ON RINGS WITH CENTRAL INNER DERIVATIONS

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We establish properties of rings with central inner derivations.

Let R be an associative ring. As usual, a map $\delta : R \to R$ is called a derivation of R if

$$\delta(a+b) = \delta(a) + \delta(b) \text{ and } \delta(ab) = \delta(a)b + a\delta(b)$$

for all $a, b \in R$. The set of all derivations of R is denoted by Der$R$. The map $\partial_x : R \to R$ defined by the rule

$$\partial_x(r) = xr - rx \ (r \in R)$$

is a derivation, which is called an inner derivation of R generated by an element $x \in R$. An inner derivation ∂_x will be called central if $\partial_x d = d\partial_x$ for each $d \in$ Der$R$.

In this note we study the properties of rings in which all inner derivations are central.

For a ring R, let $[R, R]$ stand for the two-sided ideal generated by all $\partial_x(y)$, where $x, y \in R$. Also let $Z(R)$, and $N(R)$ denote the center, and the set of all nilpotent elements of R, respectively. Any unexplained terminology is standard as in [5] and [2].

1. In this section we study the basic properties of rings with central inner derivations.

Lemma 1. *All inner derivations of a ring R are central if and only if $d(a) \in Z(R)$ for any $a \in R$ and $d \in Der\, R$.*

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Proof. Let a and b be the elements of R and $d \in \mathrm{Der}R$.

($\Rightarrow$) Since $ad(b) - d(b)a = \partial_a(d(b)) = (\partial_a d)(b) = (d\partial_a)(b) = d(ab - ba) = d(a)b + ad(b) - d(b)a - bd(a)$, we deduce that $d(a)b - bd(a) = 0$. This means that $d(a) \in Z(R)$.

($\Leftarrow$) If $d(a) \in Z(R)$, then $\partial_b(d(a)) = 0$ and therefore $d(\partial_b(a)) = d(ba - ab) = [d(b)a - ad(b)] + [bd(a) - d(a)b] = 0$. As a consequence, $\partial_b d = 0 = d\partial_b$.

Lemma 2. *Let R be a ring with all inner derivations central. Then*

$$a^k b - ba^k = ka^{k-1}(ab - ba)$$

for any $a, b \in R$ and an integer $k \geq 1$.

Proof. We go by induction on k. In view of Lemma 1 for $k = 2$ we have $a^2b - ba^2 = a(ab - ba) + (ab - ba)a = 2a(ab - ba)$. Now assuming that

$$a^{k-1}b - ba^{k-1} = (k-1)a^{k-2}(ab - ba),$$

we obtain that $a^k b - ba^k = a(a^{k-1}b - ba^{k-1}) + (a^{k-1}b - ba^{k-1})a + aba^{k-1} - a^{k-1}ba = 2a(a^{k-1}b - ba^{k-1}) - a(a^{k-2}b - ba^{k-2})a = 2(k-1)a^{k-1}(ab - ba) - (k-2)a^{k-2}(ab - ba)a = a^{k-1}(ab - ba)(2k - 2 - k + 2) = ka^{k-1}(ab - ba)$, as required.

Lemma 3. *Let R be a ring with all inner derivations central, $a, x, y, r \in R$ and $d \in DerR$. Then the following statements hold:*

1) $\partial_x d = 0 = d\partial_x$;

2) $d(\partial_x(a)r) = \partial_x(ad(r)) = \partial_x(a)d(r) = -d(a)\partial_x(r)$ *and, in particular,* $[R, R]$ *is a d-ideal;*

3) $d(a)\partial_a(y) = 0$;

4) $ad(a) \in Z(R)$;

5) $\partial_x(y) \in Z(R)$;

6) $\partial_x(y)^2 = 0$;

7) *if* $c \in Z(R)$, *then* $d(c) \in Ann[R, R]$.

Proof. 1) In view of Lemma 1 $\partial_x(d(a)) = xd(a) - d(a)x = 0$ and $d(\partial_x(a)) = [d(x)a - ad(x)] + [xd(a) - d(a)x] = 0$ and so $\partial_x d = 0 = d\partial_x$.

2) Of course, we have $d(\partial_x(a)r) = (d\partial_x)(a)r + \partial_x(a)d(r) = \partial_x(a)d(r) = \partial_x(ad(r)) = \partial_x(d(ar) - d(a)r) = -\partial_x(d(a)r) = -(\partial_x d)(a)r - d(a)\partial_x(r) = -d(a)\partial_x(r)$.

3) Since $0 = (d\partial_x)(ay) = d(\partial(a)y + a\partial_x(y)) = (d\partial_x)(a)y + \partial_x(a)d(y) + d(a)\partial_x(y) + a(d\partial_x)(y) = \partial_x(a)d(y) + d(a)\partial_x(y)$, we obtain for $x = a$ that

$$d(a)\partial_a(y) = 0.$$

4) follows from 3).

5) Clearly that by the property 1) $\partial_a(\partial_x(y)) = (\partial_a\partial_x)(y) = 0$. This gives that $\partial_x(y) \in Z(R)$.

6) follows from the equality $\partial_x(y)^2 = \partial_x(y\partial_x(y))$ and the property 4).

7) Inasmuch as $c\partial_a(y) \in Z(R)$, we have $c\partial_a(xy) = (c\partial_a)(x)y + x(c\partial_a)(y)$ and therefore $c\partial_a \in \mathrm{Der}R$. Moreover, the derivation $c\partial_a$ is inner and so $d(c\partial_a) = (c\partial_a)d$. This implies that $0 = ((c\partial_a)d)(x) = (d(c\partial_a))(x) = d(c(ax - xa)) = d(c)(ax - xa) + cd(ax - xa) = d(c)(ax - xa)$ and $d(c) \in \mathrm{Ann}[R, R]$.

Recall that a ring R is said to be J-semisimple if its Jacobson radical $J(R)$ is zero.

Theorem 1. *Let R be a reduced (respectively semiprime or J-semisimple) ring with an identity element. Then all inner derivations of R are central if and only if R is commutative.*

Proof. ($\Leftarrow$) is immediate.

($\Rightarrow$) If $\mathrm{char}R = n$ for some positive integer n, then by Lemma 2 $x^n \in Z(R)$ for any $x \in R$ and so by Theorem from [3] $[R, R] \subseteq N(R)$. If $\mathrm{char}R = 0$, then by Lemma 3 $\partial_x(y) \in N(R)$ for all $x, y \in R$. Since $N(R) = \{0\}$, we deduce that a ring R is commutative.

A ring R is said to be strongly 2-primal if $P(R/I) = N(R/I)$ for every proper ideal I of R, where $P(R)$ is the prime radical of R.

Lemma 4. *If all inner derivations of a ring R are central, then R is strongly 2-primal.*

Proof. Let P be a prime ideal of R. Then by Lemmas 1 and 3 $[R, R] \subseteq P$ and consequently R/P is a commutative domain. Hence P is a completely prime ideal. By Proposition 1.11 of [6] and Proposition 1.2 of [4] R is a strongly 2-primal ring.

From Lemma 4 it follows that $[R, R]$ is a nil ideal for any ring R with the central inner derivations.

2. In this section we investigate the right Artinian rings with the central inner derivations. Recall that a ring having no non-zero derivations will be called differentially trivial [1].

2.1. Let R be a $\frac{3}{4}$-perfect rings (that is a semilocal rings with the nil Jacobson radical $J(R)$) with the central inner derivations and $e = e^2 \in R$. Then $d(e) = 2ed(e)$ for any $d \in \mathrm{Der}R$ and so $ed(e) = 2e^2d(e)$. As a consequence, $d(e) = 0$. This means that each idempotent e of R is central. Hence R is a ring direct product of the local $\frac{3}{4}$-perfect rings.

2.2. If R is a local right Artinian ring with the central inner derivations and $J(R)^2 = \{0\}$, then its unit group $U(R)$ is nilpotent.

Indeed $\partial_a(r\partial_y(x)) = \partial_a(r)\partial_y(x) = 0$ for any elements $a, y, x, r \in R$. This gives that $[R, R] \subseteq Z(R)$.

If R° is the adjoint group of R, then $R^\circ/[R,R]^\circ \cong (R/[R,R])^\circ$ is an Abelian group and that R° is a nilpotent group. In view of a group isomorphism of $U(R)$ and R° it follows that $U(R)$ is nilpotent.

2.3. Let R be a local right Artinian ring with the central inner derivations.

2.3.*a*) Suppose that $\mathrm{char}(R/J(R)) = 0$ and $R/J(R)$ is a differentially trivial field. Then by Lemma 2 of [8] $[R, J(R)] \leq J(R)^2$ and by Corollary of Theorem 3 from [8] $R = C + J(R)$ for some subfield C of R such that $C \leq Z(R)$. If $J(R)^2 = \{0\}$, then R is a commutative ring.

2.3.*b*) Now suppose that $\mathrm{char}(R/J(R)) = p > 0$ and $R/J(R)$ is an algebraic over $\mathbb{Z}_p$. Then by Theorem 3.2 of [7] R contains unique subring S such that $S/pS \cong R/M$. By Theorem 2.2 of [7] S has a sequence $\{S_i\}_{i=1}^\infty$ of subrings S_i of S such that $S_i \subseteq S_{i+1}$, $S_i \cong GR(p^n, r_i)$ is a Galois ring ($i \geq 1$) and $S = \bigcup_{i=1}^\infty S_i$, where $\{r_i\}_{i=1}^\infty$ is a sequence of positive integers such that $r_i \mid r_{i+1}$. Moreover, $GR(p^n, r_i)$ is a ring isomorphic to $(\mathbb{Z}/p^n\mathbb{Z})[x]/f(x)(\mathbb{Z}/p^n\mathbb{Z})[x]$, where $f(x) \in (\mathbb{Z}/p^n\mathbb{Z})[x]$ is a monic polynomial of degree r, and is irreducible modulo $p\mathbb{Z}/p^n\mathbb{Z}$. By Lemma 2 of [8] $[R, J(R)] \leq J(R)^2$. If $J(R)^2 = \{0\}$, then $J(R) \subseteq Z(R)$. Since $R = S + J(R)$, then R is a commutative ring.

Hence we prove the following

Proposition 1. *Let R be a local right Artinian ring and $J(R)^2 = \{0\}$ and $R/J(R)$ has one of the following properties:*

(a) $R/J(R)$ is a differentially trivial field of characteristic 0, or

(b) $R/J(R)$ is a field of characteristic p, which is algebraic over its prime subfield.

Then all inner derivations of R are central if and only if R is a commutative ring.

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ПРО КІЛЬЦЯ З ЦЕНТРАЛЬНИМИ ВНУТРІШНІМИ ДИФЕРЕНЦІЮВАННЯМИ

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