CONTROLLABILITY RESULTS FOR SEMILINEAR FUNCTIONAL DIFFERENTIAL INCLUSIONS WITH UNBOUNDED DELAY

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In this paper, by using semigroup of evolution operators and fixed point argument we establish existence results for the controllability of semilinear functional and neutral functional differential inclusions in a Banach space with infinite delay when the right hand side has convex as well as nonconvex values.

1. INTRODUCTION

The problem of controllability of linear and nonlinear systems represented by ODEs in finite dimensional space has been extensively studied. Several authors have extended the controllability concept to infinite dimensional systems in Banach space with unbounded operators see the monographs [8, 14, 28, 30] and the references therein. Lasiecka and Triggiani [26] established sufficient conditions for controllability of linear and nonlinear systems in Banach space. With the aid of Sadovskii's fixed point theory and the semigroup theory, Fu [19, 20] considered the controllability of two classes of abstract neutral functional differential equations with infinite delay. During the last decade Balachandran and his collaborators have considered various classes of first and second order semilinear ordinary, functional and neutral functional differential equations on Banach spaces. These works and others

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due to other authors are listed in the survey paper by Balachandran and Dauer [2]. By means of fixed point arguments some of the previous works were extended to the multivalued case by Benchohra *et al* (see the book [6] and the references therein, and the papers [3–5, 7]).

In this paper, we are concerned with the controllability of some classes of first order semilinear functional and neutral functional differential inclusions in a Banach space. Initially, in Section 3 we will consider the first order functional differential inclusion

$$y'(t) \in A(t)y(t) + Cu(t) + F(t, y_t), \quad a.e. \quad t \in [0, b], \tag{1}$$

$$y_0 = \varphi \in B, \tag{2}$$

$F : [0, b] \times B \to \mathcal{P}(E)$ is a multivalued map with nonempty compact values, $\varphi \in B$, B is the phase space to be specified later, the control function $u(\cdot)$ is given in $L^2([0, b], E)$, the Banach space of admissible control function with E is a real separable Banach space with the norm $|\cdot|$, C is a bounded linear operator from E into E, the family $\{A(t) : 0 \leq t \leq b\}$ of unbounded linear operators generates a linear evolution system and $\mathcal{P}(E)$ is the family of all nonempty subsets of E.

To study the system (1), (2), we assume that the histories $y_t : (-\infty, 0] \to E$, $y_t(\theta) = y(t + \theta)$ belong to some abstract space B.

Later, in Section 4, we study the neutral functional differential inclusions of the form

$$\frac{d}{dt}[y(t) + g(t, y_t)] \in A(t)y(t) + Cu(t) + F(t, y_t), \ a.e. \ t \in [0, b], \tag{3}$$

$$y_0 = \varphi \in B, \tag{4}$$

where F and φ are as in problem (1), (2), $g : [0, b] \times B \to E$. For each of the above problems we shall present three existence theorems. The first one when the right hand side is convex valued relies of the Bohnenblust–Karlin's fixed point theorem. The second and the third one which are both in the nonconvex valued right hand side rely on a combination of Schauder's fixed point theorem with a selection theorem due to Bressan and Colombo [10] for lower semicontinuous multivalued operators with nonempty closed and decomposable values and on a fixed point theorem for contraction multivalued maps due to Covitz and Nadler, respectively. Our results extend to the multivalued case those considered by Balachandran and Dauer [2], Fu [19,20] and those considered by Benchohra *et al* for finite delay.

2. PRELIMINARIES

In this section, we introduce notations, definitions, and preliminary facts from multivalued analysis which are used throughout this paper.

In this article, we will employ an axiomatic definition of the phase space B introduced by Hale and Kato [22] and follow the terminology used in [23]. Thus, B will be a linear space of functions mapping $(-\infty, 0]$ into E endowed with a seminorm $\|\cdot\|_B$. We will assume that B satisfies the following axioms:

(A_1) If $y : (-\infty, \sigma + b) \to E, b > 0$, is continuous on $[\sigma, \sigma + b)$ and $y_\sigma \in B$, then for every $t \in [\sigma, \sigma + b)$ the following conditions hold:
(i) y_t is in B;
(ii) $\|y(t)\| \leq H\|y_t\|_B$;
(iii) $\|y_t\|_B \leq K(t-\sigma)\sup\{\|y(s)\| : \sigma \leq s \leq t\} + \mathcal{M}(t-\sigma)\|y_\sigma\|_B$.

(A_2) For the function $y(\cdot)$ in $(A_1), y_t$ is a B-valued continuous function on $[\sigma, \sigma + b]$.

(A_3) The space B is complete.

Here $H \geq 0$ is a constant, $K, \mathcal{M} : [0, \infty) \to [0, \infty), K$ is continuous and $\mathcal{M}$ is locally bounded, and $H, K, \mathcal{M}$ are independent of $y(t)$.

We say that a family $\{A(t) : t \in \mathbb{R}\}$ generates a unique linear evolution system $\{U(t,s) : 0 \leq s \leq t \leq b\}$ satisfying the following properties :

(1) $U(t,t) = I$ (I is the identity operator in E),

(2) $U(t,s)U(s,\tau) = U(t,\tau)$ for $0 \leq \tau \leq s \leq t \leq b$,

(3) $U(t,s) \in L(E)$ the spaces of bounded linear operators on E, where every $0 \leq s \leq t \leq b$ and for each $y \in E$, the mapping $(t,s) \to U(t,s)y$ is continuous.

More details on evolution systems and their properties can be found in [1,17]. For the family $\{A(t) : 0 \leq t \leq b\}$ of linear operators, we introduce the following conditions:

(B_1) the domain $D(A)$ of $\{A(t) : 0 \leq t \leq b\}$ is dense and independent of $t, A(t)$ is closed linear operator;

(B_2) For each $t \in [0,b]$, the resolvent $R(\lambda, A(t))$ exists for all λ with $\operatorname{Re}\lambda \leq 0$ and there exists $K > 0$ so that $\|R(\lambda, A(t))\| \leq \frac{K}{|\lambda|+1}$;

(B_3) For all $t, s, \tau \in [0, b]$ there exists $0 < \alpha < 1$ such that

$$\|(A(t) - A(s)A^{-1}(\tau))\| \leq K|t - s|^{\alpha};$$

(B_4) For each $t \in [0, b]$ and some $\lambda \in \rho(A(t))$, the resolvent set of $A(t)$, the resolvent $R(\lambda, A(t))$ is compact operators.

Let $(X, \|\cdot\|)$ be a Banach space. A multi-valued map $G : X \to \mathcal{P}(X)$ has *convex (closed) values* if $G(x)$ is convex (closed) for all $x \in X$. We say that G is *bounded on bounded sets* if $G(B)$ is bounded in X for each bounded set B of X, i.e.,

$$\sup_{x \in B}\{\sup\{\|y\| : y \in G(x)\}\} < \infty.$$

The map G is called *upper semi-continuous (u.s.c.)* on X if for each $x_0 \in X$ the set $G(x_0)$ is a nonempty, closed subset of X, and if for each open set N of X containing $G(x_0)$, there exists an open neighborhood M of x_0 such that $G(M) \subseteq N$. Also, G is said to be *completely continuous* if $G(B)$ is relatively compact for every bounded subset $B \subseteq X$. If the multi-valued map G is completely continuous with nonempty compact values, then G is u.s.c. if and only if G has a closed graph (i.e., $x_n \to x_*$, $y_n \to y_*$, $y_n \in G(x_n)$ imply $y_* \in G(x_*)$). Finally, we say that G has a *fixed point* if there exists $x \in X$ such that $x \in G(x)$.

In the following, let $\mathcal{P}(X) = \{Y \subset X : Y \neq \emptyset\}$, $\mathcal{P}_{cl}(X) = \{Y \in P(X) : Y \text{closed}\}$, $\mathcal{P}_b(X) = \{Y \in \mathcal{P}(X) : Y \text{ bounded}\}, \mathcal{P}_c(X) = \{Y \in P(X) : Y \text{ convex}\}$, and $\mathcal{P}_{cp}(X) = \{Y \in \mathcal{P}(X) : Y \text{ compact}\}$. A multi-valued map $G : [0, b] \to \mathcal{P}_{cl}(X)$ is said to be *measurable* if for each $x \in E$, the function $Y : [0, b] \to X$ defined by

$$Y(t) = d(x, G(t)) = \inf\{|x - z| : z \in G(t)\}$$

is measurable where d is the metric induced by the normed Banach space X. For more details on multivalued maps we refer to the books of Deimling [15], Górniewicz [21], Hu and Papageorgiou [24] and Tolstonogov [29].

For each $y \in C((-\infty, b], E)$ let the set $S_{F,y}$ known as the set of selectors from F defined by

$$S_{F,y} = \{v \in L^1([0, b], E) : v(t) \in F(t, y_t)\}, \text{ a.e. } t \in [0, b]\}.$$

Lemma 1. [27] *Let X be a Banach space. Let $F : [0, b] \times X \to \mathcal{P}_{cp,c}(X)$ be an L^1-Carathéodory multivalued map with and let Γ be a linear continuous mapping from $L^1([0, b], X)$ to $C([0, b], X)$. Then the operator*

$$\Gamma \circ S_F : C([0, b], X) \to P_{cp,c}(C([0, b], X)), \; y \mapsto (\Gamma \circ S_F)(y) := \Gamma(S_{F,y})$$

is a closed graph operator in $C([0,b],X) \times C([0,b],X)$.

Our considerations in the convex case are based on the Bohnenblust–Karlin's fixed point theorem.

Lemma 2. (Bohnenblust–Karlin [9], see also [31, p. 452]). *Let* X *be a Banach space and* $K \in P_{cl,c}(X)$ *and suppose that the operator* $G : K \to P_{cl,c}(K)$ *is upper semicontinuous and the set* $G(K)$ *is relatively compact in* X. *Then* G *has a fixed point in* K.

3. FUNCTIONAL DIFFERENTIAL INCLUSIONS

Before stating and proving the main result, we give the definition of the mild solution of the IVP (1), (2).

Definition 1. *We say that the function* $y(\cdot) : (-\infty, b] \to E$ *is a mild solution of system* (1), (2) *if* $y(t) = \varphi(t)$ *for all* $t \in (-\infty, 0]$, *the restriction of* $y(\cdot)$ *to the interval* $[0,b]$ *is continuous and there exists* $v(\cdot) \in L^1([0,b],E)$: $v(t) \in F(t, y_t)$ *a.e* $[0,b]$ *such that* y *satisfies the following integral equation:*

$$y(t) = U(t,0)\varphi(0) + \int_0^t U(t,s)v(s)ds + \int_0^t U(t,s)Cu(s)ds, \quad 0 \le t \le b. \quad (5)$$

Definition 2. *The IVP* (1), (2) *is said to be controllable on the interval* $[0,b]$ *if for every initial function* $\varphi \in B$ *and* $y_1 \in E$ *there exists a control* $u \in L^2([0,b],E)$ *such that the mild solution* $y(\cdot)$ *of* (1), (2) *satisfies* $y(b) = y_1$.

Let us introduce the following hypotheses which are assumed here after:

(H1) The linear operator $W : L^2([0,b],E) \to C([0,b],E)$ is defined by

$$Wu = \int_0^b U(b,s)Cu(s)ds,$$

and there exists a bounded invertible operator W^{-1} defined on $L^2([0,b],E)/kerW$, and positive constants M, $\overline{M}$, $\overline{M}_1$ such that

$$\|U(t,s)\| \le M, \ 0 \le s \le t \le b, \ \|C\| \le \overline{M} \text{ and } \|W^{-1}\| \le \overline{M}_1.$$

(H2) The function $F : [0,b] \times B \to \mathcal{P}(E)$ is a nonempty compact multivalued map, with convex valued, such that:
a) $(t,x) \mapsto F(t,x)$ is measurable;
b) $x \mapsto F(t,x)$ is upper semi-continuous for a.e. $t \in [0,b]$;

(H3) For each $k > 0$, there exists a positive function $h_k \in L^1([0,b], \mathbb{R}^+)$ such that

$$\|F(t,z)\|_{\mathcal{P}} := \sup\{\|v\| : v \in F(t,z)\} \le h_k(t) \text{ for a.e. } t \in [0,b],$$

and for $z \in B$ with $\|z\|_B \le k$; and

$$\lim_{k\to\infty} \inf \frac{1}{k}\int_0^b h_k(s)ds \le \frac{1}{\max\{b\overline{M}M^2\overline{M_1}, M\}}.$$

(H4) $U(t,s)$ is a compact operator wherever $t - s > 0$.

Remark 1. *For the construction of* W *see* [11].

Theorem 1. *Suppose that hypotheses* (H1)–(H4) *are satisfied. Then the initial value problem* (1), (2) *is controllable on* $(-\infty, b]$.

Proof. Transform the problem into a fixed point problem. Consider the operator, $N : C((-\infty,b],E) \to \mathcal{P}(C((-\infty,b],E))$ defined by:

$$N(y) = \left\{ h \in C((-\infty,b],E) : \right.$$

$$h(t) = \left\{ \begin{array}{ll} \varphi(t), & \text{if } t \in (-\infty,0]; \\ U(t,0)\varphi(0) + \int_0^t U(t,s)v(s)ds+ & \\ + \int_0^t U(t,s)Cu_y(s)ds, & \text{if } t \in [0,b], \end{array} \right\}$$

where $v \in S_{F,y} = \{v \in L^1([0,b],E) : v(t) \in F(t,y_t) \text{ for a.e.} t \in [0,b]\}$. Using the assumption (H1), for arbitrary function $y(\cdot)$ define the control

$$u_y(t) = W^{-1}\left[y_1 - U(b,0)\varphi(0) - \int_0^b U(b,s)v(s)ds\right](t).$$

Next we will prove that N has a fixed point.

Let $x(\cdot) : (-\infty,b) \to E$ be the function defined by

$$x(t) = \begin{cases} U(t,0)\varphi(0), & \text{if } t \in [0,b], \\ \varphi(t), & \text{if } t \in (-\infty,0], \end{cases}$$

then $x_0 = \varphi$. For each $z \in C([0,b],E)$ with $z(0) = 0$, we denote by $\bar{z}$ the function defined by

$$\bar{z}(t) = \begin{cases} z(t), & \text{if } t \in [0,b], \\ 0, & \text{if } t \in (-\infty,0]. \end{cases}$$

If $y(\cdot)$ satisfies (5), we can decompose it as $y(t) = z(t) + x(t), 0 \leq t \leq b$, which implies $y_t = \bar{z}_t + x_t$, for every $0 \leq t \leq b$ and the function $z(\cdot)$ satisfies

$$z(t) = \int_0^t U(t,s)v(s)ds + \int_0^t U(t,s)Cu(s)ds, \tag{6}$$

where $v(t) \in F(t, \bar{z}_t + x_t)$ a.e $t \in [0,b]$. Let the operator $P : C([0,b], E) \to \mathcal{P}(C([0,b], E))$ defined by

$$P(z) = \left\{ h \in C([0,b], E) : \right.$$

$$\left. h(t) = \int_0^t U(t,s)v(s)ds + \int_0^t U(t,s)Cu_z(s)ds, \quad t \in [0,b] \right\},$$

where $v \in S_{F,z} = \{v \in L^1([0,b], E) : v(t) \in F(t, \bar{z}_t + x_t) \text{ for a.e. } t \in [0,b]\}$. Obviously the operator N has a fixed point is equivalent to P has one, so it turns to prove that P has a fixed point. We shall use the fixed point theorem of Bohnenblust–Karlin to prove that P has fixed point. For each positive number k, let

$$B_k = \{z \in C([0,b], E) : z(0) = 0, \ \|z(t)\| \leq k, \ t \in [0,b]\}.$$

It is clear that B_k is closed and convex set in $C([0,b], E)$. We claim that there exists positive number k such that $P(B_k) \subset B_k$. If is not true, then for each positive number k, there is function $z_k \in B_k$ and $h \in P(z_k)$ such that $\|h(t)\| > k$ for some $t \in [0,b]$. Then we have

$$\begin{aligned}
k < |h(t)| &\leq \int_0^t |U(t,s)||(Cu_z)(s)|ds + \int_0^t |U(t,s)||v(s)|ds \leq \\
&\leq M \int_0^t \|C\||u_z(s)|ds + M \int_0^t h_k(s)ds \leq \\
&\leq b\overline{M}M\overline{M_1}\left[\|y_1\| + M\|\phi\| + M\int_0^b h_k(s)ds + M\int_0^b h_k(s)ds\right] \leq \\
&\leq \ b\overline{M}M\overline{M_1}[\|y_1\| + M\|\phi\|] + \max\{b\overline{M}M^2\overline{M_1}, M\} \int_0^b h_k(s)ds.
\end{aligned}$$

Dividing both sides by k and taking the lower limit and from (H3), we get

$$\begin{aligned}
1 < \lim_{k\to\infty} &\left(\frac{b\overline{M}M\overline{M_1}\,[\|y_1\| + M\|\phi\|]}{k} + \right. \\
&\left. + \max\{b\overline{M}M^2\overline{M_1}, M\} \inf \frac{1}{k}\int_0^b h_k(s)ds \right) = \\
&= \max\{b\overline{M}M^2\overline{M_1}, M\} \lim_{k\to\infty} \inf \frac{1}{k}\int_0^b h_k(s)ds \leq 1,
\end{aligned}$$

which yields to a contradiction. Hence there exists a positive number k_0 such that $P(B_{k_0}) \subset B_{k_0}$.

Step 1: $P(z)$ is convex for each $z \in C([0,b],E)$.

Indeed, if h_1, h_2 belong to $P(z)$, then there exist $v_1, v_2 \in S_{F,z}$ such that for each $t \in [0,b]$ we have

$$h_i(t) = \int_0^t U(t,s)v_i(s)ds + \int_0^t U(t,s)Cu_z(s)ds, \quad i = 1,2.$$

Let $0 \leq d \leq 1$. Then for each $t \in [0,b]$ we have

$$(dh_1+(1-d)h_2)(t) = \int_0^t U(t,s)[dv_1(s)+(1-d)v_2(s)]ds + \int_0^t U(t,s)Cu_z(s)ds,$$

where

$$u_z(t) = W^{-1}\left[y_1 - U(b,0)\varphi(0) - \int_0^b U(b,s)[dv_1(s) + (1-d)v_2(s)]ds\right](t).$$

Since $S_{F,z}$ is convex (because F has convex values) then

$$dh_1 + (1-d)h_2 \in P(z).$$

Step 2: $P(B_{k_0})$ is relatively compact.

Since B_{k_0} is bounded and $P(B_{k_0}) \subset B_{k_0}$, it is clear that $P(B_{k_0})$ is bounded. It remains to show that $P(B_{k_0})$ is equicontinuous.

Let $\tau_1, \tau_2 \in [0,b]$, $\tau_1 < \tau_2$ and $z \in B_{k_0}$. Then

$$\begin{aligned}
|h(\tau_2) - h(\tau_1)| = \Big| &\int_0^{\tau_2} U(\tau_2,s)v(s)ds + \int_0^{\tau_2} U(\tau_2,s)Cu_z(s)ds - \\
&- \int_0^{\tau_1} U(\tau_1,s)v(s)ds - \int_0^{\tau_1} U(\tau_1,s)Cu_z(s)ds\Big| \leq \\
&\leq \int_0^{\tau_1} \|U(\tau_2,s) - U(\tau_1,s)\|\|v(s)\|ds + \int_{\tau_1}^{\tau_2} \|U(\tau_1,s)\|\|v(s)\|ds + \\
&+ \int_0^{\tau_1} \|U(\tau_2,s) - U(\tau_1,s)\|\|C\|\|u_z(s)\|ds + \\
&+ \int_{\tau_1}^{\tau_2} \|U(\tau_1,s)\|\|C\|\|u_z(s)\|ds \leq \\
&\leq \int_0^{\tau_1} \|U(\tau_2,s) - U(\tau_1,s)\|h_{k_0}(s)ds + \int_{\tau_1}^{\tau_2} \|U(\tau_1,s)\|h_{k_0}(s)ds + \\
&+ M\int_{\tau_2}^{\tau_1} \|C\|\|u_z(s)\|ds + \int_0^{\tau_1} \|U(\tau_2,s) - U(\tau_1,s)\|\|C\|\|u_z(s)\|ds.
\end{aligned}$$

Noting that

$$\begin{aligned}\|u_z(s)\| &\leq \|W^{-1}\| \left[\|y_1\| + M\|\varphi\| + M\int_0^b \|v(s)\|ds\right] \leq \\ &\leq \|W^{-1}\| \left[\|y_1\| + M\|\varphi\| + M\int_0^b h_{k_0}(s)ds\right].\end{aligned}$$

We see that $\|h(\tau_2) - h(\tau_1)\|$ tend to zero independently of $z \in B_{k_0}$ as $(\tau_2 - \tau_1) \to 0$. The right-hand side tends to zero as $\tau_2 - \tau_1 \to 0$, and ϵ sufficiently small, since $U(t,s)(t-s>0)$ is a strongly continuous operator and the compactness implies the continuity in the uniform operator topology. As a consequence of the Arzelá–Ascoli theorem it suffices to show that the multivalued P maps B_{k_0} into a precompact set in E. Let $0 < t \leq b$ be fixed and let ϵ be a real number satisfying $0 < \epsilon < t$. For $z \in B_{k_0}$ we define

$$h_\epsilon(t) = U(t, t-\epsilon)\int_0^{t-\epsilon} U(t-\epsilon, s)[v(s) + (Cu)(s)]ds$$

where $v \in S_{F,z}$. Since $U(t,s)$ is a compact operator, the set $H_\epsilon(t) = \{h_\epsilon(t) : h_\epsilon \in P(y)\}$ is precompact in E for every ϵ, $0 < \epsilon < t$. Moreover, for every $h \in P(z)$ we have

$$|h_\epsilon(t) - h(t)| \leq \|C\|_{B(E)}k^* \int_{t-\epsilon}^t \|U(t,s)\|_{B(E)}ds + \int_{t-\epsilon}^t \|U(t,s)\|_{B(E)}h_{k_0}(s)ds,$$

where

$$k^* = \|W^{-1}\| \left[\|y_1\| + M\|\varphi\| + M\int_0^b h_{k_0}(s)ds\right].$$

Therefore there are precompact sets arbitrarily close to the set $\{h(t) : h \in P(z)\}$. Hence the set $\{h(t) : h \in P(z)\}$ is precompact in E.

Step 3: P has a closed graph.

Let $z_n \to z_*, h_n \in P(z_n)$ and $h_n \to h_*$. We shall prove that $h_* \in P(z_*)$. $h_n \in P(z_n)$ means that there exists $v_n \in S_{F,z_n}$ such that, for each $t \in [0,b]$,

$$h_n(t) = \int_0^t U(t,s)v_n(s)ds + \int_0^t U(t,s)(Cu_{z_n})(s)ds.$$

We have to prove that there exists $v_* \in S_{F,z_*}$ such that, for each $t \in [0,b]$,

$$h_*(t) = \int_0^t U(t,s)v_*(s)ds + \int_0^t U(t,s)(Cu_{z_*})(s)ds.$$

We have

$$\left\|\left(h_n - \int_0^t U(t,s)CW^{-1}(y_1 - U(b,0)\phi(0))\right) - \right.$$
$$\left. -\left(h_* - \int_0^t U(t,s)CW^{-1}(y_1 - U(b,0)\phi(0))\right)\right\|_\infty \to 0, \text{ as } n \to \infty.$$

Consider the operator

$$\Gamma : L^1([0,b],E) \to C([0,b],E),$$
$$v \mapsto \Gamma(v)(t) = \int_0^t U(t,s)\left[v(s) + CW^{-1}\left(\int_0^b U(b,s)v(\gamma)d\gamma\right)\right] ds.$$

We can see that the operator Γ is linear and continuous. Indeed, one has

$$\|\Gamma v\|_\infty \le L_* \|v\|_{L^1},$$

where $L_* = M + b\overline{M}_1 M\overline{M}$. From Lemma 1, it follows that $\Gamma \circ S_F$ is a closed graph operator. Moreover, we have that

$$h_n(t) - \int_0^t U(t,s)CW^{-1}\left[y_1 - U(b,0)\phi(0)\right] \in \Gamma(S_{F,z_n}).$$

Since $z_n \to z_*$, it follows, from Lemma 1, that

$$\int_0^t U(t,s)\left[v_*(s) - CW^{-1}\left(\int_0^b U(b,s)v_*(\gamma)d\gamma\right)\right] ds =$$
$$= h_*(t) - U(t,0)\phi(0) - \int_0^t U(t,s)CW^{-1}(y_1 - U(b,0)\phi(0))ds,$$

for some $v_* \in S_{F,z_*}$. As a consequence of Lemma 2, we deduce that P has a fixed point which is a mild solution of (1), (2).

Remark 2. *Assume that* (H1), (H2), (H4) *are satisfied, then a slight modification of the proof (i.e. use the usual Leray-Schauder alternative) guarantees that* (H3) *could be replaced by*

*(H3)** *There exist a continuous non-decreasing function $\psi_F : [0,\infty) \to (0,\infty)$, a function $p \in L^1([0,b],\mathbb{R}_+)$ such that*

$$\|F(t,x)\|_{\mathcal{P}} \le p(t)\psi_F(\|x\|_B) \quad for \quad t \in [0,b], \quad x \in B,$$

with

$$\max\{b\overline{M}M^2\overline{M}_1, M\} \int_0^t p(s)ds < \int_c^\infty \frac{ds}{\psi_F(s)},$$

where

$$c = \max(M\|\varphi\|, \|\varphi\||) + b\overline{M}M\overline{M_1}\left[\||y_1\| + M\|\phi\||\right].$$

Let z be solutions of the inclusion $z \in \lambda P(z)$, for some $\lambda \in (0,1)$, then there exists $v \in S_{F,z}$ such that

$$\begin{aligned}
|z(t)| &\leq \int_0^t |U(t,s)||(Cu_z)(s)|ds + \int_0^t |U(t,s)||v(s)|ds \leq \\
&\leq M\int_0^t \|C\||u_z(s)|ds + M\int_0^t p(s)\psi_F(\|\bar{z}_s + x_s\|)ds \leq \\
&\leq b\overline{M}M\overline{M_1}[\|y_1\| + M\|\phi\|] + b\overline{M}M^2\overline{M_1}\int_0^t p(s)\psi_F(\|\bar{z}_s + x_s\|)ds] + \\
&+ M\int_0^t p(s)\psi_F(\|\bar{z}_s + x_s\|)ds \leq \\
&\leq b\overline{M}M\overline{M_1}[\|y_1\| + M\|\phi\|] + b\overline{M}M^2\overline{M_1}\int_0^t p(s)\psi_F(\|z(s)\| + \\
&+ \max(M\|\varphi\|, \|\varphi\|\|)ds] + M\int_0^t p(s)\psi_F(\|z(s)\| + \max(M\|\varphi\|, \|\varphi\|\|))ds \leq \\
&\leq \max\{b\overline{M}M^2\overline{M_1}, M\}\int_0^t p(s)\psi_F(\|z(s)\| + \max(M\|\varphi\|, \|\varphi\|\|))ds + \\
&+ b\overline{M}M\overline{M_1}[\|y_1\| + M\|\phi\|].
\end{aligned}$$

We consider the function μ defined by

$$\mu(t) = \sup\{\|z(s)\| + \max(M\|\varphi\|, \|\varphi\|\|) : 0 \leq s \leq t\}, \quad 0 \leq t \leq b.$$

Let $t^* \in [0,t]$ be such that $\mu(t) = |y(t^*)|$. Then

$$\begin{aligned}
\mu(t) &\leq \max\{b\overline{M}M^2\overline{M_1}, M\}\int_0^t p(s)\psi_F(\mu(s)|)ds + \\
&+ b\overline{M}M\overline{M_1}[\|y_1\| + M\|\phi\|] + \max(M\|\varphi\|, \|\varphi\|\|).
\end{aligned}$$

Let us take the right-hand side of the above inequality as $v(t)$. Then we have

$$c = v(0) = \max(M\|\varphi\|, \|\varphi\|\|) + b\overline{M}M\overline{M_1}[\|y_1\| + M\|\phi\|], \mu(t) \leq v(t), t \in [0,b],$$

and

$$v'(t) = \max\{b\overline{M}M^2\overline{M}_1, M\}p(t)\psi_F(\mu(t)), \ \text{a.e. } t \in [0,b].$$

Using the nondecreasing character of ψ_F we get

$$v'(t) \leq \max\{b\overline{M}M^2\overline{M}_1, M\}p(t)\psi_F(v(t)) \quad \text{a.e.} \quad t \in [0,n].$$

This implies that for each $t \in [0,b]$

$$\int_c^{v(t)} \frac{ds}{\psi_F(s)} \leq \max\{b\overline{M}M^2\overline{M}_1, M\}\int_0^t p(s)ds < \int_c^\infty \frac{ds}{\psi_F(s)}.$$

Thus from $(H3)^*$ there exists a constant K_* such that $v(t) \leq K_*$, $t \in [0,b]$, and hence $\mu(t) \leq K_*$, $t \in [0,b]$. Since for every $t \in [0,b]$, $\|z(t)\| \leq \mu(t)$, we have $\|z\|_\infty \leq K_*$. Set

$$U_* = \{z \in C([0,b],E) : \sup\{|z(t)| : 0 \leq t \leq b\} < K_* + 1\}.$$

As in Theorem 1, the operator $P : U_* \to \mathcal{P}(C([0,b],E))$ is completely continuous. From the choice of U_*, there is no $z \in \partial U_*$ such that $z \in \lambda P(z)$, for some $\lambda \in (0,1)$. As a consequence of the nonlinear alternative of Leray–Schauder type [16], we deduce that P has a fixed point z in U_*. Then the problem (1), (2) has at least one mild solution on $(-\infty, b]$.

In this part, by using Schauder's fixed point theorem [31] combined with a selection theorem due to Bressan and Colombo [10] for lower semicontinuous multivalued maps with decomposables values we establish the controlability of the problem (1), (2).

Let $\mathcal{A}$ be a subset of $[0,b] \times B$. $\mathcal{A}$ is $\mathcal{L} \otimes \mathcal{B}$ measurable if $\mathcal{A}$ belongs to the σ-algebra generated by all sets of the form $N \times D$ where N is Lebesgue measurable in $[0,b]$ and D is Borel measurable in B. A subset $\mathcal{A}$ of $L^1([0,b],E)$ is decomposable if for all $u, v \in \mathcal{A}$ and $N \subset L^1([0,b],E)$ measurable the function $u\chi_N + v\chi_{J-N} \in \mathcal{A}$, where χ stands for the characteristic function.

Let X a nonempty closed subset of E and $G : X \to \mathcal{P}(E)$ a multivalued operator with nonempty closed values. G is lower semi-continuous (l.s.c.) if the set $\{x \in X : G(x) \cap B \neq \emptyset\}$ is open for any open set B in E.

Definition 3. *Let Y be a separable metric space and let*

$$N : Y \to \mathcal{P}(L^1([0,b],E))$$

be a multivalued operator. We say that N has property (BC) if

1) N is lower semi-continuous (l.s.c.);

2) N has nonempty closed and decomposable values.

Let $F : [0,b] \times B \to \mathcal{P}(E)$ be a multivalued map with nonempty compact values. Assign to F the multivalued operator

$$\mathcal{F} : C((-\infty,b],E) \to \mathcal{P}(L^1([0,b],E))$$

by letting

$$\mathcal{F}(y) = \{v \in L^1([0,b],E) : v(t) \in F(t,y_t) \text{ for a.e. } t \in [0,b]\}.$$

The operator $\mathcal{F}$ is called the Niemytzki operator associated to F.

Definition 4. *Let $F : [0,b] \times B \to \mathcal{P}(E)$ be a multivalued function with nonempty compact values. We say that F is of lower semi-continuous type (l.s.c. type) if its associated Niemytzki operator $\mathcal{F}$ is lower semi-continous and has nonempty closed and decomposable values.*

Next we state a selection theorem due to Bressan and Colombo.

Theorem 2. [10] *Let Y be separable metric space and let $N : Y \to \mathcal{P}(L^1([0,b],E))$ be a multivalued operator which has property (BC). Then N has a continuous selection, i.e. there exists a continuous function (single-valued) $f : Y \to L^1([0,b],E)$ such that $f(x) \in N(x)$ for every $x \in Y$.*

Let us introduce the following hypotheses which are assumed hereafter:

(A1) $F : [0,b] \times B \to \mathcal{P}(E)$ is a nonempty compact valued multivalued map such that:
a) $(t,x) \mapsto F(t,x)$ is $\mathcal{L} \otimes \mathcal{B}$ measurable;
b) $x \mapsto F(t,x)$ is lower semi-continuous for a.e. $t \in [0,b]$.

The following lemma is crucial in the proof of our main theorem:

Lemma 3. [18]. *Let $F : [0,b] \times B \to \mathcal{P}(E)$ be a multivalued map with nonempty, compact values. Assume* (A1) *and* (H2) *hold. Then F is of l.s.c. type.*

Theorem 3. *Suppose that hypotheses* (H1), (H3) *and* (A1), *hold. Then the problem* (1), (2) *has at least one solution.*

Proof. (H3) and (A1) imply by Lemma 3 that F is of lower semi-continuous type. Then from Theorem 2 there exists a continuous function

$$f : C((-\infty,b],E) \to L^1([0,b],E)$$

such that $f(y) \in \mathcal{F}(y)$ for all $y \in C((-\infty,b],E)$. Consider the following problem

$$y'(t) = A(t)y(t) + Cu(t) + f(y_t), \quad t \in [0,b], \tag{7}$$

$$y(t) = \varphi(t), \quad t \in (-\infty,0]. \tag{8}$$

We consider the operator $N : C((-\infty,b],E) \to C((-\infty,b],E)$ defined by:

$$(N_1 y)(t) = \begin{cases} \varphi(t), & \text{if } t \in (-\infty,0]; \\ U(t,0)\varphi(0) + \displaystyle\int_0^t U(t,s)f(y_s)ds + & \\ + \displaystyle\int_0^t U(t,s)Cu_y(s)ds, & \text{if } t \in [0,b]. \end{cases}$$

As in Theorem 1, let the operator $P_1 : C(([0,b],E) \to C([0,b],E)$ defined by:

$$(P_1 z)(t) = \int_0^t U(t,s)f(\bar{z}_s + y_s)ds + \int_0^t U(t,s)Cu_z(s)ds, \quad t \in [0,b],$$

where u is the control defined in Theorem 1.

Remark 3. *If $y \in C((-\infty,b],E)$ is a solution of the problem* (7), (8), *then y is a solution to the problem* (1), (2).

It is clear that $P_1 : B_{k_0} \to B_{k_0}$ is continuous and completely continuous. As a consequence of the theorem of Schauder [31] we deduce that P_1 has a fixed point z in B_{k_0}, then the operator N has a fixed point y which is a mild solution of the problem (1), (2).

We present, now, a second result for the problem (1), (2) with a non-convex valued right hand side.

Let (X,d) be a metric space induced from the normed space $(X,|\cdot|)$. Consider $H_d : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R}_+ \cup \{\infty\}$, given by

$$H_d(\mathcal{A},\mathcal{B}) = \max\left\{\sup_{a\in\mathcal{A}} d(a,\mathcal{B}), \sup_{b\in\mathcal{B}} d(\mathcal{A},b)\right\},$$

where $d(\mathcal{A},b) = \inf_{a\in\mathcal{A}} d(a,b)$, $d(a,\mathcal{B}) = \inf_{b\in\mathcal{B}} d(a,b)$. Then $(\mathcal{P}_{b,cl}(X), H_d)$ is a metric space and $(\mathcal{P}_{cl}(X), H_d)$ is a generalized (complete) metric space (see [25]).

Definition 5. *A multivalued operator $G : X \to \mathcal{P}_{cl}(X)$ is called*

a) γ-Lipschitz if and only if there exists $\gamma > 0$ such that

$$H_d(G(x),G(y)) \le \gamma d(x,y) \quad \textit{for each } x, \ y \in X,$$

b) a contraction if and only if it is γ-Lipschitz with $\gamma < 1$.

Our considerations are based on the following fixed point theorem for contraction multivalued operators given by Covitz and Nadler in 1970 [13] (see also Deimling, [15] Theorem 11.1).

Lemma 4. *Let (X,d) be a complete metric space. If $G : X \to \mathcal{P}_{cl}(X)$ is a contraction then $FixG \neq \emptyset$.*

Let us introduce the following hypotheses:

(A2) $F : [0,b] \times B \to \mathcal{P}_{cp}(E)$; $(t,x) \mapsto F(t,x)$ is measurable for each $x \in B$.

(A3) There exists a function $l \in L^1([0,b], \mathbb{R}^+)$ such that

$$H_d(F(t,x), F(t,\overline{x})) \leq l(t)\|x - \overline{x}\| \text{ for a.e. } t \in [0,b] \text{ and all } x, \overline{x} \in B,$$

and $d(0, F(t,0)) \leq l(t)$ $hboxfora.e.$ $t \in [0,b]$.

Theorem 4. *Suppose that hypotheses $(A2)-(A3)$ are satisfied. Then the IVP (1), (2) has at least one mild solution.*

Remark 4. *For each $z \in C([0,b], E)$ the set $S_{F,z}$ is nonempty since by (A2), F has a measurable selection (see [12], Theorem III.6).*

Proof. Let $P : C([0,b], E) \to \mathcal{P}(C([0,b], E))$ where P is defined in Theorem 1 are solutions of the problem (1), (2). We shall show that P satisfies the assumptions of Lemma 4. The proof will be given in two steps.

Step 1: $P(z) \in \mathcal{P}_{cl}(C([0,b], E))$ for each $z \in C([0,b], E)$.

Indeed, let $(z_n)_{n \geq 0} \in P(z)$ be such that $z_n \to \tilde{z}$ in $C([0,b], E)$. Then $\tilde{z} \in C([0,b], E)$ and there exists $v_n \in S_{F,z}$ such that, for each $t \in [0,b]$,

$$z_n(t) = \int_0^t U(t,s)v_n(s)ds + \int_0^t U(t,s)(Cu_z)(s)ds.$$

Using the fact that F has compact values and from (A2), (A3), we may pass to a subsequence if necessary to get that v_n converges to v in $L^1([0,b], E)$ and hence $v \in S_{F,z}$. Then, for each $t \in [0,b]$,

$$z_n(t) \to \tilde{z}(t) = \int_0^t U(t,s)v(s)ds + \int_0^t U(t,s)(Cu_z)(s)ds.$$

So, $\tilde{z} \in P(z)$.

Step 2: There exists $\gamma < 1$ such that

$$H_d(P(z), P(z_*)) \leq \gamma\|z - z_*\|_\infty \text{ for each } z, z_* \in C([0,b], E).$$

Let $z, z_* \in C([0,b], E)$ and $h \in P(z)$. Then there exists $v(t) \in F(t, \bar{z}_t + x_t)$ such that, for each $t \in [0,b]$,

$$h(t) = \int_0^t U(t,s)v(s)ds \int_0^t U(t,s)(Cu_z)(s)ds.$$

From (A3), it follows that $H_d(F(t, \bar{z}_t + x_t), F(t, \bar{z}_{*_t} + x_t)) \leq l(t)\|z(t) - z_*(t)\|$. Hence, there is $w \in F(t, \bar{z}_{*_t} + x_t)$ such that $|v(t) - w| \leq l(t)\|z(t) - z_*(t)\|$, $t \in [0,b]$. Consider $U : [0,b] \to \mathcal{P}(E)$, given by

$$U(t) = \{w \in E : |v(t) - w| \leq l(t)\|z(t) - z_*(t)\|\}.$$

Since the multivalued operator $V(t) = U(t) \cap F(t, \bar{z}_{*t} + x_t)$ is measurable (see Proposition III.4 in [12]), there exists a function $t \to \overline{v}(t)$, which is a measurable selection for V. So, $\overline{v}(t) \in F(t, \bar{z}_{*t} + x_t)$ and

$$|v(t) - \overline{v}(t)| \leq l(t)\|z_(t) - z_*(t)\| \quad \text{for each} \quad t \in [0, b].$$

Let us define, for each $t \in [0, b]$,

$$\overline{h}(t) = \int_0^t U(t,s)\overline{v}(s)ds + \int_0^t U(t,s)(Cu_{z_*}(s)ds.$$

Then we have

$$\begin{aligned}
|h(t) \ - \ \overline{h}(t)| =& \\
= &\left|\int_0^t U(t,s)[(Cu_z)(s) - (Cu_{z_*})(s)]ds + \int_0^t U(t,s)[v(s) - \overline{v}(s)]ds\right| \leq \\
\leq & \ M\int_0^t \|C\||u_z(s) - u_{z_*}(s)|ds + \int_0^t l(s)M\|z(s) - z_*(s)\|ds \leq \\
\leq & \ M\int_0^t l(s)|z(s) - z_*(s)|ds + \\
+ & \ M\overline{M}\int_0^t |W^{-1}|\left[\int_0^b |U(b,s)||v(\omega) - \overline{v}(\omega)d\omega\right] ds \leq \\
\leq & \ \overline{M}M^2\overline{M}_1\int_0^t \left[\int_0^b |v(\omega) - \overline{v}(\omega))|d\omega\right] ds + \\
+ & \ M\int_0^t l(s)\|z(s) - z_*(s)\|ds \leq \\
\leq & \ \overline{M}M^2\overline{M}_1 b\int_0^t l(s)\|z(s) - z_*(s)\|ds + M\int_0^t l(s)\|z(s) - z_*(s)\|ds \leq \\
\leq & \int_0^t \overline{l}(s)\|z(s) - z_*(s)\|ds = \frac{1}{\tau}\int_0^t (e^{\tau L(s)})'ds\|z - z_*\|_{\overline{B}} \leq \\
\leq & \ \ \frac{1}{\tau}e^{\tau L(t)}\|z - z_*\|_{\overline{B}},
\end{aligned}$$

where $\tau > 1$, $L(t) = \int_0^t \overline{l}(s)ds$, $\overline{l}(t) = \max(Ml(t), \overline{M}M^2\overline{M}_1 bl(t)$ and $\|\cdot\|_{\overline{B}}$ is the Bielecki-type norm on $C([0, b], E)$ defined by

$$\|z\|_{B_1} = \max_{t\in[0,b]}\{\|z(t)\|e^{-\tau L(t)}\}.$$

Therefore, $\|h - \overline{h}\|_{\overline{B}} \leq \frac{1}{\tau}\|z - z_*\|_{\overline{B}}$. By an analogous relation, obtained by interchanging the roles of y and $\overline{y}$, it follows that $H_d(P(z), P(z_*)) \leq \frac{1}{\tau}\|z - z_*\|_{\overline{B}}$. So, P is a contraction, and thus, by Lemma 4 it has a fixed point z, which is a mild solution to (1), (2).

4. NEUTRAL FUNCTIONAL DIFFERENTIAL INCLUSIONS

Let us introduce the following hypotheses which are assumed here after:

(A4) There exist $L_1 > 0$ such that

$$\|A(t)g(s,\varphi)\| \leq L_1(\|\varphi\|_B + 1), \ 0 \leq t, s \leq b, \ \varphi \in B.$$

(A5) The function g is completely continuous and for any bounded set $Q \subseteq C((-\infty, b], E)$ the set $\{t \to g(t, x_t) : x \in Q\}$ is equicontinuous in $C([0,b], E)$.

(A6) There is $M_0 > 0$ such that $\|A^{-1}(t)\| \leq M_0$, for all $0 \leq t \leq b$.

(A7) There exist $L_* > 0$ such that

$$\|A(t)g(s_1,\varphi) - A(t)g(s_2,\overline{\varphi})\| \leq L_*(|s_1 - s_2| + \|\varphi - \overline{\varphi}\|)$$

for $0 \leq t, s_1, s_2 \leq b, \varphi, \overline{\varphi} \in B$.

Theorem 5. *Assume that hypothesis* (H1)–(H4) *and* (A4)–(A7) *hold. If*

$$\begin{aligned} &L_0 = L_* K_b (M_0 + bM) < 1, \\ &(1 + bM\overline{M}M_1)\,(L_1 M_0 K_b + bML_1K_b + MK_b\gamma) \leq 1, \end{aligned} \tag{9}$$

where $K_b = sup\{K(t) : 0 \leq t \leq b\}$. Then the problem (3), (4) *is controllable on* $(-\infty, b]$.

Proof. Consider the operator $\overline{N}_1 : C((-\infty, b], E) \to \mathcal{P}(C((-\infty, b], E))$ defined by:

$$\overline{N}_1(y) = \left\{ \ h \in C((-\infty, b], E) : \right.$$

$$h(t) = \left\{ \begin{array}{ll} \varphi(t), & \text{if } t \in (-\infty, 0]; \\ U(t,0)[\varphi(0) + g(0,\varphi)] - g(t, y_t) + & \\ + \displaystyle\int_0^t U(t,s)A(s)g(s,y_s)ds + & \\ + \displaystyle\int_0^t U(t,s)[Cu(s) + v(s)]ds, & \text{if } t \in [0,b], \end{array} \right\}$$

where $v \in S_{F,y}$. Using the assumption (H1), for arbitrary function $y(\cdot)$ define the control

$$u(t) = W^{-1}\Big[y_1 - U(b,0)(\varphi(0) + g(0,\varphi)) + g(b, y_b) -$$

$$-\int_0^b U(b,s)A(s)g(s,y_s)ds - \int_0^b U(b,s)v(s)ds\Big](t).$$

It shall be show that when using this control the operator $\overline{N}_1$ has a fixed point $y(\cdot)$. Then $y(\cdot)$ is a mild solution of system (3), (4). By analogue of Theorem 1 we consider the operator $\overline{P}_1 : C([0,b],E) \to \mathcal{P}(C([0,b],E))$ defined by

$$\overline{P}_1(z) = \{h \in C([0,b],E) : h(t) = U(t,0)g(0,\varphi) - g(t,\overline{z}_t + x_t) +$$

$$+ \int_0^t U(t,s)A(s)g(s,\overline{z}_s + x_s)ds + \int_0^t U(t,s)(Cu(s)ds + v(s))ds, v \in S_{F,z\cdot}\}$$

As in Theorem 1 we can show that there exists $k_0 > 0$ such that $\overline{P}_1(B_{k_0}) \subset B_{k_0}$ and $\overline{P}_1 : B_{k_0} \to \mathcal{P}_{cl,c}(B_{k_0})$ is completely continuous. As a consequence the fixed point theorem of Bohnenblust–Karlin we deduce that $\overline{P}_1$ has a fixed point z in B_{k_0}, then the problem (3), (4) has at least mild one solution.

Now we consider nonconvex version of the problem (3), (4).

Theorem 6. *Assume that the hypotheses* (H1), (H3), (H4), (A1), (A4), (A5) *and the condition* (9) *are satisfied then the problem* (3), (4) *has at least mild one solution.*

Proof. (A1) and (H3) imply by Lemma 1 that F is of lower semi-continuous type. Then from Theorem 2 there exists a continuous function

$$f : C((-\infty,b],E) \to L^1([0,b],E)$$

such that $f(y) \in \mathcal{F}(y)$ for all $y \in C((-\infty,b],E)$. Consider the following problem

$$\frac{d}{dt}[y(t) - g(t,y_t)] = f(y_t), \quad t \in [0,b], \tag{10}$$

$$y(t) = \varphi(t) \in B. \tag{11}$$

Consider the operator $\overline{N}_* : C((-\infty,b],E) \to C((-\infty,b],E))$ defined by:

$$(\overline{N}_* y)(t) = \begin{cases} \varphi(t), & \text{if } t \in (-\infty,0]; \\ U(t,0)[\varphi(0) + g(0,\varphi)] - g(t,y_t) + & \\ + \int_0^t U(t,s)A(s)g(s,y_s)ds + & \\ + \int_0^t U(t,s)[Cu(s) + f(y_s)]ds, & \text{if } t \in [0,b]. \end{cases}$$

Let $\overline{P}_* : C([0,b],E) \to C([0,b],E))$ defined by

$$(\overline{P}_* z)(t) = U(t,0)g(0,\varphi) - g(t,\overline{z}_t + x_t) + \int_0^t U(t,s)A(s)g(s,\overline{z}_s + x_s)ds + \\ + \int_0^t U(t,s)Cu_z(s)ds + \int_0^t U(t,s)f(\overline{z}_s + x_t)ds.$$

By Schauder's fixed point we can show that $\overline{P}_*$ has at least one fixed point, this implies that $\overline{N}_*$ has fixed point which is mild solution of the problem (3), (4).

Now we present a result for the problem (3), (4) by using fixed point theorem for contraction multivalued operators given by Covitz and Nadler.

Theorem 7. *Suppose that hypotheses* (A2)–(A3), (A7) *and the condition*

(A8) There exists $L > 0$ *such that*

$$\|A(t)g(s,\varphi) - A(t)g(s,\overline{\varphi})\| \le L\|\varphi - \overline{\varphi}\| \quad \textit{for} \quad 0 \le t,s \le b, \ \varphi, \overline{\varphi} \in B,$$

are satisfied. If $\tau^{-1} + L < 1$, *then the the problem* (3), (4) *has at least one mild solution.*

Proof. Let $\overline{P}_1 : C([0,b],E) \to \mathcal{P}(C([0,b],E))$ where $\overline{P}_1$ is defined in Theorem 5. We shall show that $\overline{P}_1$ satisfies the assumptions of Lemma 4 will be given in two steps. As in Theorem 4, we show that for each $z \in C([0,b],E)$, $\overline{P}_1(z) \in \mathcal{P}_{cl}(C([0,b],E))$. Now we prove that there exist $\gamma < 1$ such that

$$H_d(\overline{P}_1(z), \overline{P}_1(z_*)) \le \gamma\|z - z_*\|_\infty \ \text{ for each } \ z, z_* \in C([0,b],E).$$

Let $z, z_* \in C([0,b],E)$ and $h \in \overline{P}_1(z)$. Then there exists $v(t) \in F(t, \bar{z}_t + x_t)$ such that, for each $t \in [0,b]$,

$$h(t) = U(t,0)g(0,\varphi) - g(t,\overline{z}_t + x_t) + \int_0^t U(t,s)A(s)g(s,\overline{z}_s + x_s)ds + \\ + \int_0^t U(t,s)Cu_z(s)ds + \int_0^t U(t,s)v(s)ds.$$

From (A3), it follows that

$$H_d(F(t,\bar{z}_t + x_t), F(t,\bar{z}_{*_t} + x_t) \le l(t)\|z(t) - z_*(t)\|.$$

Hence, there is $w \in F(t, \bar{z}_{*_t} + x_t)$ such that

$$|v(t) - w| \le l(t)\|z(t) - z_*(t)\|, \quad t \in [0,b].$$

Consider $U : [0,b] \to \mathcal{P}(E)$, given by

$$U(t) = \{w \in E : |v(t) - w| \leq l(t)\|z(t) - z_*(t)\|\}.$$

Since the multivalued operator $V(t) = U(t) \cap F(t, \overline{z}_{*t} + x_t)$ is measurable (see Proposition III.4 in [12]), there exists a function $t \to \overline{v}(t)$, which is a measurable selection for V. So, $\overline{v}(t) \in F(t, \overline{z}_{*t} + x_t)$ and

$$|v(t) - \overline{v}(t)| \leq l(t)\|z_(t) - z_*(t)\| \text{ for each } t \in [0,b].$$

Let us define, for each $t \in [0,b]$,

$$\overline{h}(t) = U(t,0)g(0,\varphi) - g(t, \overline{z}_t + x_t) + \int_0^t U(t,s)A(s)g(s, \overline{z}_s + x_s)ds +$$
$$+ \int_0^t U(t,s)Cu_z(s)ds + \int_0^t U(t,s)\overline{v}(s)ds.$$

Then we have

$$\begin{aligned}
|h(t) - \overline{h}(t)| &= \Big| A^{-1}(t)[A(t)g(t, \overline{z}_t + x_t) - A(t)g(t, \overline{z}_{*t} + x_t)] + \\
&+ \int_0^t U(t,s)[A(s)g(s, \overline{z}_s + x_s) - A(s)g(s, \overline{z}_{*s} + x_s)]ds + \\
&+ \int_0^t U(t,s)[(Cu_z)(s) - (Cu_{z_*})(s)]ds + \int_0^t U(t,s)[v(s) - \overline{v}(s)]ds \Big| \leq \\
&\leq LM_0\|z(t) - \overline{z}_*(t)\| + \int_0^t ML\|\overline{z}(s) - \overline{z}_*(s)\|ds + \\
&+ M \int_0^t \|C\||u_z(s) - u_{z_*}(s)|ds + \int_0^t l(s)M\|z(s) - z_*(s)\|ds \leq \\
&\leq L\|z(t) - \overline{z}_*(t)\| + \int_0^t MLM_0\|\overline{z}(s) - \overline{z}_*(s)\|ds + \\
&+ M \int_0^t l(s)\|z(s) - z_*(s)\|ds + M\overline{M} \int_0^t |W^{-1}|L\|z(s) - z_*(s)\|ds + \\
&+ M^2\overline{M}bL \int_0^t |W^{-1}|\|z(s) - z_*(s)\|ds + \\
&+ M\overline{M} \int_0^t |W^{-1}| \left[\int_0^b |U(b,s)||v(\omega) - \overline{v}(\omega)d\omega \right] ds \leq \\
&\leq LM_0\|z(t) - \overline{z}_*(t)\| + \int_0^t MLM_0\|\overline{z}(s) - \overline{z}_*(s)\|ds + \\
&+ M \int_0^t l(s)\|z(s) - z_*(s)\|ds + \overline{M}_1 M\overline{M}L \int_0^t \|z(s) - z_*(s)\|ds + \\
&+ \overline{M}_1 M^2\overline{M}bL \int_0^t \|z(s) - z_*(s)\|ds +
\end{aligned}$$

$$+ \overline{M}_1 M^2 \overline{M} b \int_0^t l(s)\|z(s) - z_*(s)\|ds \leq$$
$$\leq \left(L + \frac{1}{\tau}\right) \int_0^t (e^{\tau L_*(s)})' ds \|z - z_*\|_{\overline{B}} \leq \left(L + \frac{1}{\tau}\right) e^{\tau L_*(t)} \|z - z_*\|_{\overline{B}},$$

where $L_*(t) = \int_0^t l_*(s)ds$,

$$l_*(t) = \max\{Ml(t), \overline{M}M^2\overline{M}_1 bl(t), MLM_0, \overline{M}_1 M\overline{M}LM_0, \overline{M}_1 M^2 \overline{M} bL\}$$

and $\|\cdot\|_{\overline{B}}$ is the Bielecki-type norm on $C([0,b],E)$ defined by

$$\|z\|_{\overline{B}} = \max_{t\in[0,b]} \{\|z(t)\| e^{-\tau L_*(t)}\}.$$

Therefore,

$$\|h - \overline{h}\|_{\overline{B}} \leq \left(L + \frac{1}{\tau}\right) \|z - z_*\|_{\overline{B}}.$$

By an analogous relation, obtained by interchanging the roles of y and $\overline{y}$, it follows that

$$H_d(\overline{P}_1(z), \overline{P}_1(z_*)) \leq \left(L + \frac{1}{\tau}\right) \|z - z_*\|_{\overline{B}}.$$

So, $\overline{P}_1$ is a contraction, and thus, by Lemma 4 it has a fixed point z, which is a mild solution to (3), (4).

Remark 5. *Assume that* (H1), (H2), (H3)*, (H4), (A5)–(A7) *are satisfied, then a slight modification of the proof and use the usual Leray–Schauder alternative guarantees that* (A4) *could be replaced by*

(A4)* *There exist a continuous non-decreasing function* $\psi_g : [0,\infty) \to (0,\infty)$, *a function* $q \in L^1([0,b], \mathbb{R}_+)$ *such that*

$$\|A(t)g(t,x)\| \leq q(t)\psi_g(\|x\|_B) \text{ for } t \in [0,b], \quad x \in B.$$

Then the problem (3), (4) *has at least one mild solution on* $(-\infty, b]$.

5. AN EXAMPLE

As an application of our results we consider the following partial neutral functional differential inclusions of the form

$$\frac{\partial}{\partial t}\left[z(t,x) + \int_{-\infty}^t \int_0^\pi b(s-t,y,x)z(s,y)dyds\right] - a(t,x)\frac{\partial^2 z(t,x)}{\partial x^2} \in \tag{12}$$
$$Q(t, z(t-r,x), z_x(t-r,x))) + d(x)u(t), \quad 0 \leq x \leq \pi, \quad t \in [0,b],$$

$$\begin{aligned} z(t,0) = z(t,\pi) = 0, \quad & t \in [0,b] \\ z(t,x) = \phi(t,x), \quad & t \leq 0, \quad 0 \leq x \leq \pi, \end{aligned} \tag{13}$$

where $a(t,x)$ is a continuous function and is uniformly Hölder continuous in t. Let

$$g(t,w_t)(x) = \int_{-\infty}^{t} \int_{-\infty}^{\pi} b(s-t,y,x)z(s,y)dyds, \quad 0 \leq x \leq \pi,$$

and

$$F(t,w_t)(x) = Q\left(t, w(t-x), \frac{\partial}{\partial x} w(t-x)\right), \quad 0 \leq x \leq \pi.$$

Take $E = L^2[0,\pi]$ and $A(t)$ defined by $A(t)w = -a(t,x)w''$ with domain $D(A) = \{w \in E, w, w'$ are absolutely continuous, $w'' \in E, w(0) = w(\pi) = 0\}$. Then $A(t)$ generates an evolution system $U(t,s)$ satisfying assumptions $(B_1) - (B_4)$ (see [17]). Here we take the phase space $B = C_0 \times L^2(f;E)$, which contains all classes of function $\varphi : (-\infty, 0] \to E$ such that φ is Lebesgue measurable and $f\|\varphi\|^2$ is Lebesgue integrable on $(-\infty, 0]$ where $f : (-\infty, 0) \to \mathbb{R}$ is a positive integrable function. The norm in B is defined by

$$\|\varphi\| = \|\phi(0)\| + \left(\int_{-\infty}^{0} f(s)\|\varphi(s)\|^2 ds\right)^{1/2}.$$

The general case of phase space $B_r \times L^p(f;E)$, $r \geq 0$, $1 \leq p < \infty$, has been discussed in (here we only let $(r = 0, p = 2)$. From [23], under some conditions B is a phase space verifying axioms $(A_1) - (A_3)$, $(B_1) - (B_4)$ and in this case $K(t) = 1 + \left(\int_{-\infty}^{0} f(s)ds\right)^{1/2}$ (see [23]).

(i) The function b is measurable and

$$\int_0^{\pi} \int_{-\infty}^{t} \int_0^{\pi} \frac{b^2(s,y,x)}{f(s)} dsdydx < \infty.$$

(ii) The function $\frac{\partial}{\partial x} b(s,y,x)$ and $\frac{\partial^2}{\partial x^2} b(s,y,x)$ are measurable, $b(s,y,0) = b(s,y,\pi)) = 0$ and $\sup_{t\in[0,b]} N(t) < \infty$, where

$$N(t) = \int_0^{\pi} \int_{-\infty}^{t} \int_0^{\pi} \frac{1}{f(s)} \left(a(s,x) \frac{\partial^2}{\partial x^2} b(s,y,x)\right)^2 dsdydx.$$

Finally let $C \in L(\mathbb{R}, E)$ be defined as

$$(Cu)(x) = d(x)u, \quad 0 \leq x \leq \pi, \quad u \in \mathbb{R}, \quad d(x) \in E.$$

Thus, under the above definitions of F, g, $A(\cdot)$ and C, system (12), (13) can be represented by the abstract Cauchy problem (3), (4). Furthermore, more appropriate conditions on b, Q, d ensure the controllability of system (12), (13) by Theorems 5, 6, 7.

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ТЕОРЕМИ КЕРУВАННЯ ДЛЯ НАПІВЛІНІЙНИХ ФУНКЦІОНАЛЬНИХ ДИФЕРЕНЦІАЛЬНИХ ВКЛЮЧЕНЬ З НЕОБМЕЖЕНИМ ЗАГАЮВАННЯМ

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На основі результатів про півгрупи операторів еволюції та теорем про нерухому точку у роботі встановлено твердження про керування для напівлінійних та нейтральних функціональних диференціальних включень у банаховому просторі з необмеженим загаюванням.