of the system (1). From the equating left sides of equations to zero, we get

$$\begin{cases} a_1(a_2Y - a_3X) = 0, \\ b_1(b_2X - b_3Y) - b_4XZ = 0, \\ c_1XY - c_2Z = 0, \end{cases}$$

Solve obtained system

$$\begin{cases} Y = \dfrac{a_3}{a_2}X, \\ b_1(b_2X - b_3\dfrac{a_3}{a_2}X) - b_4XZ = 0, \\ c_1\dfrac{a_3}{a_2}X^2 - c_2Z = 0, \end{cases}$$

$$\begin{cases} Y = \dfrac{a_3}{a_2}X, \\ X(b_1b_2 - \dfrac{a_3b_1b_3}{a_2} - b_4Z) = 0, \\ c_1\dfrac{a_3}{a_2}X^2 - c_2Z = 0. \end{cases}$$

From second equation $X = 0$ or $b_1b_2 - \dfrac{a_3b_1b_3}{a_2} - b_4Z = 0.$

Thus point $K_1(0,0,0)$ — stationary point of the system (1).

Under $b_1b_2 - \dfrac{a_3b_1b_3}{a_2} - b_4Z = 0,$ have

$$\begin{cases} Y = \dfrac{a_3}{a_2}X, \\ Z = \dfrac{a_2b_1b_2 - a_3b_1b_3}{b_4a_2}, \\ c_1\dfrac{a_3}{a_2}X^2 = c_2\dfrac{a_2b_1b_2 - a_3b_1b_3}{b_4a_2}, \end{cases}$$

$$\begin{cases} Y = \dfrac{a_3}{a_2} X, \\ Z = \dfrac{a_2 b_1 b_2 - a_3 b_1 b_3}{b_4 a_2}, \\ X^2 = c_2 \dfrac{a_2 b_1 b_2 - a_3 b_1 b_3}{b_4 c_1 a_3}. \end{cases}$$

Taking into accunt, that all coefficients of the system are positive, the following condition should be held

$$a_2 b_1 b_2 - a_3 b_1 b_3 > 0 \qquad (11)$$

(under $a_2 b_1 b_2 = a_3 b_1 b_3$ get stationary point K_1).

Under this condition, for the considered system there exist two more stationary points

$$\begin{cases} Y = \pm \dfrac{a_3}{a_2} \sqrt{c_2 \dfrac{a_2 b_1 b_2 - a_3 b_1 b_3}{b_4 c_1 a_3}}, \\ Z = \dfrac{a_2 b_1 b_2 - a_3 b_1 b_3}{b_4 a_2}, \\ X = \pm \sqrt{c_2 \dfrac{a_2 b_1 b_2 - a_3 b_1 b_3}{b_4 c_1 a_3}}. \end{cases}$$

Therefore, stationary points of the system are $K_1(0,0,0)$ and, under condition (11),

$$K_{2,3} = \left(\pm \sqrt{c_2 \frac{a_2 b_1 b_2 - a_3 b_1 b_3}{b_4 c_1 a_3}}, \pm \frac{a_3}{a_2} \sqrt{c_2 \frac{a_2 b_1 b_2 - a_3 b_1 b_3}{b_4 c_1 a_3}}, \frac{a_2 b_1 b_2 - a_3 b_1 b_3}{b_4 a_2} \right).$$

Now, from second inequality of asymptotic dissipativity conditions (10), the following condition is obviously holds

$$a_1 a_3 + b_1 b_3 - \sqrt{(a_1 a_3 - b_1 b_3)^2 + (a_1 a_2 + b_1 b_2)^2} > 0.$$

Hence,

$$\sqrt{(a_1 a_3 - b_1 b_3)^2 + (a_1 a_2 + b_1 b_2)^2} < a_1 a_3 + b_1 b_3,$$

$$(a_1 a_3 - b_1 b_3)^2 + (a_1 a_2 + b_1 b_2)^2 < (a_1 a_3 + b_1 b_3)^2,$$

$$a_1^2 a_3^2 - 2 a_1 a_3 b_1 b_3 + b_1^2 b_3^2 + (a_1 a_2 + b_1 b_2)^2 < a_1^2 a_3^2 + a_1 a_3 b_1 b_3 + b_1^2 b_3^2,$$

$$(a_1 a_2 + b_1 b_2)^2 < 4 a_1 a_3 b_1 b_3.$$

According to (11)

$$(a_1 a_2 + b_1 b_2)^2 < 4 a_1 a_3 b_1 b_3 < 4 a_1 a_2 b_1 b_2,$$

$$a_1^2 a_2^2 + 2a_1 a_2 b_1 b_2 + b_1^2 b_2^2 < 4a_1 a_2 b_1 b_2,$$
$$a_1^2 a_2^2 - 2a_1 a_2 b_1 b_2 + b_1^2 b_2^2 < 0.$$

Therefore

$$(a_1 a_2 - b_1 b_2)^2 < 0.$$

Thus, under conditions of asymptotic stability of (3), the only stationary point is $K_1(0,0,0)$.

Conclusions. In the article the dissipativity conditions were obtain for the generalized Lorenz model, which describes the city economics. The city system is considered provided the existence of the external perturbations on the rent value. Also, the stationary points were studied for the considered system under condition of the asymptotic diddipativity.

References:

1. Olems'koi O. I. Synergetic model of the economic structure of society / O. I. Olems'koi, O. V. Yushchenko, S. V. Kokhan // Journal of Physical Research. — 2004. — Vol. 8, № 3. — P. 268–278. (in Ukrainian).
2. Kinash A. V. Asymptotic dissipativity of the diffusion process. процесу / A. V. Kinash, Ya. M. Chabanyuk, U. T. Khimka // Mathematical and computer modeling. A series of physical and mathematical sciences. Collected Works. V.M. Glushkov Institute of Cybernetics, Kamyanets-Podilsky national university of I. Ogiienko. — 2014. — Issue. 11. — P. 77–87. (in Ukrainian).
3. Beliavskii S. S. The generalized model of urban system (generalized Lorenz system) / S. S. Beliavskii, T. A. Orlianin // Third International Scientific Conference «Mathematical Modeling and Differential Equations»: a collection of articles of the Third International Scientific Conference. Brest, September 17–22, 2012. — Minsk : BGU, 2012. — P. 70-78. (in Russian).
4. Bondarev B. V. Stochastic calculus in problems of financial and actuarial mathematics. Risk assessment for insurance / B. V. Bondarev, O. E. Sosnitzkii. — 2013. — 228 p. (in Russian).

У роботі розглянуто діяльність системи міста, що описується узагальненою моделлю Лоренца. Припускається наявність зовнішнього впливу на зміну величини земельної ренти в місті, який описується дифузійним доданком та ергодичним марковським процесом. За зроблених припущень, встановлено умови асимптотичної дисипативності вихідної системи. Досліджено стаціонарні точки моделі економічної діяльності міста на основі результатів, отриманих для асимптотичної дисипативності узагальненої моделі Лоренца.

Ключові слова: *модель Лоренца, система міста, асимптотична дисипативність, стаціонарна точка.*

Отримано: 17.03.2016