FRACTIONAL ORDER IMPULSIVE PARTIAL HYPERBOLIC FUNCTIONAL DIFFERENTIAL EQUATIONS WITH VARIABLES TIMES AND STATE-DEPENDENT DELAY

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In this paper we study existence and uniqueness of solutions of two classes of partial impulsive hyperbolic differential equations with variable time impulses and state-dependent delay involving the Caputo fractional derivative. Suitable fixed point theorems are used.

1 Introduction

In this paper we provide sufficient conditions for existence and uniqueness of solutions to the following impulsive partial hyperbolic differential equations

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with variable times:

$$({}^cD_0^r u)(x,y) = f(x,y,u_{(\rho_1(x,y,u_{(x,y)}),\rho_2(x,y,u_{(x,y)}))}),$$

$$\text{if } (x,y) \in J := [0,a] \times [0,b], \quad x \neq x_k(u(x,y)),\ k = 1,\dots,m, \tag{1}$$

$$u(x^+,y) = I_k(u(x,y)), \quad \text{if } (x,y) \in J,\ x = x_k(u(x,y)),\ k = 1,\dots,m, \tag{2}$$

$$u(x,y) = \phi(x,y),\ \text{if } (x,y) \in \tilde{J} := [-\alpha,a] \times [-\beta,b] \backslash (0,a] \times (0,b], \tag{3}$$

$$u(x,0) = \varphi(x),\ x \in [0,a],\ u(0,y) = \psi(y);\ y \in [0,b], \tag{4}$$

where $a,b,\alpha,\beta > 0$, ${}^cD_0^r$ is the fractional Caputo derivative of order $r = (r_1,r_2) \in (0,1] \times (0,1]$, $0 = x_0 < x_1 < \cdots < x_m < x_{m+1} = a$, $\phi : \tilde{J} \to \mathbb{R}^n$ is a given function, $\varphi : [0,a] \to \mathbb{R}^n$, $\psi : [0,b] \to \mathbb{R}^n$ are given absolutely continuous functions such that $\varphi(x) = \phi(x,0)$ for each $x \in [0,a]$, $\psi(y) = \phi(0,y)$ for each $y \in [0,b]$, $f : J \times C \to \mathbb{R}^n$, $\rho_1,\rho_2 : J \times C \to \mathbb{R}$, $I_k : \mathbb{R}^n \to \mathbb{R}^n$, $k = 1,\dots,m$, are given functions and C is the space defined by

$$\begin{aligned} C = C_{(\alpha,\beta)} = \{u : [-\alpha,0] \times [-\beta,0] \to \mathbb{R}^n :\ &\text{continuous and there exist} \\ \tau_k \in (-\alpha,0)\ \text{ such that } \tau_k = \tau_k(u(\tau_k,.)),\ &\text{with } u(\tau_k^-,\tilde{y}) \text{ and } u(\tau_k^+,\tilde{y}), \\ k = 1,\dots,m,\ \text{exist for any } \tilde{y} \in [-\beta,0] &\text{ with } u(\tau_k^-,\tilde{y}) = u(\tau_k,\tilde{y})\big\}. \end{aligned}$$

Here C is a Banach space with norm

$$\|u\|_C = \sup_{(x,y)\in[-\alpha,0]\times[-\beta,0]} \|u(x,y)\|.$$

We also denote by $u_{(x,y)}$ an element of C defined as

$$u_{(x,y)}(s,t) = u(x+s,y+t);\ (s,t) \in [-\alpha,0] \times [-\beta,0],$$

where $u_{(x,y)}(.,.)$ represents the history of the state from time $(x-\alpha,y-\beta)$ up to the present time (x,y).

Below we consider the following system of partial hyperbolic differential equations of fractional order with infinite delay

$$({}^cD_0^r u)(x,y) = f(x,y,u_{(\rho_1(x,y,u_{(x,y)}),\rho_2(x,y,u_{(x,y)}))}),$$

$$\text{if } (x,y) \in J := [0,a] \times [0,b], \quad x \neq x_k(u(x,y)),\ k = 1,\dots,m, \tag{5}$$

$$u(x^+,y) = I_k(u(x,y)), \quad \text{if } (x,y) \in J,\ x = x_k(u(x,y)),\ k = 1,\dots,m, \tag{6}$$

$$u(x,y) = \phi(x,y),\ \text{if } (x,y) \in \tilde{J}' := (-\infty,a] \times (-\infty,b] \backslash (0,a] \times (0,b], \tag{7}$$

$$u(x,0) = \varphi(x),\ x \in [0,a],\ u(0,y) = \psi(y),\ y \in [0,b], \tag{8}$$

where $\varphi,\ \psi,\ I_k$ are as in problem (1) – (4), $f : J \times B \to \mathbb{R}^n,\ \rho_1, \rho_2 : J \times B \to \mathbb{R},\ \phi : \tilde{J}' \to \mathbb{R}^n$ are given functions and B is called a *phase space* which will be specified in Section 5.

Numerous applications of differential equations of fractional order have been indicated in viscoelasticity, electrochemistry, control, porous media, electromagnetic, etc. There has been a significant development in ordinary and partial fractional differential equations with or without impulses in recent years; see the monographs of Kilbas *et al.* [27], Lakshmikantham *et al.* [30], Podlubny [33], Samko *et al.* [34], the papers of Abbas and Benchohra [1–3], Agarwal *et al.* [5], Benchohra *et al.* [6, 7, 9, 10], Vityuk and Golushkov [36] and the references therein.

The theory of impulsive integer order differential equations have become important in some mathematical models of real processes and phenomena studied in physics, chemical technology, population dynamics, biotechnology and economics. There has been a significant development in this theory in recent years, especially in the area of impulsive differential equations and inclusions with fixed moments; see the monographs of Benchohra *et al.* [8], Lakshmikantham *et al.* [29], and Samoilenko and Perestyuk [35]. The theory of impulsive differential equations with variable time is relatively less developed due to the difficulties created by the state-dependent impulses.

Functional differential equations with state-dependent delay appear frequently in applications as model of equations and for this reason the study of this type of equations has received great attention in the last year, see, for instance [13, 17, 18], and the references therein. The literature related to partial functional differential equations with state-dependent delay is limited, see for instance [21, 22]. The literature related to ordinary and partial functional differential equations with delay for which $\rho(s,t,.) = (s,t)$ is very extensive, see for instance Abbas and Benchohra [1, 2], Hale [16], Hale and Verduyn Lunel [15], Kolmanovskii and Myshkis [28] and Wu [37] and the papers therein. Some classes of hyperbolic fractional order differential equations with finite delay are considered by Abbas and Benchohra [1, 2]. In [3] hyperbolic fractional order differential equations with impulses are considered. Czlapinski [11, 12] considered some classes of integer order hyperbolic functional differential equations with infinite delay.

In this paper we present the existence results for our problems which based on the Schaefer's fixed point approach. The present results extend those considered before by Abbas and Benchohra [1–4], and those related with integer order derivative [11,12,25,26,32].

2 Preliminaries

In this section we introduce notations and definitions which are used throughout this paper. By $AC(J,\mathbb{R}^n)$ we denote the space of absolutely continuous functions from J to $\mathbb{R}^n$ and $L^1(J,\mathbb{R}^n)$ is the space of Lebesgue-integrable functions $w : J \to \mathbb{R}^n$ with the norm

$$\|w\|_1 = \int_0^a \int_0^b \|w(x,y)\| dydx,$$

where $\|.\|$ denotes a suitable complete norm on $\mathbb{R}^n$.

Let $a_1 \in [0,a]$, $z^+ = (a_1^+,0) \in J$, $J_z = [a_1,a] \times [0,b]$, $r_1,r_2 > 0$ and $r = (r_1,r_2)$. For $w \in L^1(J_z,\mathbb{R}^n)$, the expression

$$(I^r_{z^+}w)(x,y) = \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{a_1^+}^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1}w(s,t)\,dt\,ds,$$

where $\Gamma(.)$ is the Euler gamma function, is called the left-sided mixed Riemann-Liouville integral of order r.

Definition 2.1. *[36]. For $w \in L^1(J_z,\mathbb{R}^n)$, the Caputo fractional-order derivative of order r is defined by the expression*

$$({}^cD^r_{z^+}w)(x,y) = (I^{1-r}_{z^+}\frac{\partial^2}{\partial x \partial y}w)(x,y).$$

Theorem 2.2. ***(Schaefer's theorem)*** *[19]. Let X be a Banach space and $N : X \to X$ completely continuous operator. If the set*

$$E(N) = \{u \in X : \ u = \lambda N(u) \ \textit{for some} \ \lambda \in [0,1]\}$$

is bounded, then N has fixed points.

In the sequel we will make use of the following generalization of Gronwall's lemma for two independent variables and singular kernel.

Lemma 2.3. *[20]. Let $\upsilon : J \to [0,\infty)$ be a real function and $\omega(.,.)$ be a nonnegative, locally integrable function on J. If there are constants $c > 0$ and $0 < r_1,r_2 < 1$ such that*

$$\upsilon(x,y) \le \omega(x,y) + c\int_0^x \int_0^y \frac{\upsilon(s,t)}{(x-s)^{r_1}(y-t)^{r_2}}dtds,$$

then there exists a constant $\delta = \delta(r_1, r_2)$ *such that*

$$\upsilon(x,y) \leq \omega(x,y) + \delta c \int_0^x \int_0^y \frac{\omega(s,t)}{(x-s)^{r_1}(y-t)^{r_2}} dtds,$$

for every $(x,y) \in J$.

3 Auxiliary results

Let $h \in C([x_k, x_{k+1}] \times [0,b], \mathbb{R}^n)$, $z_k = (x_k, 0)$, and

$$\mu_k(x,y) = u(x,0) + u(x_k^+, y) - u(x_k^+, 0), \quad k = 0, \ldots, m.$$

For the existence of solutions for the problem (1) – (3), we need the following lemma:

Lemma 3.1. *[3]. A function* $u \in AC([x_k, x_{k+1}] \times [0,b], \mathbb{R}^n)$, $k = 0, \ldots, m$, *is a solution of the differential equation*

$$({}^cD^r_{z_k} u)(x,y) = h(x,y), \ (x,y) \in [x_k, x_{k+1}] \times [0,b],$$

if and only if $u(x,y)$ *satisfies*

$$u(x,y) = \mu_k(x,y) + (I^r_{z_k} h)(x,y), \ (x,y) \in [x_k, x_{k+1}] \times [0,b]. \tag{9}$$

Lemma 3.2. *Let* $0 < r_1, r_2 \leq 1$, $h : J \to \mathbb{R}^n$ *be a continuous function, and denote* $\mu(x,y) := \mu_0(x,y)$; $(x,y) \in J$. *A function* $u(x,y)$ *is a solution of the fractional integral equation*

$$u(x,y) = \begin{cases} \mu(x,y) + \\ + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} h(s,t) dtds, \\ \textit{if } (x,y) \in [0, x_1] \times [0,b], \\ \varphi(x) + I_k(u(x_k, y)) - I_k(u(x_k, 0)) + \\ + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{x_k}^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} h(s,t) dtds, \\ \textit{if } (x,y) \in (x_k, x_{k+1}] \times [0,b], \ k = 1, \ldots, m, \end{cases} \tag{10}$$

if and only if $u(x,y)$ *is a solution of the fractional IVP*

$$ {}^cD^r u(x,y) = h(x,y), \quad (x,y) \in J', \tag{11}$$

$$u(x_k^+, y) = I_k(u(x_k, y)), \quad k = 1, \ldots, m. \tag{12}$$

Proof: Assume $u(x,y)$ satisfies (11) – (12). If $(x,y) \in [0,x_1] \times [0,b]$ then
$$^cD^r u(x,y) = h(x,y).$$

Lemma 3.1 implies
$$u(x,y) = \mu(x,y) + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} h(s,t)dtds.$$

If $(x,y) \in (x_1,x_2] \times [0,b]$ then Lemma 3.1 implies
$$\begin{aligned}
u(x,y) &= \mu_1(x,y) + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{x_1}^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} h(s,t)\,dtds = \\
&= \varphi(x) + u(x_1^+,y) - u(x_1^+,0) + \\
&+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{x_1}^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} h(s,t)\,dtds = \\
&= \varphi(x) + I_1(u(x_1,y)) - I_1(u(x_1,0)) + \\
&+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{x_1}^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} h(s,t)dtds.
\end{aligned}$$

If $(x,y) \in (x_2,x_3] \times [0,b]$ then from Lemma 3.1 we get
$$\begin{aligned}
u(x,y) &= \mu_2(x,y) + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{x_2}^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} h(s,t)\,dtds = \\
&= \varphi(x) + u(x_2^+,y) - u(x_2^+,0) + \\
&+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{x_2}^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} h(s,t)\,dtds = \\
&= \varphi(x) + I_2(u(x_2,y)) - I_2(u(x_2,0)) + \\
&+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{x_2}^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} h(s,t)dtds.
\end{aligned}$$

If $(x,y) \in (x_k,x_{k+1}] \times [0,b]$ then again from Lemma 3.1 we get (10).

Conversely, assume that $u(x,y)$ satisfies the impulsive fractional integral equation (10). Using the fact that $^cD^r$ is the left inverse of I^r we get
$$^cD^r u(x,y) = h(x,y) \quad \text{for each } (x,y) \in [0,x_1] \times [0,b].$$

Using the fact that $^cD^r C = 0$, where C is a constant, we get
$$^cD^r u(x,y) = h(x,y) \quad \text{for each } (x,y) \in [x_k,x_{k+1}) \times [0,b],\ k = 1,\dots,m.$$

Also, we can easily show that

$$u(x_k^+, y) = I_k(u(x_k, y)), \quad y \in [0, b],\ k = 1, \dots, m.$$

In all what follows set

$$J_k := (x_k, x_{k+1}] \times [0, b],\ k = 1, \dots, m,$$

$J' := J \backslash \{(x_1, y), \dots, (x_m, y),\, y \in [0, b]\}$ and consider the space

$$\begin{aligned} PC &:= PC(J, \mathbb{R}^n) = \\ &= \big\{u : J \to \mathbb{R}^n : u \in C(J_k, \mathbb{R}^n);\ k = 1, \dots, m, \text{ and there exist} \\ &\quad 0 = x_0 < x_1 < x_2 < \dots < x_m < x_{m+1} = a \text{ such that } x_k = x_k(u(x_k, .)), \\ &\quad \text{and } u(x_k^-, y) \text{ and } u(x_k^+, y); k = 1, \dots, m, \text{ exist for any} \\ &\quad y \in [0, b] \text{ with } u(x_k^-, y) = u(x_k, y)\big\}. \end{aligned}$$

This set is a Banach space with the norm

$$\|u\|_{PC} = \sup_{(x,y)\in J} \|u(x, y)\|.$$

4 Impulsive functional hyperbolic differential equations with finite delay

Let us start in this section by defining what we mean by a solution of the problem (1) – (4). Set

$$\widetilde{PC} := PC([-\alpha, a] \times [-\beta, b], \mathbb{R}^n).$$

$\widetilde{PC}$ is a Banach space with the norm

$$\|u\|_{\widetilde{PC}} = \sup\{\|u(x, y)\| : (x, y) \in [-\alpha, a] \times [-\beta, b]\}.$$

Definition 4.1. *A function $u(x, y) \in \widetilde{PC}$ whose r-derivative exists on J' is said to be a solution of (1) – (4) if $u(x, y)$ satisfies the condition (3) on $\tilde{J}$, the equation (1) on J' and conditions (2) and (4) are satisfied on J.*

Set $\mathcal{R} := \mathcal{R}_{(\rho_1^-, \rho_2^-)} =$

$$= \{(\rho_1(s, t, u), \rho_2(s, t, u)) : (s, t, u) \in J \times C,\ \rho_i(s, t, u) \le 0;\ i = 1, 2\}.$$

We always assume that $\rho_i : J \times C \to \mathbb{R};\ i = 1, 2,$ are continuous and the function $(s, t) \longmapsto u_{(s,t)}$ is continuous from $\mathcal{R}$ into C.

Theorem 4.2. *Assume that*

(H1) The function $f : J \times C \to \mathbb{R}^n$ *is continuous.*

(H2) There exists a constant $M > 0$ *such that*

$$\|f(x,y,u)\| \le M(1+\|u\|), \text{ for each } (x,y) \in J, \; u \in C.$$

(H3) The function $x_k \in C^1(\mathbb{R}^n, \mathbb{R})$ *for* $k = 1, \ldots, m$. *Moreover,*

$$0 = x_0(u) < x_1(u) < \ldots < x_m(u) < x_{m+1}(u) = a, \quad \text{for all } u \in \mathbb{R}^n.$$

(H4) There exists a constant $M^* > 0$ *such that*

$$\|I_k(u)\| \le M^*(1+\|u\|), \; k = 1, \ldots, m, \text{ for each } u \in C.$$

(H5) For all $u \in C$, $x_k(I_k(u)) \le x_k(u) < x_{k+1}(I_k(u))$ *for* $k = 1, \ldots, m$.

(H6) For all $(s,t,u) \in J \times C$, *we have*

$$x_k'(u)[\varphi'(s) + \frac{r_1 - 1}{\Gamma(r_1)\Gamma(r_2)} \int_{x_k}^{s} \int_0^t (s-\theta)^{r_1-2}(t-\eta)^{r_2-1} \times$$

$$\times f(\theta, \eta, u_{(\rho_1(\theta,\eta,u_{(\theta,\eta)}), \rho_2(\theta,\eta,u_{(\theta,\eta)}))}) d\eta d\theta] \ne 1, \quad k = 1, \ldots, m.$$

Then (1) – (4) has at least one solution on $[-\alpha, a] \times [-\beta, b]$.

Proof: The proof will be given in several steps.

Step 1: Set $PC_0 =$

$$= \big\{u : [-\alpha, a]\times[-\beta, b] \to \mathbb{R}^n : u_{(x,y)} \in C \text{ for } (x,y) \in J \text{ and } u \in PC(J, \mathbb{R}^n)\big\}.$$

Consider the following problem

$$({}^cD_0^r u)(x,y) = f(x,y,u_{(\rho_1(x,y,u_{(x,y)}), \rho_2(x,y,u_{(x,y)}))}), \text{ if } (x,y) \in J, \tag{13}$$

$$u(x,y) = \phi(x,y), \text{ if } (x,y) \in \tilde{J}, \tag{14}$$

$$u(x,0) = \varphi(x), \; x \in [0,a], \; u(0,y) = \psi(y), \; y \in [0,b]. \tag{15}$$

Transform problem (13) – (15) into a fixed point problem. Consider the operator $N : PC_0 \to PC_0$ defined by

$$N(u)(x,y) = \begin{cases} \phi(x,y), & (x,y) \in \tilde{J}, \\ \mu(x,y) + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} \times \\ \times f(s,t,u_{(\rho_1(s,t,u_{(s,t)}),\rho_2(s,t,u_{(s,t)}))})dtds, & (x,y) \in J. \end{cases}$$

Lemma 3.1 implies that the fixed points of operator N are solutions of problem (13) – (15) . We shall show that the operator N is continuous and completely continuous.

Claim 1: N is continuous.

Let $\{u_n\}$ be a sequence such that $u_n \to u$ in PC_0. Let $\eta > 0$ be such that $\|u_n\| \leq \eta$. Then for each $(x,y) \in J$, we have

$$\begin{aligned} &\|N(u_n)(x,y) - N(u)(x,y)\| \leq \\ &\leq \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} \times \\ &\times \|f(s,t,u_{n(\rho_1(s,t,u_{(s,t)}),\rho_2(s,t,u_{(s,t)}))}) - \\ &- f(s,t,u_{(\rho_1(s,t,u_{(s,t)}),\rho_2(s,t,u_{(s,t)}))})\| dtds \leq \\ &\leq \frac{\|f(.,.,u_{n(.,.)}) - f(.,.,u)\|_\infty}{\Gamma(r_1)\Gamma(r_2)} \int_0^a \int_0^b (x-s)^{r_1-1}(y-t)^{r_2-1}\, dsdt \leq \\ &\leq \frac{a^{r_1} b^{r_2} \|f(.,.,u_{n(.,.)}) - f(.,.,u_{(.,.)})\|_\infty}{\Gamma(r_1+1)\Gamma(r_2+1)}. \end{aligned}$$

Since f is a continuous function, we have

$$\|N(u_n) - N(u)\|_\infty \to 0 \quad \text{as } n \to \infty.$$

Claim 2: N maps bounded sets into bounded sets in PC_0.

Indeed, it is enough to show that for any $\eta^* > 0$, there exists a positive constant ℓ such that for each $u \in B_{\eta^*} = \{u \in PC_0 : \|u\|_\infty \leq \eta^*\}$, we have $\|N(u)\|_\infty \leq \ell$.

By (H2) for each $(x,y) \in J$ we have

$$\begin{aligned} \|N(u)(x,y)\| &\leq \|\mu(x,u)\| + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} \times \\ &\times \|f(s,t,u_{(\rho_1(s,t,u_{(s,t)}),\rho_2(s,t,u_{(s,t)}))})\|\, dtds \leq \\ &\leq \|\mu(x,u)\| + \frac{M(1+\eta^*)a^{r_1}b^{r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)}. \end{aligned}$$

Thus

$$\|N(u)\|_\infty \leq \|\mu\|_\infty + \frac{M(1+\eta^*)a^{r_1}b^{r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)} := \ell.$$

Claim 3: N maps bounded sets into equicontinuous sets of PC_0.

Let $(\tau_1, y_1), (\tau_2, y_2) \in J$, $\tau_1 < \tau_2$ and $y_1 < y_2$, B_{η^*} be a bounded set of PC_0 as in Claim 2, and let $u \in B_{\eta^*}$. Then for each $(x, y) \in J$, we have

$$\|N(u)(\tau_2,y_2) - N(u)(\tau_1,y_1)\| = \Big\|\mu(\tau_1,y_1) - \mu(\tau_2,y_2) +$$
$$+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^{\tau_1}\int_0^{y_1}[(\tau_2-s)^{r_1-1}(y_2-t)^{r_2-1} - (\tau_1-s)^{r_1-1}\times$$
$$\times(y_1-t)^{r_2-1}]f(s,t,u_{(\rho_1(s,t,u_{(s,t)}),\rho_2(s,t,u_{(s,t)}))})dtds +$$
$$+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{\tau_1}^{\tau_2}\int_{y_1}^{y_2}(\tau_2-s)^{r_1-1}(y_2-t)^{r_2-1}\times$$
$$\times f(s,t,u_{(\rho_1(s,t,u_{(s,t)}),\rho_2(s,t,u_{(s,t)}))})dtds +$$
$$+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^{\tau_1}\int_{y_1}^{y_2}(\tau_2-s)^{r_1-1}(y_2-t)^{r_2-1}\times$$
$$\times f(s,t,u_{(\rho_1(s,t,u_{(s,t)}),\rho_2(s,t,u_{(s,t)}))})dtds +$$
$$+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{\tau_1}^{\tau_2}\int_0^{y_1}(\tau_2-s)^{r_1-1}(y_2-t)^{r_2-1}\times$$
$$\times f(s,t,u_{(\rho_1(s,t,u_{(s,t)}),\rho_2(s,t,u_{(s,t)}))})dtds\Big\| \leq$$

$$\leq \|\mu(\tau_1,y_1) - \mu(\tau_2,y_2)\| +$$
$$+\frac{M(1+\eta^*)}{\Gamma(r_1)\Gamma(r_2)}\int_0^{\tau_1}\int_0^{y_1}[(\tau_1-s)^{r_1-1}(y_1-t)^{r_2-1} -$$
$$-(\tau_2-s)^{r_1-1}(y_2-t)^{r_2-1}]dtds +$$

$$+\frac{M(1+\eta^*)}{\Gamma(r_1)\Gamma(r_2)}\int_{\tau_1}^{\tau_2}\int_{y_1}^{y_2}(\tau_2-s)^{r_1-1}(y_2-t)^{r_2-1}\,dtds+$$

$$+\frac{M(1+\eta^*)}{\Gamma(r_1)\Gamma(r_2)}\int_{0}^{\tau_1}\int_{y_1}^{y_2}(\tau_2-s)^{r_1-1}(y_2-t)^{r_2-1}\,dtds+$$

$$+\frac{M(1+\eta^*)}{\Gamma(r_1)\Gamma(r_2)}\int_{\tau_1}^{\tau_2}\int_{0}^{y_1}(\tau_2-s)^{r_1-1}(y_2-t)^{r_2-1}dtds\leq$$

$$\leq\|\mu(\tau_1,y_1)-\mu(\tau_2,y_2)\|+\frac{M(1+\eta^*)}{\Gamma(r_1+1)\Gamma(r_2+1)}[2y_2^{r_2}(\tau_2-\tau_1)^{r_1}+$$
$$+2\tau_2^{r_1}(y_2-y_1)^{r_2}+\tau_1^{r_1}y_1^{r_2}-\tau_2^{r_1}y_2^{r_2}-2(\tau_2-\tau_1)^{r_1}(y_2-y_1)^{r_2}].$$

As $\tau_1\longrightarrow\tau_2$ and $y_1\longrightarrow y_2$ the right-hand side of the above inequality tends to zero. As a consequence of Claims 1 – 3 together with the Arzelá-Ascoli theorem, we can conclude that N is completely continuous.

Claim 4: A priori bounds.

Now it remains to show that the set $\mathcal{E}=\{u\in PC_0 : u=\lambda N(u)$ for some $0<\lambda<1\}$ is bounded.

Let $u\in\mathcal{E}$, then $u=\lambda N(u)$ for some $0<\lambda<1$. Thus, for each $(x,y)\in J$ we have

$$\begin{aligned}\|u(x,y)\| &\leq \|\mu(x,y)\|+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^x\int_0^y(x-s)^{r_1-1}(y-t)^{r_2-1}\times\\ &\times \|f(s,t,u_{(\rho_1(s,t,u_{(s,t)}),\rho_2(s,t,u_{(s,t)}))})\|\,dtds\leq\\ &\leq \|\mu\|_\infty+\frac{Ma^{r_1}b^{r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)}+\\ &+ \frac{M}{\Gamma(r_1)\Gamma(r_2)}\int_0^x\int_0^y(x-s)^{r_1-1}(y-t)^{r_2-1}\|u_{(s,t)}\|\,dtds.\end{aligned}$$

Set
$$\omega=\|\mu\|_\infty+\frac{Ma^{r_1}b^{r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)}.$$
Then Lemma 2.3 implies that for each $(x,y)\in J$, there exists $\delta=\delta(r_1,r_2)$ such that

$$\begin{aligned}\|u(x,y)\| &\leq \omega\left[1+\frac{M\delta}{\Gamma(r_1)\Gamma(r_2)}\int_0^x\int_0^y(x-s)^{r_1-1}(y-t)^{r_2-1}dtds\right]\leq\\ &\leq \omega\left[1+\frac{M\delta a^{r_1}b^{r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)}\right]:=R.\end{aligned}$$

This shows that the set $\mathcal{E}$ is bounded. As a consequence of Schaefer's fixed point theorem (Theorem 2.2), we deduce that N has a fixed point which is a solution of the problem (13) – (15). Denote this solution by u_1. Define the function

$$r_{k,1}(x,y) = x_k(u_1(x,y)) - x, \qquad \text{for } x \geq 0, y \geq 0.$$

Hypothesis (H3) implies that $r_{k,1}(0,0) \neq 0$ for $k = 1, \dots, m$.
If $r_{k,1}(x,y) \neq 0$ on J for $k = 1, \dots, m$, i.e.,

$$x \neq x_k(u_1(x,y)) \quad \text{on } J \quad \text{for } k = 1, \dots, m,$$

then u_1 is a solution of the problem (1) – (4).

It remains to consider the case when $r_{1,1}(x,y) = 0$ for some $(x,y) \in J$. Since $r_{1,1}(0,0) \neq 0$ and $r_{1,1}$ is continuous, there exists $x_1 > 0, y_1 > 0$ such that $r_{1,1}(x_1,y_1) = 0$, and $r_{1,1}(x,y) \neq 0$, for all $(x,y) \in [0,x_1) \times [0,y_1)$.

Thus by (H6) we have

$$r_{1,1}(x_1,y_1) = 0 \text{ and } r_{1,1}(x,y) \neq 0 \text{ for all } (x,y) \in [0,x_1) \times [0,y_1] \cup (y_1,b].$$

Suppose that there exist $(\bar{x},\bar{y}) \in [0,x_1) \times [0,y_1] \cup (y_1,b]$ such that $r_{1,1}(\bar{x},\bar{y}) = 0$. The function $r_{1,1}$ attains a maximum at some point $(s,t) \in [0,x_1) \times [0,b]$. Since

$$({}^cD_0^r u_1)(x,y) = f(x,y,u_{1(\rho_1(x,y,u_{(x,y)}),\rho_2(x,y,u_{(x,y)}))}) \text{ for } (x,y) \in J,$$

then

$$\frac{\partial u_1(x,y)}{\partial x} \text{ exists, and } \frac{\partial r_{1,1}(s,t)}{\partial x} = x_1'(u_1(s,t))\frac{\partial u_1(s,t)}{\partial x} - 1 = 0.$$

Since

$$\begin{aligned} \frac{\partial u_1(x,y)}{\partial x} \quad &= \varphi'(x) + \frac{r_1 - 1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-2}(y-t)^{r_2-1} \times \\ &\times f(s,t,u_{1(\rho_1(s,t,u_{(s,t)}),\rho_2(s,t,u_{(s,t)}))})\, dtds, \end{aligned}$$

then

$$\begin{aligned} &x_1'(u_1(s,t))[\varphi'(s) + \frac{r_1 - 1}{\Gamma(r_1)\Gamma(r_2)} \int_0^s \int_0^t (s-\theta)^{r_1-2}(t-\eta)^{r_2-1} \\ &\times f(\theta,\eta,u_{(\rho_1(\theta,\eta,u_{(\theta,\eta)}),\rho_2(\theta,\eta,u_{(\theta,\eta)}))}) d\theta d\eta] = 1, \end{aligned}$$

which contradicts (H6).

From (H3) we have

$$r_{k,1}(x,y) \neq 0 \text{ for all } (x,y) \in [0,x_1) \times [0,b] \text{ and } k = 1,...m.$$

Step 2: In what follows set

$$\begin{aligned} PC_k = \big\{ & u : [-\alpha,a] \times [-\beta,b] \to \mathbb{R}^n : u_{(x,y)} \in C \text{ for } (x,y) \in J, \text{ and there} \\ & \text{exist } 0 = x_0 < x_1 < x_2 < ... < x_m < x_{m+1} = a \text{ such that} \\ & x_k = x_k(u(x_k,.)), \text{ and } u(x_k^-,.), \ u(x_k^+,.) \text{ exist with} \\ & u(x_k^-,.) = u(x_k,.), \ k = 1,\dots,m, \text{ and } u \in C(X_k,\mathbb{R}^n), k = 0,\dots,m \big\}, \end{aligned}$$

where

$$X_k := [x_k,a] \times [0,b], \ k = 1,...,m.$$

Consider now the problem

$$({}^cD_0^r u)(x,y) = f(x,y,u_{(\rho_1(x,y,u_{(x,y)}),\rho_2(x,y,u_{(x,y)}))}), \text{ if}(x,y) \in X_1, \tag{16}$$

$$u(x_1^+,y) = I_1(u_1(x_1,y)), \tag{17}$$

$$u(x,y) = u_1(x,y), \text{ if } (x,y) \in \tilde{J} \cup [0,x_1) \times [0,b]. \tag{18}$$

Consider the operator $N_1 : PC_1 \to PC_1$ defined as

$$N_1(u)(x,y) = \begin{cases} u_1(x,y), & (x,y) \in \tilde{J} \cup [0,x_1) \times [0,b], \\ \varphi(x) + I_1(u_1(x_1,y)) - I_1(u_1(x_1,0)) + \\ +\frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{x_1}^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} \times \\ \times f(s,t,u_{(\rho_1(s,t,u_{(s,t)}),\rho_2(s,t,u_{(s,t)}))})dtds, & (x,y) \in X_1. \end{cases}$$

As in Step 1 we can show that N_1 is completely continuous. Now it remains to show that the set $\mathcal{E}^* = \{u \in PC_1 : u = \lambda N_1(u)$ for some $0 < \lambda < 1\}$ is bounded.

Let $u \in \mathcal{E}^*$, then $u = \lambda N_1(u)$ for some $0 < \lambda < 1$. Thus, from (H2) and (H4) we get for each $(x,y) \in X_1$,

$$\begin{aligned} \|u(x,y)\| &\leq \|\varphi(x)\| + \|I_1(u_1(x_1,y))\| + \|I_1(u_1(x_1,0))\| + \\ &+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{x_1}^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} \times \\ &\times \|f(s,t,u_{(\rho_1(s,t,u_{(s,t)}),\rho_2(s,t,u_{(s,t)}))})\| \, dtds \leq \\ &\leq \|\varphi\|_\infty + 2M^*(1 + \|u_1\|) + \frac{Ma^{r_1}b^{r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)} + \\ &+ \frac{M}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} \|u_{(s,t)}\| \, dtds. \end{aligned}$$

Set
$$\omega^* = \|\varphi\|_\infty + 2M^*(1+\|u_1\|) + \frac{Ma^{r_1}b^{r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)}.$$
Then Lemma 2.3 implies that for each $(x,y)\in X_1$, there exists $\delta=\delta(r_1,r_2)$ such that
$$\begin{aligned}\|u(x,y)\| \quad & \leq \omega^*\left[1+\frac{M\delta}{\Gamma(r_1)\Gamma(r_2)}\int_0^x\int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1}dtds\right] \leq \\ & \leq \omega^*\left[1+\frac{M\delta a^{r_1}b^{r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)}\right] := R^*.\end{aligned}$$
This shows that the set $\mathcal{E}^*$ is bounded.

As a consequence of Schaefer's fixed point theorem (Theorem 2.2), we deduce that N_1 has a fixed point u which is a solution to problem (16) – (18). Denote this solution by u_2. Define
$$r_{k,2}(x,y) = x_k(u_2(x,y)) - x, \quad \text{for } (x,y)\in X_1.$$
If $r_{k,2}(x,y)\neq 0$ on $(x_1,a]\times[0,b]$ and for all $k=1,\ldots,m$, then
$$u(x,y) = \begin{cases} u_1(x,y), & \text{if } (x,y)\in \tilde{J}\cup[0,x_1)\times[0,b], \\ u_2(x,y), & \text{if } (x,y)\in [x_1,a]\times[0,b], \end{cases}$$
is a solution of the problem (1) – (3).

It remains to consider the case when $r_{2,2}(x,y)=0$, for some $(x,y)\in(x_1,a]\times[0,b]$. By (H5), we have
$$\begin{aligned} r_{2,2}(x_1^+,y_1) &= x_2(u_2(x_1^+,y_1) - x_1 = x_2(I_1(u_1(x_1,y_1))) - x_1 > \\ &> x_1(u_1(x_1,y_1)) - x_1 = r_{1,1}(x_1,y_1) = 0.\end{aligned}$$

Since $r_{2,2}$ is continuous, there exists $x_2>x_1$, $y_2>y_1$ such that $r_{2,2}(x_2,y_2)=0$, and $r_{2,2}(x,y)\neq 0$ for all $(x,y)\in(x_1,x_2)\times[0,b]$.

It is clear by (H3) that
$$r_{k,2}(x,y)\neq 0 \quad \text{for all } (x,y)\in(x_1,x_2)]\times[0,b],\ k=2,\ldots,m.$$

Now suppose that there are $(s,t)\in(x_1,x_2)\times[0,b]$ such that $r_{1,2}(s,t)=0$. From (H5) it follows that
$$\begin{aligned} r_{1,2}(x_1^+,y_1) &= x_1(u_2(x_1^+,y_1) - x_1 = x_1(I_1(u_1(x_1,y_1))) - x_1 \leq \\ &\leq x_1(u_1(x_1,y_1)) - x_1 = r_{1,1}(x_1,y_1) = 0.\end{aligned}$$

Thus $r_{1,2}$ attains a nonnegative maximum at some point $(s_1, t_1) \in (x_1, a) \times [0, x_2) \cup (x_2, b]$. Since

$$({}^cD_0^r u_2)(x,y) = f(x, y, u_2(x,y)), \text{ for } (x,y) \in X_1,$$

then we get

$$\begin{aligned} u_2(x,y) &= \varphi(x) + I_1(u_1(x_1,y)) - I_1(u_1(x_1,0)) + \\ &+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{x_1}^{x} \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} \times \\ &\times f(s,t,u_{2(\rho_1(s,t,u_{(s,t)}),\rho_2(s,t,u_{(s,t)}))})\, dtds, \end{aligned}$$

hence

$$\frac{\partial u_2}{\partial x}(x,y) = \varphi'(x) + \frac{r_1 - 1}{\Gamma(r_1)\Gamma(r_2)} \int_{x_1}^{x} \int_0^y (x-s)^{r_1-2}(y-t)^{r_2-1} \times$$

$$\times f(s,t,u_{2(\rho_1(s,t,u_{(s,t)}),\rho_2(s,t,u_{(s,t)}))})\, dtds,$$

then

$$\frac{\partial r_{1,2}(s_1,t_1)}{\partial x} = x_1'(u_2(s_1,t_1)) \frac{\partial u_2}{\partial x}(s_1,t_1) - 1 = 0.$$

Therefore

$$\begin{aligned} & x_1'(u_2(s_1,t_1))[\varphi'(s_1) + \frac{r_1-1}{\Gamma(r_1)\Gamma(r_2)} \int_{x_1}^{s_1} \int_0^{t_1} (s_1-\theta)^{r_1-2}(t_1-\eta)^{r_2-1} \times \\ & \times f(\theta,\eta,u_{2(\rho_1(\theta,\eta,u_{(\theta,\eta)}),\rho_2(\theta,\eta,u_{(\theta,\eta)}))})\, d\eta d\theta] = 1, \end{aligned}$$

which contradicts (H6).

Step 3: We continue this process and take into account that $u_{m+1} := u\Big|_{X_m}$ is a solution to the problem

$$\begin{cases} ({}^cD_0^r u)(x,y) = f(x,y,u_{(\rho_1(x,y,u_{(x,y)}),\rho_2(x,y,u_{(x,y)}))}), \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{a.e. } (x,y) \in (x_m, a] \times [0,b], \\ u(x_m^+, y) = I_m(u_{m-1}(x_m, y)), \\ u(x,y) = u_1(x,y), \text{ if } (x,y) \in \tilde{J} \cup [0,x_1) \times [0,b], \\ u(x,y) = u_2(x,y), \text{ if } (x,y) \in [x_1,x_2) \times [0,b], \\ \ldots \\ u(x,y) = u_m(x,y), \text{ if } (x,y) \in [x_{m-1},x_m) \times [0,b]. \end{cases}$$

The solution $u(x,y)$ of the problem (1) – (3) is then defined by

$$u(x,y)=\begin{cases} u_1(x,y), & \text{if } (x,y)\in[0,x_1]\times[0,b],\\ u_2(x,y), & \text{if } (x,y)\in(x_1,x_2]\times[0,b],\\ \dots & \\ u_{m+1}(x,y), & \text{if } (x,y)\in(x_m,a]\times[0,b].\end{cases}$$

5 The phase space B

The notation of the phase space B plays an important role in the study of both qualitative and quantitative theory for functional differential equations. A usual choice is a semi-normed space satisfying suitable axioms, which was introduced by Hale and Kato [14] (see also [15, 24, 31]).

For any $(x,y)\in J$ denote $E_{(x,y)}:=[0,x]\times\{0\}\cup\{0\}\times[0,y]$, furthermore in case $x=a,\ y=b$ we write simply E. Consider the space $(B,\|(.,.)\|_B)$ is a seminormed linear space of functions mapping $(-\infty,0]\times(-\infty,0]$ into $\mathbb{R}^n$, and satisfying the following fundamental axioms which were adapted from those introduced by Hale and Kato for ordinary differential functional equations:

(A_1) If $z:(-\infty,a]\times(-\infty,b]\to\mathbb{R}^n$, $z_{(x,y)}\in B$ for all $(x,y)\in E$ and $z\in PC$, then for every $(x,y)\in J$ the following conditions hold:

- (i) $z_{(x,y)}$ is in B.
- (ii) There exists a positive constant H such that $\|z(x,y)\|\le H\|z_{(x,y)}\|_B$.
- (iii) There exist two functions $K,M:\mathbb{R}_+\times\mathbb{R}_+\to\mathbb{R}_+$ independent of u, with K continuous and M locally bounded such that

$$\|z_{(x,y)}\|_B\le K(x,y)\sup_{(s,t)\in[0,x]\times[0,y]}\|z(s,t)\|+M(x,y)\sup_{(s,t)\in E_{(x,y)}}\|z_{(s,t)}\|_B.$$

(A_2) The space B is complete.

Denote $K=\sup_{(x,y)\in J}K(x,y)$ and $M=\sup_{(x,y)\in J}M(x,y)$.

Now, we present some examples of phase spaces (see [11, 12]).

Example 5.1. *Let B be the set of all functions $\phi : (-\infty, 0] \times (-\infty, 0] \to \mathbb{R}^n$ such that for each $\alpha, \beta \geq 0$ we define in C the semi-norms by*

$$\|\phi\|_B = \sup_{(s,t)\in[-\alpha,0]\times[-\beta,0]} \|\phi(s,t)\|.$$

Then we have $H = K = M = 1$. The quotient space $\widehat{B} = B/\|.\|_B$ is isometric to the space $C([-\alpha, 0]\times[-\beta, 0], \mathbb{R}^n)$ of all continuous functions from $[-\alpha, 0]\times[-\beta, 0]$ into $\mathbb{R}^n$ with the supremum norm, this means that partial differential functional equations with finite delay are included in our axiomatic model.

Example 5.2. *Let C_γ be the set of all functions $\phi : (-\infty, 0]\times(-\infty, 0] \to \mathbb{R}^n$ for which a limit $\lim\limits_{\|(s,t)\|\to\infty} e^{\gamma(s+t)}\phi(s, t)$ exists, with the norm*

$$\|\phi\|_{C_\gamma} = \sup_{(s,t)\in(-\infty,0]\times(-\infty,0]} e^{\gamma(s+t)}\|\phi(s,t)\|.$$

Then we have $H = 1$ and $K = M = \max\{e^{-(a+b)}, 1\}$.

Example 5.3. *Let $\alpha, \beta, \gamma \geq 0$ and let*

$$\|\phi\|_{CL_\gamma} = \sup_{(s,t)\in[-\alpha,0]\times[-\beta,0]} \|\phi(s,t)\| + \int_{-\infty}^{0}\int_{-\infty}^{0} e^{\gamma(s+t)}\|\phi(s,t)\|dtds.$$

be the seminorm for the space CL_γ of all functions $\phi : (-\infty, 0] \times (-\infty, 0] \to \mathbb{R}^n$ which are measurable on $(-\infty, -\alpha] \times (-\infty, 0] \cup (-\infty, 0] \times (-\infty, -\beta]$, and such that $\|\phi\|_{CL_\gamma} < \infty$. Then

$$H = 1,\ K = \int_{-\alpha}^{0}\int_{-\beta}^{0} e^{\gamma(s+t)}dtds,\ M = 2.$$

6 Impulsive functional hyperbolic differential equations with infinite delay

Now we present an existence result for the problem (5) – (8). Consider the space $\Omega = \big\{u : (-\infty, a] \times (-\infty, b] \to \mathbb{R}^n : u_{(x,y)} \in B$ for $(x, y) \in E$ and $u|_J \in PC\big\}$. Let $\|u\|_\Omega$ be the seminorm in Ω defined by

$$\|u\|_\Omega = \|\phi\|_B + \sup\{\|u_k\|,\ k = 0, ..., m\},$$

where u_k is the restriction of u to $J_k,\ k = 0, ..., m$.
Let us define what we mean by a solution of problem (5) – (8).

Definition 6.1. *A function $u(x,y)\in\Omega$ whose r-derivative exists on J' is said to be a solution of (5) – (8) if $u(x,y)$ satisfies $({}^cD_0^r u)(x,y) = f(x,y,u(x,y))$ on J' and conditions* (6), (7) *and* (8) *are satisfied.*

Set $\mathcal{R}' := \mathcal{R}'_{(\rho_1^-,\rho_2^-)} =$

$$= \{(\rho_1(s,t,u), \rho_2(s,t,u)) : (s,t,u)\in J\times B,\ \rho_i(s,t,u)\leq 0,\ i=1,2\}.$$

We always assume that $\rho_i : J\times B\to\mathbb{R}$, $i=1,2$, are continuous and the function $(s,t)\mapsto u_{(s,t)}$ is continuous from $\mathcal{R}'$ into B.

We will need to introduce the following hypothesis:

(H_ϕ) There exists a continuous bounded function $L : \mathcal{R}'_{(\rho_1^-,\rho_2^-)} \to (0,\infty)$ such that

$$\|\phi_{(s,t)}\|_B \leq L(s,t)\|\phi\|_B, \text{ for any} (s,t)\in\mathcal{R}'.$$

In the sequel we will make use of the following generalization of a consequence of the phase space axioms (Lemma 2.1, [21]).

Lemma 6.2. *If $u\in\Omega$, then*

$$\|u_{(s,t)}\|_B = (M+L')\|\phi\|_B + K \sup_{(\theta,\eta)\in[0,\max\{0,s\}]\times[0,\max\{0,t\}]} \|u(\theta,\eta)\|,$$

where

$$L' = \sup_{(s,t)\in\mathcal{R}'} L(s,t).$$

Theorem 6.3. *Let $f : J\times B\to\mathbb{R}^n$ be an Carathé odory function. Assume that*

(H'1) The function $x_k\in C^1(\mathbb{R}^n,\mathbb{R})$ for $k=1,\ldots,m$. Moreover,

$$0 = x_0(u) < x_1(u) < \ldots < x_m(u) < x_{m+1}(u) = a \quad \textit{for all } u\in\mathbb{R}^n.$$

(H'2) There exists a constant $M'>0$ such that

$$\|f(x,y,u)\| \leq M'(1+\|u\|_B) \textit{ for each } (x,y)\in J \textit{ and each } u\in B.$$

(H'3) For all $(s,t,u) \in J \times \mathbb{R}^n$ *and* $u_{(.,.)} \in B$ *we have*

$$x'_k(u)[\varphi'(s) + \frac{r_1 - 1}{\Gamma(r_1)\Gamma(r_2)} \int_{x_k}^{s} \int_0^t (s-\theta)^{r_1-2}(t-\eta)^{r_2-1} \times$$

$$\times f(\theta, \eta, u_{(\rho_1(\theta,\eta,u_{(\theta,\eta)}),\rho_2(\theta,\eta,u_{(\theta,\eta)}))}) d\eta d\theta] \neq 1, \quad k = 1, \ldots, m.$$

(H'4) For all $u \in \mathbb{R}^n$, $x_k(I_k(u)) \leq x_k(u) < x_{k+1}(I_k(u))$ *for* $k = 1, \ldots, m$.

(H'5) There exists a constant $M^* > 0$ *such that*

$$\|I_k(u)\| \leq M^*(1 + \|u\|_B), \text{ for each } u \in B, \ k = 1, \ldots, m.$$

Then the initial value problem (5) – (8) has at least one solution on $(-\infty, a] \times (-\infty, b]$,

Proof: The proof will be given in several steps.

Step 1: Set $\Omega_0 = \{u : (-\infty, a] \times (-\infty, b] \to \mathbb{R}^n : u_{(x,y)} \in B \text{ for } (x,y) \in E$ and $u \in C(J, \mathbb{R}^n)\}$.

Consider the following problem

$$({}^cD_0^r u)(x,y) = f(x, y, u_{(\rho_1(x,y,u_{(x,y)}),\rho_2(x,y,u_{(x,y)}))}), \text{ if } (x,y) \in J, \tag{19}$$

$$u(x,y) = \phi(x,y), \text{ if } (x,y) \in \tilde{J}, \tag{20}$$

$$u(x,0) = \varphi(x), \ x \in [0,a], \ u(0,y) = \psi(y), \ y \in [0,b]. \tag{21}$$

Transform problem (19) – (21) into a fixed point problem. Consider the operator $N : \Omega_0 \to \Omega_0$ defined by

$$N(u)(x,y) = \begin{cases} \phi(x,y), & (x,y) \in \tilde{J}, \\ \mu(x,y) + & \\ + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} \times & \\ \times f(s,t,u_{(\rho_1(s,t,u_{(s,t)}),\rho_2(s,t,u_{(s,t)}))}) dt ds, & (x,y) \in J. \end{cases} \tag{22}$$

Let $v(.,.) : (-\infty, a] \times (-\infty, b] \to \mathbb{R}^n$ be function defined by

$$v(x,y) = \begin{cases} \phi(x,y), & (x,y) \in \tilde{J}, \\ \mu(x,y), & (x,y) \in J. \end{cases}$$

Then $v_{(x,y)} = \phi$ for all $(x,y) \in E$.

For each $w \in C(J, \mathbb{R}^n)$ with $w(0,0) = 0$, we denote by $\overline{w}$ the function defined by

$$\overline{w}(x,y) = \begin{cases} 0, & (x,y) \in \tilde{J}, \\ w(x,y) & (x,y) \in J. \end{cases}$$

If $u(.,.)$ satisfies the integral equation

$$\begin{aligned} u(x,y) \quad &= \quad \mu(x,y) + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} \times \\ &\times f(s,t,u_{(\rho_1(s,t,u_{(s,t)}),\rho_2(s,t,u_{(s,t)}))})dtds, \end{aligned}$$

we can decompose $u(.,.)$ as $u(x,y) = \overline{w}(x,y) + v(x,y)$, $(x,y) \in J$, which implies $u_{(x,y)} = \overline{w}_{(x,y)} + v_{(x,y)}$ for every $(x,y) \in J$, and the function $w(.,.)$ satisfies

$$\begin{aligned} w(x,y) \quad &= \quad \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} \times \\ &\times f(s,t,\overline{w}_{(\rho_1(s,t,u_{(s,t)}),\rho_2(s,t,u_{(s,t)}))} + v_{(\rho_1(s,t,u_{(s,t)}),\rho_2(s,t,u_{(s,t)}))})\, dtds. \end{aligned}$$

Set $C_0 = \{w \in \Omega_0 : \ w(x,y) = 0 \ \text{ for } (x,y) \in E\}$, and let $\|.\|_{(a,b)}$ be the seminorm in C_0 defined by

$$\|w\|_{(a,b)} = \sup_{(x,y)\in E} \|w_{(x,y)}\|_B + \sup_{(x,y)\in J} \|w(x,y)\| = \sup_{(x,y)\in J} \|w(x,y)\|, \ w \in C_0.$$

C_0 is a Banach space with norm $\|.\|_{(a,b)}$.

Let the operator $P: \ C_0 \to C_0$ be defined by

$$P(w)(x,y) = \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} \times \qquad (23)$$

$$\times f(s,t,\overline{w}_{(\rho_1(s,t,u_{(s,t)}),\rho_2(s,t,u_{(s,t)}))} + v_{(\rho_1(s,t,u_{(s,t)}),\rho_2(s,t,u_{(s,t)}))})dtds, \quad (x,y) \in J.$$

That the property "operator N has a fixed point"is equivalent to property "operator P has a fixed point", and so we turn to proving that P has a fixed point.

We shall use Schaefer's fixed point theorem to prove that P has fixed point.

We shall show that the operator P is continuous and completely continuous.

Claim 1: P is continuous.

Let$\{w_n\}$ be a sequence such that $w_n \to w$ in C_0. Then

$$\begin{aligned}
& \|P(w_n)(x,y) - P(w)(x,y)\| \leq \\
& \leq \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^a \int_0^b (x-s)^{r_1-1}(y-t)^{r_2-1} \times \\
\times \quad & \|f(s,t,\overline{w}_{n(\rho_1(s,t,u_{(s,t)}),\rho_2(s,t,u_{(s,t)}))} + v_{n(\rho_1(s,t,u_{(s,t)}),\rho_2(s,t,u_{(s,t)}))}) - \\
- \quad & f(s,t,\overline{w}_{(\rho_1(s,t,u_{(s,t)}),\rho_2(s,t,u_{(s,t)}))} + v_{(\rho_1(s,t,u_{(s,t)}),\rho_2(s,t,u_{(s,t)}))})\| \, dtds.
\end{aligned}$$

Since f is a Carathéodory function, then we have

$$\|P(w_n) - P(w)\|_\infty \to 0 \text{ as } n \to \infty.$$

Claim 2: P maps bounded sets into bounded sets in C_0.

Indeed, it is enough show that for any $\eta > 0$ there exists a positive constant $\widetilde{\ell}$ such that for each $w \in B_\eta = \{w \in C_0 : \|w\|_{(a,b)} \leq \eta\}$, we have $\|P(w)\|_\infty \leq \widetilde{\ell}$.

Lemma 6.2 implies that

$$\begin{aligned}
& \|\overline{w}_{(\rho_1(s,t,u_{(s,t)}),\rho_2(s,t,u_{(s,t)}))} + v_{(\rho_1(s,t,u_{(s,t)}),\rho_2(s,t,u_{(s,t)}))}\|_B \leq \\
& \leq \|\overline{w}_{(\rho_1(s,t,u_{(s,t)}),\rho_2(s,t,u_{(s,t)}))}\|_B + \|v_{(\rho_1(s,t,u_{(s,t)}),\rho_2(s,t,u_{(s,t)}))}\|_B \leq \\
& \leq K\eta + K\|\phi(0,0)\| + (M + L')\|\phi\|_B.
\end{aligned}$$

Set $\eta^* := K\eta + K\|\phi(0,0)\| + (M + L')\|\phi\|_B$.

Let $w \in B_\eta$. By $(H'2)$ for each $(x,y) \in J$ we have

$$\begin{aligned}
& \|P(w)(x,y)\| \leq \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} \times \\
& \times \|f(s,t,\overline{w}_{(\rho_1(s,t,u_{(s,t)}),\rho_2(s,t,u_{(s,t)}))} + v_{(\rho_1(s,t,u_{(s,t)}),\rho_2(s,t,u_{(s,t)}))})\| dtds \leq \\
& \leq \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} \times \\
& \times M'(1 + \|\overline{w}_{(\rho_1(s,t,u_{(s,t)}),\rho_2(s,t,u_{(s,t)}))} + v_{(\rho_1(s,t,u_{(s,t)}),\rho_2(s,t,u_{(s,t)}))}\|_B) \, dtds \leq \\
& \leq \frac{M'^*)}{\Gamma(r_1)\Gamma(r_2)} \int_0^a \int_0^b (x-s)^{r_1-1}(y-t)^{r_2-1} \, dtds \leq \\
& \leq \frac{M'^{r_1} b^{r_2}(1+\eta^*)}{\Gamma(r_1+1)\Gamma(r_2+1)} := \ell^*.
\end{aligned}$$

Hence $\|P(w)\|_\infty \le \ell^*$.

Claim 3: P maps bounded sets into equicontinuous sets in C_0.

Let $(x_1,y_1),(x_2,y_2) \in (0,a]\times(0,b]$, $x_1<x_2$, $y_1<y_2$, B_η be a bounded set as in Claim 2, and let $w \in B_\eta$. Then

$$\begin{aligned}
&\|P(w)(x_2,y_2)-P(w)(x_1,y_1)\| \le \\
&\le \frac{1}{\Gamma(r_1)\Gamma(r_2)}\|\int_0^{x_1}\int_0^{y_1}[(x_2-s)^{r_1-1}(y_2-t)^{r_2-1}- \\
&-(x_1-s)^{r_1-1}(y_1-t)^{r_2-1}]f(s,t,u_{(\rho_1(s,t,u_{(s,t)}),\rho_2(s,t,u_{(s,t)}))})dtds+ \\
&+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{x_1}^{x_2}\int_{y_1}^{y_2}(x_2-s)^{r_1-1}(y_2-t)^{r_2-1}\times \\
&\times f(s,t,\overline{w}_{(\rho_1(s,t,u_{(s,t)}),\rho_2(s,t,u_{(s,t)}))}+v_{(\rho_1(s,t,u_{(s,t)}),\rho_2(s,t,u_{(s,t)}))})\,dtds+ \\
&+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^{x_1}\int_{y_1}^{y_2}(x_2-s)^{r_1-1}(y_2-t)^{r_2-1}\times \\
&\times f(s,t,\overline{w}_{(\rho_1(s,t,u_{(s,t)}),\rho_2(s,t,u_{(s,t)}))}+v_{(\rho_1(s,t,u_{(s,t)}),\rho_2(s,t,u_{(s,t)}))})\,dtds+ \\
&+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{x_1}^{x_2}\int_0^{y_1}(x_2-s)^{r_1-1}(y_2-t)^{r_2-1}\times \\
&\times f(s,t,\overline{w}_{(\rho_1(s,t,u_{(s,t)}),\rho_2(s,t,u_{(s,t)}))}+v_{(\rho_1(s,t,u_{(s,t)}),\rho_2(s,t,u_{(s,t)}))})dtds\| \le \\
&\le \frac{M'(1+\eta)}{\Gamma(r_1)\Gamma(r_2)}\int_0^{x_1}\int_0^{y_1}[(x_1-s)^{r_1-1}(y_1-t)^{r_2-1}- \\
&-(x_2-s)^{r_1-1}(y_2-t)^{r_2-1}]\,dtds+ \\
&+\frac{M'(1+\eta)}{\Gamma(r_1)\Gamma(r_2)}\int_{x_1}^{x_2}\int_{y_1}^{y_2}(x_2-s)^{r_1-1}(y_2-t)^{r_2-1}\,dtds+ \\
&+\frac{M'(1+\eta)}{\Gamma(r_1)\Gamma(r_2)}\int_0^{x_1}\int_{y_1}^{y_2}(x_2-s)^{r_1-1}(y_2-t)^{r_2-1}\,dtds+ \\
&+\frac{M'(1+\eta)}{\Gamma(r_1)\Gamma(r_2)}\int_{x_1}^{x_2}\int_0^{y_1}(x_2-s)^{r_1-1}(y_2-t)^{r_2-1}\,dtds \le \\
&\le \frac{M'(1+\eta)}{\Gamma(r_1+1)\Gamma(r_2+1)}[2y_2^{r_2}(x_2-x_1)^{r_1}+2x_2^{r_1}(y_2-y_1)^{r_2}+ \\
&+x_1^{r_1}y_1^{r_2}-x_2^{r_1}y_2^{r_2}-2(x_2-x_1)^{r_1}(y_2-y_1)^{r_2}].
\end{aligned}$$

As $x_1 \to x_2$, $y_1 \to y_2$ the right-hand side of the above inequality tends to zero. The equicontinuity for the cases $x_1<x_2<0$, $y_1<y_2<0$ and $x_1 \le 0 \le x_2$, $y_1 \le 0 \le y_2$ is obvious.

As a consequence of Claims 1 – 3 together with the Arzela-Ascoli theorem, we can conclude that $P : C_0 \to C_0$ is continuous and completely continuous.

Claim 4: (A priori bounds)

Now it remains to show that the set $\mathcal{F} = \{u \in C_0 : u = \lambda P(u)$ for some $0 < \lambda < 1\}$ is bounded.

Let $u \in \mathcal{F}$, then $u = \lambda P(u)$ for some $0 < \lambda < 1$. Thus, for each $(x, y) \in J$ we have

$$w(x,y) = \frac{\lambda}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} \times$$

$$\times f(s,t,u_{(\rho_1(s,t,u_{(s,t)}),\rho_2(s,t,u_{(s,t)}))})\, dtds.$$

This implies by $(H'2)$ that for each $(x, y) \in J$ we have

$$\|w(x,y)\| \leq \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} \times$$

$$\times M'[1 + \|\overline{w}_{(\rho_1(s,t,u_{(s,t)}),\rho_2(s,t,u_{(s,t)}))} + v_{(\rho_1(s,t,u_{(s,t)}),\rho_2(s,t,u_{(s,t)}))}\|_B]\, dtds.$$

But

$$\begin{aligned} &\|\overline{w}_{(\rho_1(s,t,u_{(s,t)}),\rho_2(s,t,u_{(s,t)}))} + v_{(\rho_1(s,t,u_{(s,t)}),\rho_2(s,t,u_{(s,t)}))}\|_B \leq \\ \leq\ &\|\overline{w}_{(\rho_1(s,t,u_{(s,t)}),\rho_2(s,t,u_{(s,t)}))}\|_B + \|v_{(\rho_1(s,t,u_{(s,t)}),\rho_2(s,t,u_{(s,t)}))}\|_B \leq \\ \leq\ &K \sup\{w(\tilde{s},\tilde{t}) : (\tilde{s},\tilde{t}) \in [0,s] \times [0,t]\} + \\ &+(M + L')\|\phi\|_B + K\|\phi(0,0)\|. \end{aligned} \tag{24}$$

If we name $z(s, t)$ the right hand side of (24), then we have

$$\|\overline{w}_{(\rho_1(s,t,u_{(s,t)}),\rho_2(s,t,u_{(s,t)}))} + v_{(\rho_1(s,t,u_{(s,t)}),\rho_2(s,t,u_{(s,t)}))}\|_B \leq z(s,t),$$

and therefore for each $(x, y) \in J$ we obtain

$$\|w(x,y)\| \leq \frac{M'}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1}(1 + z(s,t))dtds. \tag{25}$$

Using the above inequality and the definition of z, for each $(x, y) \in J$ we have

$$\begin{aligned} z(x,y) \ &\leq\ (M + L')\|\phi\|_B + K\|\phi(0,0)\| + \\ &+\ \frac{KM'}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1}(1 + z(s,t))dtds. \end{aligned}$$

Then for Lemma 2.3 there exists $\delta = \delta(r_1, r_2)$ such that we have

$$\|z\|_\infty \quad \leq \quad R[1 + \frac{K\delta M'^{r_1} b^{r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)}] := M^*,$$

where

$$R = (M + L')\|\phi\|_B + K\|\phi(0,0)\| + \frac{KM'^{r_1} b^{r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)}.$$

Then (25) implies that

$$\|w\|_\infty \leq \frac{KM'^{r_1} b^{r_2}(1 + \widetilde{M})}{\Gamma(r_1+1)\Gamma(r_2+1)} := M^*.$$

This shows that the set $\mathcal{F}$ is bounded. As a consequence of Schaefer's fixed point theorem (Theorem 2.2), we deduce that P has a fixed point u which is a solution to problem (19) – (21). Denote this solution by u_1.

Define the functions $r_{k,1}(x,y) = x_k(u_1(x,y)) - x$ for $x \geq 0, y \geq 0$.

Hypothesis (H'1) implies that $r_{k,1}(0,0) \neq 0$ for $k = 1, \ldots, m$.

If $r_{k,1}(x,y) \neq 0$ on J for $k = 1, \ldots, m$, i.e.,

$$x \neq x_k(u_1(x,y)) \quad \text{on } J \quad \text{for } k = 1, \ldots, m,$$

then u_1 is a solution of the problem (5) – (8).

It remains to consider the case when $r_{1,1}(x,y) = 0$ for some $(x,y) \in J$. Now since $r_{1,1}(0,0) \neq 0$ and $r_{1,1}$ is continuous, there exists $x_1 > 0, y_1 > 0$ such that $r_{1,1}(x_1, y_1) = 0$ and $r_{1,1}(x,y) \neq 0$ for all $(x,y) \in [0, x_1) \times [0, y_1)$.

Thus, we have

$$r_{1,1}(x_1, y_1) = 0 \text{ and } r_{1,1}(x,y) \neq 0 \text{ for all } (x,y) \in [0, x_1) \times [0, y_1] \cup (y_1, b].$$

Suppose that there exist $(\bar{x}, \bar{y}) \in [0, x_1) \times [0, y_1] \cup (y_1, b]$ such that $r_{1,1}(\bar{x}, \bar{y}) = 0$. The function $r_{1,1}$ attains a maximum at some point $(s,t) \in [0, x_1) \times [0, b]$. Since

$$({}^cD_0^r u_1)(x,y) = f(x, y, u_{1(\rho_1(x,y,u_{(x,y)}), \rho_2(x,y,u_{(x,y)}))}) \text{ for} (x,y) \in J,$$

then

$$\frac{\partial u_1(x,y)}{\partial x} \quad \text{exists, and} \quad \frac{\partial r_{1,1}(s,t)}{\partial x} = x_1'(u_1(s,t)) \frac{\partial u_1(s,t)}{\partial x} - 1 = 0.$$

Since

$$\frac{\partial u_1(x,y)}{\partial x} = \varphi'(x) + \frac{r_1-1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-2}(y-t)^{r_2-1} \times$$
$$\times f(s,t,u_{1(\rho_1(s,t,u_{(s,t)}),\rho_2(s,t,u_{(s,t)}))})dtds,$$

then

$$x_1'(u_1(s,t))[\varphi'(s) + \frac{r_1-1}{\Gamma(r_1)\Gamma(r_2)} \int_0^s \int_0^t (s-\theta)^{r_1-2}(t-\eta)^{r_2-1} \times$$
$$\times f(\theta,\eta,u_{1(\rho_1(\theta,\eta,u_{(\theta,\eta)}),\rho_2(\theta,\eta,u_{(\theta,\eta)}))})d\eta d\theta] = 1,$$

witch contradicts (H'3).

From (H'1) we have

$$r_{k,1}(x,y) \neq 0 \text{ for all } (x,y) \in [0,x_1) \times [0,b] \text{ and } k = 1,...m.$$

Step 2: In what follows set

$$\begin{aligned}\Omega_k = \{ & u : (-\infty,a] \times (-\infty,b] \to \mathbb{R}^n : u_{(x,y)} \in B \text{ for } (x,y) \in E \ \text{ and} \\ & \text{there exist } 0 = x_0 < x_1 < x_2 < ... < x_m < x_{m+1} = a \text{ such that} \\ & x_k = x_k(u(x_k,.)), \text{ and } u(x_k^-,.),\ u(x_k^+,.) \text{ exist with} \\ & u(x_k^-,.) = u(x_k,.);\ k = 1,\ldots,m, \text{ and } u \in C(X_k,\mathbb{R}^n), k = 0,\ldots,m\},\end{aligned}$$

where $X_k := [x_k,a] \times [0,b],\ k = 1,...,m.$

Consider now the problem

$$({}^cD_0^r u)(x,y) = f(x,y,u_{(\rho_1(x,y,u_{(x,y)}),\rho_2(x,y,u_{(x,y)}))}), \text{ if } (x,y) \in X_1, \tag{26}$$
$$u(x_1^+,y) = I_1(u_1(x_1,y)), \tag{27}$$
$$u(x,y) = u_1(x,y), \text{ if } (x,y) \in \tilde{J} \cup [0,x_1) \times [0,b]. \tag{28}$$

Consider the operator $N_1 : \Omega_1 \to \Omega_1$ defined as

$$N_1(u)(x,y) = \begin{cases} u_1(x,y), & (x,y) \in \tilde{J} \cup [0,x_1) \times [0,b], \\ \varphi(x) + I_1(u_1(x_1,y)) - I_1(u_1(x_1,0)) + & \\ +\frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{x_1}^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} \times & \\ \times f(s,t,u_{(\rho_1(s,t,u_{(s,t)}),\rho_2(s,t,u_{(s,t)}))})\, dtds, & (x,y) \in X_1. \end{cases}$$

As in Step 1 we can show that N_1 is completely continuous. Now it remains to show that the set $\mathcal{F}^* = \{u \in C(X_1,\mathbb{R}^n) : u = \lambda N_1(u)$ for some $0 < \lambda < 1\}$ is bounded.

Let $u \in \mathcal{F}^*$, then $u = \lambda N_1(u)$ for some $0 < \lambda < 1$. Thus, from (H'2) and (H'5) we get for each $(x,y) \in X_1$,

$$\begin{aligned}
\|w(x,y)\| &\leq \|\varphi(x)\| + \|I_1(u_1(x_1,y))\| + \|I_1(u_1(x_1,0))\| + \\
&+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{x_1}^{x} \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} \times \\
&\times \|f(s,t,u_{(\rho_1(s,t,u_{(s,t)}),\rho_2(s,t,u_{(s,t)}))})\|\, dtds \leq \\
&\leq \|\varphi\|_\infty + 2M^*(1+\|u_1\|) + \\
&+ \frac{M}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1}(1+\|z(s,t)\|)\, dtds.
\end{aligned}$$

Set

$$C^* = \|\varphi\|_\infty + 2M^*(1+\|u_1\|) + \frac{KM'^{r_1}b^{r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)}.$$

Then Lemma 2.3 implies that there exists $\delta = \delta(r_1, r_2) > 0$ such that for each $(x,y) \in X_1$,

$$\|w(x,y)\| \leq C^*[1 + \delta \frac{KM'^*)a^{r_1}b^{r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)}] := R^*.$$

This shows that the set $\mathcal{F}^*$ is bounded.

As a consequence of Schaefer's fixed point theorem (Theorem 2.2), we deduce that N_1 has a fixed point u which is a solution to problem (26) – (28). Denote this solution by u_2.

Define $r_{k,2}(x,y) = x_k(u_2(x,y)) - x$, for $(x,y) \in X_1$. If $r_{k,2}(x,y) \neq 0$ on $(x_1, a] \times [0,b]$ and for all $k = 1, \ldots, m$, then

$$u(x,y) = \begin{cases} u_1(x,y), & \text{if } (x,y) \in \tilde{J} \cup [0,x_1) \times [0,b], \\ u_2(x,y), & \text{if } (x,y) \in [x_1,a] \times [0,b], \end{cases}$$

is a solution of the problem (5) – (8).

It remains to consider the case when $r_{2,2}(x,y) = 0$ for some $(x,y) \in (x_1,a] \times [0,b]$. By (H4) we have

$$\begin{aligned}
r_{2,2}(x_1^+, y_1) = x_2(u_2(x_1^+, y_1) - x_1 = x_2(I_1(u_1(x_1,y_1))) - x_1 > \\
> x_1(u_1(x_1,y_1)) - x_1 = r_{1,1}(x_1,y_1) = 0.
\end{aligned}$$

Since $r_{2,2}$ is continuous, there exists $x_2 > x_1$, $y_2 > y_1$ such that $r_{2,2}(x_2,y_2) = 0$, and $r_{2,2}(x,y) \neq 0$ for all $(x,y) \in (x_1,x_2) \times [0,b]$.

It is clear by (H1) that

$$r_{k,2}(x,y) \neq 0 \quad \text{for all } (x,y) \in (x_1,x_2) \times [0,b],\ k = 2,\dots,m.$$

Now suppose that there are $(s,t) \in (x_1,x_2) \times [0,b]$ such that $r_{1,2}(s,t) = 0$. From (H4) it follows that

$$\begin{aligned} r_{1,2}(x_1^+, y_1) = x_1(u_2(x_1^+, y_1) - x_1 = x_1(I_1(u_1(x_1,y_1))) - x_1 \leq \\ \leq x_1(u_1(x_1,y_1)) - x_1 = r_{1,1}(x_1,y_1) = 0. \end{aligned}$$

Thus $r_{1,2}$ attains a nonnegative maximum at some point $(s_1,t_1) \in (x_1,a) \times [0,x_2) \cup (x_2,b]$. Since

$$({}^cD_0^r u_2)(x,y) = f(x,y,u_{2(\rho_1(x,y,u_{(x,y)}),\rho_2(x,y,u_{(x,y)}))}), \quad \text{for } (x,y) \in X_1,$$

then we get

$$\begin{aligned} u_2(x,y) &= \varphi(x) + I_1(u_1(x_1,y)) - I_1(u_1(x_1,0)) + \\ &+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{x_1}^{x} \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} \times \\ &\times f(s,t,u_{2(\rho_1(s,t,u_{(s,t)}),\rho_2(s,t,u_{(s,t)}))})\, dtds, \end{aligned}$$

hence

$$\begin{aligned} \frac{\partial u_2}{\partial x}(x,y) &= \varphi'(x) + \frac{r_1 - 1}{\Gamma(r_1)\Gamma(r_2)} \int_{x_1}^{x} \int_0^y (x-s)^{r_1-2}(y-t)^{r_2-1} \times \\ &\times f(s,t,u_{2(\rho_1(s,t,u_{(s,t)}),\rho_2(s,t,u_{(s,t)}))}) dtds, \end{aligned}$$

then

$$\frac{\partial r_{1,2}(s_1,t_1)}{\partial x} = x_1'(u_2(s_1,t_1)) \frac{\partial u_2}{\partial x}(s_1,t_1) - 1 = 0.$$

Therefore

$$x_1'(u_2(s_1,t_1))[\varphi'(s_1) + \frac{r_1 - 1}{\Gamma(r_1)\Gamma(r_2)} \int_{x_1}^{s_1} \int_0^{t_1} (s_1 - \theta)^{r_1-2}(t_1 - \eta)^{r_2-1} \times$$

$$\times f(\theta,\eta,u_{2(\rho_1(\theta,\eta,u_{(\theta,\eta)}),\rho_2(\theta,\eta,u_{(\theta,\eta)}))}) d\eta d\theta] = 1,$$

which contradicts (H'3).

Step 3: We continue this process and take into account that $u_{m+1} := u\Big|_{X_m}$ is a solution to the problem

$$\begin{cases} ({}^cD_0^r u)(x,y) = f(x,y,u_{(\rho_1(x,y,u_{(x,y)}),\rho_2(x,y,u_{(x,y)}))}), \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{a.e. } (x,y) \in (x_m,a]\times[0,b], \\ u(x_m^+,y) = I_m(u_{m-1}(x_m,y)), \\ u(x,y) = u_1(x,y), \text{ if } (x,y) \in \tilde{J}\cup[0,x_1)\times[0,b], \\ u(x,y) = u_2(x,y), \text{ if } (x,y) \in [x_1,x_2)\times[0,b], \\ ... \\ u(x,y) = u_m(x,y), \text{ if } (x,y) \in [x_{m-1},x_m)\times[0,b]. \end{cases}$$

Then solution $u(x,y)$ of the problem (5) – (8) is defined by

$$u(x,y) = \begin{cases} u_1(x,y), & \text{if } (x,y) \in \tilde{J}\cup[0,x_1]\times[0,b], \\ u_2(x,y), & \text{if } (x,y) \in (x_1,x_2]\times[0,b], \\ \dots \\ u_{m+1}(x,y), & \text{if } (x,y) \in (x_m,a]\times[0,b]. \end{cases}$$

7 Examples

Example 1: As an application of our results we consider the following impulsive partial hyperbolic functional differential equations of the form

$$({}^cD_0^r u)(x,y) = \frac{1+|u(x-\sigma_1(u(x,y)),y-\sigma_2(u(x,y)))|}{9+e^{x+y}},$$

$$\text{if } (x,y) \in J := [0,1]\times[0,1],\ x \neq x_k(u(x,y)),\ k=1,...,m, \tag{29}$$

$$u(x_k^+,y) = d_k u(x_k,y),\ y \in [0,1], \tag{30}$$

$$u(x,0) = x,\ x\in[0,1],\ u(0,y) = y^2,\ y\in[0,1], \tag{31}$$

$$u(x,y) = x+y^2,\ (x,y) \in [-1,1]\times[-2,1]\backslash(0,1]\times(0,1], \tag{32}$$

where $r=(r_1,r_2),\ 0<r_1,r_2\leq 1,\ x_k(u) = 1-\frac{1}{2^k(1+u^2)},\ k=1,...m,$ and $\frac{\sqrt{2}}{2} < d_k \leq 1,\ k=1,...m,\ \sigma_1 \in C(\mathbb{R},[0,1]),\ \sigma_2 \in C(\mathbb{R},[0,2])$. Set

$$\rho_1(x,y,\varphi) = x-\sigma_1(\varphi(0,0)),\quad (x,y,\varphi) \in J\times C,$$

$$\rho_2(x,y,\varphi) = y-\sigma_2(\varphi(0,0)),\quad (x,y,\varphi) \in J\times C,$$

where $C := C_{(1,2)}$.

Set

$$f(x,y,\varphi) = \frac{1+|\varphi|}{9+e^{x+y}},\ (x,y) \in [0,1] \times [0,1],\ \varphi \in C,$$

and

$$I_k(u) = d_k u,\ u \in \mathbb{R},\ k = 1,...,m.$$

Let $u \in \mathbb{R}$. Then we have

$$x_{k+1}(u) - x_k(u) = \frac{1}{2^{k+1}(1+u^2)} > 0,\ k = 1,...,m.$$

Hence $0 < x_1(u) < x_2(u) < ... < x_m(u) < 1$ for each $u \in \mathbb{R}$. Also, for each $u \in \mathbb{R}$ we have

$$x_{k+1}(I_k(u)) - x_k(u) = \frac{1+(2d_k^2-1)u^2}{2^{k+1}(1+u^2)(1+d_k^2)} > 0.$$

Finally, for all $(x,y) \in J$ and each $u \in \mathbb{R}$ we get

$$|I_k(u)| = |d_k u| \le |u| \le 3(1+|u|),\ k = 1,...,m,$$

and

$$|f(x,y,u)| = \frac{|1+u|}{9+e^{x+y}} \le \frac{1}{10}(1+|u|).$$

Since all conditions of Theorem 4.2 are satisfied, problem (29) – (32) has at least one solution on $[-1,1] \times [-2,1]$.

Example 2: As an application of our results we consider the following impulsive partial hyperbolic differential equations of the form

$$({}^cD_0^r u)(x,y) = \frac{1+|u(x-\sigma_1(u(x,y)), y-\sigma_2(u(x,y)))|}{9+e^{x+y}},$$

$$\text{if } (x,y) \in J,\ x \ne x_k(u(x,y)),\ k = 1,\dots,m, \tag{33}$$

$$u(x_k^+, y) = d_k u(x_k, y),\ y \in [0,1],\ k = 1,\dots,m, \tag{34}$$

$$u(x,y) = x + y^2,\ (x,y) \in \tilde{J} := (-\infty,1] \times (-\infty,1] \backslash (0,1] \times (0,1], \tag{35}$$

$$u(x,0) = x,\ x \in [0,1],\ u(0,y) = y^2,\ y \in [0,1], \tag{36}$$

where $J = [0,1] \times [0,1]$, $r = (r_1, r_2)$, $0 < r_1, r_2 \le 1$, $x_k(u) = 1 - \frac{1}{2^k(1+u^2)}$, $k = 1,\dots,m$, and $\frac{\sqrt{2}}{2} < d_k \le 1$, $k = 1,\dots,m$, and $\sigma_1, \sigma_2 \in C(\mathbb{R}, [0,\infty))$.

Let

$$C_\gamma = \{u : (-\infty, 0] \times (-\infty, 0] \to \mathbb{R} : \lim_{\|(\theta,\eta)\|\to\infty} e^{\gamma(\theta+\eta)} u(\theta,\eta) \text{ exists in } \mathbb{R}\}.$$

The norm of C_γ is given by

$$\|u\|_\gamma = \sup_{(\theta,\eta)\in(-\infty,0]\times(-\infty,0]} e^{\gamma(\theta+\eta)}|u(\theta,\eta)|.$$

Let $E := [0,1]\times\{0\}\cup\{0\}\times[0,1]$, and $u : (-\infty,1]\times(-\infty,1]\to\mathbb{R}$ such that $u_{(x,y)} \in C_\gamma$ for $(x,y)\in E$, then

$$\lim_{\|(\theta,\eta)\|\to\infty} e^{\gamma(\theta+\eta)} u_{(x,y)}(\theta,\eta) = \lim_{\|(\theta,\eta)\|\to\infty} e^{\gamma(\theta-x+\eta-y)} u(\theta,\eta) =$$

$$= e^{\gamma(x+y)} \lim_{\|(\theta,\eta)\|\to\infty} u(\theta,\eta) < \infty.$$

Hence $u_{(x,y)} \in C_\gamma$.

Finally we prove that

$$\|u_{(x,y)}\|_\gamma = K\sup\{|u(s,t)| : (s,t)\in[0,x]\times[0,y]\}+$$

$$+M\sup\{\|u_{(s,t)}\|_\gamma : (s,t)\in E_{(x,y)}\},$$

where $K = M = 1$ and $H = 1$.

If $x+\theta\le 0,\ y+\eta\le 0$ we get

$$\|u_{(x,y)}\|_\gamma = \sup\{|u(s,t)| : (s,t)\in(-\infty,0]\times(-\infty,0]\},$$

and if $x+\theta\ge 0,\ y+\eta\ge 0$ then we have

$$\|u_{(x,y)}\|_\gamma = \sup\{|u(s,t)| : (s,t)\in[0,x]\times[0,y]\}.$$

Thus for all $(x+\theta, y+\eta)\in[0,1]\times[0,1]$ we get

$$\|u_{(x,y)}\|_\gamma = \sup_{(s,t)\in(-\infty,0]\times(-\infty,0]} |u(s,t)| + \sup_{(s,t)\in[0,x]\times[0,y]} |u(s,t)|.$$

Then

$$\|u_{(x,y)}\|_\gamma = \sup_{(s,t)\in E} \|u_{(s,t)}\|_\gamma + \sup_{(s,t)\in[0,x]\times[0,y]} |u(s,t)|.$$

$(C_\gamma, \|.\|_\gamma)$ is a Banach space. We conclude that C_γ is a phase space.

Set

$$\rho_1(x,y,\varphi) = x - \sigma_1(\varphi(0,0)), \quad (x,y,\varphi) \in J \times C_\gamma,$$

$$\rho_2(x,y,\varphi) = y - \sigma_2(\varphi(0,0)), \quad (x,y,\varphi) \in J \times C_\gamma,$$

$$f(x,y,\varphi) = \frac{1+|\varphi|}{9+e^{x+y}}, \ (x,y) \in [0,1] \times [0,1], \ u_{(x,y)} \in C_\gamma,$$

and

$$I_k(u) = d_k u, \quad u \in \mathbb{R}, \ k = 1, \dots, m.$$

Let $u \in \mathbb{R}$. Then we have

$$x_{k+1}(u) - x_k(u) = \frac{1}{2^{k+1}(1+u^2)} > 0, \quad k = 1, \dots, m.$$

Hence $0 < x_1(u) < x_2(u) < \ldots < x_m(u) < 1$ for each $u \in \mathbb{R}$.
Also, for each $u \in \mathbb{R}$ we have

$$x_{k+1}(I_k(u)) - x_k(u) = \frac{1+(2d_k^2-1)u^2}{2^{k+1}(1+u^2)(1+d_k^2)} > 0.$$

Finally, for all $(x,y) \in J$ and every $u \in \mathbb{R}$ we get

$$|I_k(u)| = |d_k u| \leq |u| \leq 3(1+|u|), \quad k = 1, \dots, m,$$

and

$$|f(x,y,u)| = \frac{|1+u|}{9+e^{x+y}} \leq \frac{1}{10}(1+|u|).$$

Since all conditions of Theorem 6.3 are satisfied, problem (33) – (36) has at least one solution on $(-\infty, 1] \times (-\infty, 1]$.

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FRACTIONAL ORDER IMPULSIVE PARTIAL HYPERBOLIC FUNCTIONAL DIFFERENTIAL EQUATIONS WITH VARIABLES TIMES AND STATE-DEPENDENT DELAY

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In this paper we study the existence and uniqueness of solutions of two classes of partial impulsive hyperbolic differential equations with variable time impulses and state-dependent delay involving the Caputo fractional derivative. Our works will be considered by using suitable fixed point theorems.