ON σ-CONVEX SUBSETS IN SPACES OF SCATTEREDLY CONTINUOUS FUNCTIONS

©*2012 Taras BANAKH, Bohdan BOKALO, Nadiya KOLOS*

Ivan Franko National University,
1 Universytets'ka Str., Lviv 79602, Ukraine

e-mail: *tbanakh@yahoo.com, bogdanbokalo@mail.ru, nadiya_kolos@ukr.net*

Received 6 April 2012

We prove that for any topological space X of countable tightness, each σ-convex subspace $\mathcal{F}$ of the space $SC_p(X)$ of scatteredly continuous real-valued functions on X has network weight $nw(\mathcal{F}) \le nw(X)$. This implies that for a metrizable separable space X, each compact convex subset in the function space $SC_p(X)$ is metrizable. Another corollary says that two Tychonoff spaces X, Y with countable tightness and topologically isomorphic linear topological spaces $SC_p(X)$ and $SC_p(Y)$ have the same network weight $nw(X) = nw(Y)$. Also we prove that each zero-dimensional separable Rosenthal compact space is homeomorphic to a compact subset of the function space $SC_p(\omega^\omega)$ over the space ω^ω of irrationals.

This paper was motivated by the problem of studying the linear-topological structure of the space $SC_p(X)$ of scatteredly continuous real-valued functions on a topological space X, addressed in [1, 2].

A function $f : X \to Y$ between two topological spaces is called *scatteredly continuous* if for each non-empty subspace $A \subset X$ the restriction $f|A : A \to Y$ has a point of continuity. Scatteredly continuous functions were introduced in [3] (as almost continuous functions) and studied

УДК: 517.98, 517.51, 515.12; MSC 2010: 46A55, 46E99, 54C35
Key words and phrases: scatteredly continuous map, weakly discontinuous map, σ-convex subset, network weight

in details in [4], [5] and [6]. If a topological space Y is regular, then the scattered continuity of a function $f : X \to Y$ is equivalent to the weak discontinuity of f; see [3], [5, 4.4]. We recall that a function $f : X \to Y$ is *weakly discontinuous* if each subspace $A \subset X$ contains an open dense subspace $U \subset A$ such that the restriction $f|U : U \to Y$ is continuous.

For a topological space X by $SC_p(X) \subset \mathbb{R}^X$ we denote the linear space of all scatteredly continuous (equivalently, weakly discontinuous) functions on X, endowed with the topology of pointwise convergence. It is clear that the space $SC_p(X)$ contains the linear subspace $C_p(X)$ of all continuous real-valued functions on X. Topological properties of the function spaces $C_p(X)$ were intensively studied by topologists, see [7]. In particular, they studied the interplay between topological invariants of topological space X and its function space $C_p(X)$.

Let us recall [8, 9] that for a topological space X its

- *weight* $w(X)$ is the smallest cardinality of a base of the topology of X;
- *network weight* $w(X)$ is the smallest cardinality of a network of the topology of X;
- *tightness* $t(X)$ is the smallest infinite cardinal κ such that for each subset $A \subset X$ and a point $a \in \bar{A}$ in its closure there is a subset $B \subset A$ of cardinality $|B| \le \kappa$ such that $a \in \bar{B}$;
- *Lindelöf number* $l(X)$ is the smallest infinite cardinal κ such that each open cover of X has a subcover of cardinality $\le \kappa$;
- *hereditary Lindelöf number* $hl(X) = \sup\{l(Z) : Z \subset X\}$;
- *density* $d(X)$ if the smallest cardinality of a dense subset of X;
- *the hereditary density* $hd(X) = \sup\{d(Z) : Z \subset X\}$;
- *spread* $s(X) = \sup\{|D| : D \text{ is a discrete subspace of } X\}$.

By [7, §I.1], for each Tychonoff space X the function space $C_p(X)$ has weight $w(C_p(X)) = |X|$ and network weight $nw(SC_p(X)) = nw(X)$. For the function space $SC_p(X)$ the situation is a bit different.

Proposition 1. *For any T_1-space X we have*

$$s\big(SC_p(X)\big) = nw\big(SC_p(X)\big) = w\big(SC_p(X)\big) = |X|.$$

Proof. It is clear that $s\big(SC_p(X)\big) \le nw\big(SC_p(X)\big) \le w\big(SC_p(X)\big) \le w(\mathbb{R}^X) = |X|$. To see that $|X| \le s\big(SC_p(X)\big)$, observe that for each point $a \in X$ the characteristic function

$$\delta_a : X \to \mathbb{R} = \begin{cases} 1, & \text{if } x = a \\ 0, & \text{otherwise} \end{cases}$$

of the singleton $\{a\}$ is scatteredly continuous, and the subspace $\mathcal{D} = \{\delta_a : a \in X\} \subset SC_p(X)$ has cardinality $|X|$ and is discrete in $SC_p(X)$. □

The deviation of a subset $\mathcal{F} \subset SC_p(X)$ from being a subset of $C_p(X)$ can be measured with help of the cardinal number $\mathrm{dec}(\mathcal{F})$ called the *decomposition number* of $\mathcal{F}$. It is defined as the smallest cardinality $|\mathcal{C}|$ of a cover $\mathcal{C}$ of X such that for each $C \in \mathcal{C}$ and $f \in \mathcal{F}$ the restriction $f|C$ is continuous. If the function family $\mathcal{F}$ consists of a single function f, then the decomposition number $\mathrm{dec}(\mathcal{F}) = \mathrm{dec}(\{f\})$ coincides with the decomposition number $\mathrm{dec}(f)$ of the function f, studied in [10]. It is clear that $\mathrm{dec}\big(C_p(X)\big) = 1$.

Proposition 2. *For a T_1 topological space X the decomposition number* $\mathrm{dec}(SC_p(X))$ *is equal to the decomposition number* $\mathrm{dec}(\mathcal{D})$ *of the subset* $\mathcal{D} = \{\delta_a : a \in X\} \subset SC_p(X)$ *and is equal to the smallest cardinality* $\mathrm{ddec}(X)$ *of a cover of X by discrete subspaces.*

Proof. It is clear that $\mathrm{dec}(\mathcal{D}) \le \mathrm{dec}(SC_p(X)) \le \mathrm{ddec}(X)$. To prove that $\mathrm{dec}(\mathcal{D}) \ge \mathrm{ddec}(X)$, take a cover $\mathcal{C}$ of X of cardinality $|\mathcal{C}| = \mathrm{dec}(\mathcal{D})$ such that for each $C \in \mathcal{C}$ and each characteristic function $\delta_a \in \mathcal{D}$ the restriction $\delta_a|C$ is continuous. We claim that each space $C \in \mathcal{C}$ is discrete. Assuming conversely that C contains a non-isolated point $c \in C$, observe that for the characteristic function δ_c of the singleton $\{c\}$ the restriction $\delta_c|C$ is not continuous. But this contradicts the choice of the cover $\mathcal{C}$. Therefore the cover $\mathcal{C}$ consists of discrete subspaces of X and $\mathrm{ddec}(X) \le |\mathcal{C}| = \mathrm{dec}(\mathcal{D})$. □

In contrast to the whole function space $SC_p(X)$ which has large decomposition number $\operatorname{dec}(SC_p(X))$, its σ-convex subsets have decomposition numbers bounded from above by the hereditary Lindelöf number of X.

Following [11] and [12], we define a subset C of a linear topological space L to be *σ-convex* if for any sequence of points $(x_n)_{n\in\omega}$ in C and any sequence of positive real numbers $(t_n)_{n\in\omega}$ with $\sum_{n=0}^{\infty} t_n = 1$ the series $\sum_{n=0}^{\infty} t_n x_n$ converges to some point $c \in C$. It is easy to see that each compact convex subset $K \subset L$ is σ-convex. On the other hand, each σ-convex subset of a linear topological space L is necessarily convex and bounded in L.

The main result of this paper is the following:

Theorem 1. *For any topological space X of countable tightness, each σ-convex subset $\mathcal{F} \subset SC_p(X)$ has decomposition number* $\operatorname{dec}(\mathcal{F}) \le hl(X)$.

This theorem will be proved in Section 3. Now we derive some simple corollaries of this theorem.

Corollary 1. *For any topological space X of countable tightness, each σ-convex subset $\mathcal{F} \subset SC_p(X)$ has network weight $nw(\mathcal{F}) \le nw(X)$. Moreover,*

$$nw(X) = \max\{nw(\mathcal{F}) : \mathcal{F} \text{ is a } \sigma\text{-convex subset of } SC_p(X)\}$$

provided the space X is Tychonoff.

Proof. By Theorem 1, each σ-convex subset $\mathcal{F} \subset SC_p(X)$ has decomposition number $\operatorname{dec}(\mathcal{F}) \le hl(X)$. Consequently, we can find a disjoint cover $\mathcal{C}$ of X of cardinality $|\mathcal{C}| = \operatorname{dec}(\mathcal{F}) \le hl(X)$ such that for each $C \in \mathcal{C}$ and $f \in \mathcal{F}$ the restriction $f|C$ is continuous.

Let $Z = \oplus\mathcal{C} = \{(x, C) \in X \times \mathcal{C} : x \in C\} \subset X \times \mathcal{C}$ be the topological sum of the family $\mathcal{C}$, and $\pi : Z \to X$, $\pi : (x, C) \mapsto x$, be the natural projection of Z onto X. Since the cover $\mathcal{C}$ is disjoint, the map $\pi : Z \to X$ is bijective and hence induces a topological isomorphism $\pi^* : \mathbb{R}^X \to \mathbb{R}^Z$, $\pi^* : f \mapsto f \circ \pi$. The choice of the cover $\mathcal{C}$ guarantees that $\pi^*(\mathcal{F}) \subset C_p(Z)$. By (the proof of) Theorem I.1.3 of [7], $nw(C_p(Z)) \le nw(Z)$ and hence

$$\begin{aligned} nw(\mathcal{F}) = nw(\pi^*(\mathcal{F})) \le nw(C_p(Z)) \le nw(Z) \le \\ \le |\mathcal{C}| \cdot nw(X) \le hl(X) \cdot nw(X) = nw(X). \end{aligned}$$

If the space X is Tychonoff, then the "closed unit ball"

$$\mathcal{B} = \{f \in C_p(X) : \sup_{x \in X} |f(x)| \leq 1\} \subset C_p(X)$$

is σ-convex and has network weight $nw(\mathcal{B}) = nw(X)$ according to Theorem I.1.3 of [7]. So,

$$nw(X) = \max\{nw(\mathcal{F}) : \mathcal{F} \text{ is a } \sigma\text{-convex subset of } SC_p(X)\}.$$

□

In the same way we can derive some bounds on the weight of compact convex subsets in function spaces $SC_p(X)$.

Corollary 2. *For any topological space X of countable tightness, each compact convex subset $\mathcal{K} \subset SC_p(X)$ has weight $w(\mathcal{K}) \leq \max\{hl(X), hd(X)\}$. Moreover,*

$$hl(X) \leq \sup\{w(\mathcal{K}) : \mathcal{K} \textit{ is a compact convex subset of } SC_p(X)\} \leq \\ \leq \max\{hl(X), hd(X)\}.$$

Proof. Given a compact convex subset $\mathcal{K} \subset SC_p(X)$, use Theorem 1 to find a disjoint cover $\mathcal{C}$ of X of cardinality $|\mathcal{C}| = \mathrm{dec}(\mathcal{K}) \leq hl(X)$ such that for each $C \in \mathcal{C}$ and $f \in \mathcal{K}$ the restriction $f|C$ is continuous. Let $Z = \oplus\mathcal{C}$ and $\pi : \oplus\mathcal{C} \to X$ be the natural projection, which induces a linear topological isomorphism $\pi^* : \mathbb{R}^X \to \mathbb{R}^Z$, $\pi^* : f \mapsto f \circ \pi$, with $\pi^*(\mathcal{K}) \subset C_p(Z)$. It follows that the topological sum $Z = \oplus\mathcal{C}$ has density $d(Z) \leq \sum_{C \in \mathcal{C}} d(C) \leq |\mathcal{C}| \cdot hd(X) \leq \max\{hl(X), hd(X)\}$, and so we can fix a dense subset $D \subset Z$ of cardinality $|D| = d(Z) \leq \max\{hl(X), hd(X)\}$. Since the restriction operator $R : C_p(Z) \to C_p(D)$, $R : f \mapsto f|D$, is injective and continuous, we conclude that

$$w(\mathcal{K}) = w(\pi^*(\mathcal{K})) = w(R \circ \pi^*(\mathcal{K})) \leq w(\mathbb{R}^D) = \\ = |D| \cdot \aleph_0 \leq \max\{hl(X), hd(X)\}.$$

Next, we show that $hl(X) \leq \tau$ where

$$\tau = \sup\{w(\mathcal{K}) : \mathcal{K} \text{ is a compact convex subset of } SC_p(X)\}.$$

Assuming conversely that $hl(X) > \tau$ and using the equality $hl(X) = \sup\{|Z| : Z \subset X \text{ is scattered}\}$ established in [9], we can find a scattered subspace $Z \subset X$ of cardinality $|Z| > \tau$. It is easy to check that each function $f : X \to [0,1]$ with $f(X \setminus Z) \subset \{0\}$ is scatteredly continuous, which implies that the subset

$$\mathcal{K}_Z = \big\{f \in SC_p(X) : f(Z) \subset [0,1],\ f(X \setminus Z) \subset \{0\}\big\}$$

is compact, convex and homeomorphic to the Tychonoff cube $[0,1]^Z$. Then $\tau \geq w(\mathcal{K}_Z) = w([0,1]^Z) = |Z| > \tau$ and this is a desired contradiction that completes the proof. □

Corollaries 1 or 2 imply:

Corollary 3. *For a metrizable separable space X, each compact convex subspace $\mathcal{K} \subset SC_p(X)$ is metrizable.*

Finally, let us observe that Corollary 1 implies:

Corollary 4. *If for Tychonoff spaces X, Y with countable tightness the linear topological spaces $SC_p(X)$ and $SC_p(Y)$ are topologically isomorphic, then $nw(X) = nw(Y)$.*

1 Weakly discontinuous families of functions

In this section we shall generalize the notions of scattered continuity and weak discontinuity to function families.

A family of functions $\mathcal{F} \subset Y^X$ from a topological space X to a topological space Y is called

- *scatteredly continuous* if each non-empty subset $A \subset X$ contains a point $a \in A$ at which each function $f|A : A \to Y$, $f \in \mathcal{F}$ is continuous;

- *weakly discontinuous* if each subset $A \subset X$ contains an open dense subspace $U \subset A$ such that each function $f|U : U \to Y$, $f \in \mathcal{F}$ is continuous.

The following simple characterization can be derived from the corresponding definitions and Theorem 4.4 of [5] (saying that each scatteredly continuous function with values in a regular topological space is weakly discontinuous).

Proposition 3. *A function family $\mathcal{F} \subset Y^X$ is scatteredly continuous (resp. weakly discontinuous) if and only if so is the function $\Delta\mathcal{F} : X \to Y^{\mathcal{F}}$, $\Delta\mathcal{F} : x \mapsto (f(x))_{f\in\mathcal{F}}$. Consequently, for a regular topological space Y, a function family $\mathcal{F} \subset Y^X$ is scatteredly continuous if and only if it is weakly discontinuous.*

Propositions Propositions 4.7 and 4.8 [5] imply that each weakly discontinuous function $f : X \to Y$ has decomposition number $\mathrm{dec}(f) \le hl(X)$. This fact combined with Proposition 3 yields:

Corollary 5. *For any topological spaces X, Y, each weakly discontinuous function family $\mathcal{F} \subset Y^X$ has decomposition number $\mathrm{dec}(\mathcal{F}) \le hl(X)$.*

2 Weak discontinuity of σ-convex sets in function spaces

For a topological space X by $SC_p^*(X)$ we denote the space of all *bounded* scatteredly continuous real-valued functions on X. It is a subspace of the function space $SC_p(X) \subset \mathbb{R}^X$. Each function $f \in SC_p^*(X)$ has finite norm $\|f\| = \sup\limits_{x\in X} |f(x)|$.

Theorem 2. *For any topological space X with countable tightness, each σ-convex subset $\mathcal{F} \subset SC_p^*(X)$ is weakly discontinuous.*

Proof. By Proposition 3, the weak discontinuity of the function family $\mathcal{F}$ is equivalent to the scattered continuity of the function $\Delta\mathcal{F} : X \to \mathbb{R}^{\mathcal{F}}$, $\Delta\mathcal{F} : x \mapsto (f(x))_{f\in\mathcal{F}}$. Since the space X has countable tightness, the scattered continuity of $\Delta\mathcal{F}$ will follow from Proposition 2.3 of [5] as soon as we check that for each countable subset $Q = \{x_n\}_{n=1}^{\infty} \subset X$ the restriction $\Delta\mathcal{F}|Q : Q \to \mathbb{R}^{\mathcal{F}}$ has a continuity point. Assuming the converse, for each point $x_n \in Q$ we can choose a function $f_n \in \mathcal{F}$ such that the restriction $f_n|Q$ is discontinuous at x_n.

Observe that a function $f : Q \to \mathbb{R}$ is discontinuous at a point $q \in Q$ if and only if it has strictly positive oscillation

$$\mathrm{osc}_q(f) = \inf_{O_q} \sup\{|f(x) - f(y)| : x, y \in O_q\}$$

at the point q. In this definition the infimum is taken over all neighborhoods O_q of q in Q.

We shall inductively construct a sequence $(t_n)_{n=1}^{\infty}$ of positive real numbers such that for every $n \in \mathbb{N}$ the following conditions are satisfied:

1) $t_1 \le \frac{1}{2}$, $t_{n+1} \le \frac{1}{2} t_n$, and $t_{n+1} \cdot \|f_{n+1}\| \le \frac{1}{2} t_n \cdot \|f_n\|$,

2) the function $s_n = \sum_{k=1}^{n} t_k f_k$ restricted to Q is discontinuous at x_n,

3) $t_{n+1} \cdot \|f_{n+1}\| \le \frac{1}{8} \mathrm{osc}_{x_n}(s_n|Q)$.

We start the inductive construction letting $t_1 = 1/2$. Then the function $s_1|Q = t_1 \cdot f_1|Q$ is discontinuous at x_1 by the choice of the function f_1. Now assume that for some $n \in \mathbb{N}$ positive numbers $t_1 \dots, t_n$ has been chosen so that the function $s_n = \sum_{k=1}^{n} t_k f_k$ restricted to Q is discontinuous at x_n.

Choose any positive number $\tilde{t}_{n+1}$ such that

$$\tilde{t}_{n+1} \le \frac{1}{2} t_n, \ \ \tilde{t}_{n+1} \cdot \|f_{n+1}\| \le \tfrac{1}{2} t_n \cdot \|f_n\| \ \text{ and } \ \tilde{t}_{n+1} \cdot \|f_{n+1}\| \le \tfrac{1}{8} \mathrm{osc}_{x_n}(s_n|Q),$$

and consider the function $\tilde{s}_{n+1} = s_n + \tilde{t}_{n+1} f_{n+1}$. If the restriction of this function to Q is discontinuous at the point x_{n+1}, then put $t_{n+1} = \tilde{t}_{n+1}$ and finish the inductive step. If $\tilde{s}_{n+1}|Q$ is continuous at x_{n+1}, then put $t_{n+1} = \frac{1}{2} \tilde{t}_{n+1}$ and observe that the restriction of the function

$$s_{n+1} = \sum_{k=1}^{n+1} t_k f_k = s_n + \tfrac{1}{2} \tilde{t}_{n+1} f_{n+1} = \tilde{s}_{n+1} - \tfrac{1}{2} \tilde{t}_{n+1} f_{n+1}$$

to Q is discontinuous at x_{n+1}. This completes the inductive construction.

The condition (1) guarantees that $\sum_{n=1}^{\infty} t_n \le 1$ and hence the number $t_0 = 1 - \sum_{n=1}^{\infty} t_n$ is non-negative. Now take any function $f_0 \in \mathcal{F}$ and consider the function

$$s = \sum_{n=0}^{\infty} t_n f_n$$

which is well-defined and belongs to $\mathcal{F}$ by the σ-convexity of $\mathcal{F}$.

The functions $f_0, s \in \mathcal{F} \subset SC_p(X)$ are weakly discontinuous and hence for some open dense subset $U \subset Q$ the restrictions $s|U$ and $f_0|U$ are continuous. Pick any point $x_n \in U$. Observe that

$$s = t_0 f_0 + s_n + \sum_{k=n+1}^{\infty} t_k f_k$$

and hence

$$s_n = s - t_0 f_0 - \sum_{k=n+1}^{\infty} t_k f_k = s - t_0 f_0 - u_n,$$

where $u_n = \sum_{k=n+1}^{\infty} t_k f_k$. The conditions (1) and (3) of the inductive construction guarantee that the function u_n has norm

$$\|u_n\| \le \sum_{k=n+1}^{\infty} t_k \|f_k\| \le 2t_{n+1}\|f_{n+1}\| \le \frac{1}{4}\mathrm{osc}_{x_n}(s_n|Q).$$

Since $s_n = s - t_0 f_0 - u_n$, the triangle inequality implies that

$$0 < \mathrm{osc}_{x_n}(s_n|Q) \le \mathrm{osc}_{x_n}(s|Q) + \mathrm{osc}_{x_n}(t_0 f_0|Q) + \mathrm{osc}_{x_n}(u_n) \le$$

$$\le 0 + 0 + 2\|u_n\| \le \frac{1}{2}\mathrm{osc}_{x_n}(s_n|Q)$$

which is a desired contradiction, which shows that the restriction $\Delta\mathcal{F}|Q$ has a point of continuity and the family $\mathcal{F}$ is weakly discontinuous. □

3 Proof of Theorem 1

Let X be a topological space with countable tightness and $\mathcal{F}$ be a σ-convex subset in the function space $SC_p(X)$. The σ-convexity of $\mathcal{F}$ implies that for each point $x \in X$ the subset $\{f(x) : f \in \mathcal{F}\} \subset \mathbb{R}$ is bounded (in the opposite case we could find sequences $(f_n)_{n\in\omega} \in \mathcal{F}^\omega$ and $(t_n)_{n\in\omega} \in [0,1]^\omega$

with $\sum_{n=0}^{\infty} t_n = 1$ such that the series $\sum_{n=1}^{\infty} t_n f_n(x)$ is divergent). Then $X = \bigcup_{n=1}^{\infty} X_n$ where $X_n = \{x \in X : n \leq \sup_{f \in \mathcal{F}} |f(x)| < n+1\}$ for $n \in \omega$.

It follows that for every $n \in \omega$ the family $\mathcal{F}|X_n = \{f|X_n : f \in \mathcal{F}\}$ is a σ-convex subset of the function space $SC^*_p(X_n)$. By Theorem 2, the function family $\mathcal{F}|X_n$ is weakly discontinuous and by Corollary 5, $\mathrm{dec}(\mathcal{F}|X_n) \leq hl(X_n)$. Then $\mathrm{dec}(\mathcal{F}) \leq \sum_{n=0}^{\infty} \mathrm{dec}(\mathcal{F}|X_n) \leq \sum_{n=0}^{\infty} hl(X_n) \leq hl(X)$.

4 Some Open Problems

The presence of the condition of countable tightness in Theorem 1 and its corollaries suggests the following open problem.

Problem 1. *Is it true $w(\mathcal{K}) \leq nw(X)$ for each topological space X and each compact convex subset $\mathcal{K} \subset SC_p(X)$?*

By Theorem 2, for each topological space X of countable tightness, each compact convex subset $\mathcal{K} \subset SC^*_p(X)$ is weakly discontinuous.

Problem 2. *For which topological spaces X each compact convex subset $\mathcal{K} \subset SC_p(X)$ is weakly discontinuous?*

According to Corollary 3, each compact convex subset $\mathcal{K} \subset SC_p(\omega^\omega)$ is metrizable.

Problem 3. *Is a compact subset $\mathcal{K} \subset SC_p(\omega^\omega)$ metrizable if K is homeomorphic to a compact convex subset of $\mathbb{R}^{\mathfrak{c}}$.*

Let us recall that a topological space K is *Rosenthal compact* if K is homeomorphic to a compact subspace of the space $\mathcal{B}_1(X) \subset \mathbb{R}^X$ of functions of the first Baire class on a Polish space X. In this definition the space X can be assumed to be equal to the space ω^ω of irrationals.

Problem 4. *Is each Rosenthal compact space homeomorphic to a compact subset of the function space $SC_p(\omega^\omega)$?*

This problem has affirmative solution in the realm of zero-dimensional separable Rosenthal compacta.

Theorem 3. *Each zero-dimensional separable Rosenthal compact space K is homeomorphic to a compact subset of the function space $SC_p(\omega^\omega)$.*

Proof. Let $D \subset K$ be a countable dense subset in K. Let $A = C_D(K, 2)$ be the space of continuous functions $f : K \to 2 = \{0, 1\}$ endowed with the smallest topology making the restriction operator $R : C_D(K, 2) \to 2^D$, $R : f \mapsto f|D$, continuous. By the characterization of separable Rosenthal compacta [13], the space A is analytic, i.e., A is the image of the Polish space $X = \omega^\omega$ under a continuous map $\pi : X \to A$. Now consider the map $\delta : K \to 2^A$, $\delta : x \mapsto (f(x))_{f\in A}$. This map is continuous and injective by the zero-dimensionality of K. The map $\pi : X \to A$ induces a homeomorphism $\pi^* : 2^A \to 2^X$, $\pi^* : f \mapsto f \circ \pi$. Then $\pi^* \circ \delta : K \to 2^X$ is a topological embedding.

We claim that $\pi^* \circ \delta(K) \subset SC_p(X) \cap 2^X$. Given a point $x \in K$, we need to check that the function $\pi^* \circ \delta(x) \in 2^X$ is scatteredly continuous. It will be convenient to denote the function $\delta(x) \in 2^A$ by δ_x. This function assigns to each $f \in A = C_D(K)$ the number $\delta_x(f) = f(x) \in 2$.

By [14, 15], the Rosenthal compact space K is Fréchet-Urysohn, so there is a sequence $(x_n)_{n\in\omega} \in D^\omega$ with $\lim_{n\to\infty} x_n = x$. Then the function $\delta_x : A \to 2$, $\delta_x : f \mapsto f(x)$, is the pointwise limit of the continuous functions δ_{x_n}, which implies that δ_x is a function of the first Baire class on A and $\delta_x \circ \pi : X \to 2$ is a function of the first Baire class on the Polish space X. Since this function has discrete range, it is scatteredly continuous by Theorem 8.1 of [5]. Consequently, $\pi^* \circ \delta(x) \in SC_p(X)$ and K is homeomorphic to the compact subset $\pi^* \circ \delta(K) \subset SC_p(X)$. $\square$

A particularly interesting instance of Problem 4 concerns non-metrizable convex Rosenthal compacta. On of the simples spaces of this sort is the Helly space. We recall that the *Helly space* is the subspace of $B_1(I)$ consisting of all non-decreasing functions $f : I \to I$ of the unit interval $I = [0, 1]$.

Problem 5. *Is the Helly space homeomorphic to a compact subset of the function space $SC_p(\omega^\omega)$?*

[1] *Bokalo B., Kolos N.* When does $SC(X) = \mathbb{R}^X$ hold? // Topology — 2009. — **48.** — P. 178–181.

[2] *Bokalo B., Kolos N.* On normality of spaces of scattredly continuous maps // Mat. Stud. — 2011. — **35**, №2. — P. 196–204.

[3] *Архангельский А.В., Бокало Б.М.* Касание топологий и тангенциальные свойства топологических пространств // ТММО — 1992. — **54.** — С. 160–185, 278–279.

[4] *Бокало Б.М., Маланюк О.П.* Про майже неперервні відображення // Мат. Студ. — 1998. — **9**, №1. — С. 90–93.

[5] *Banakh T., Bokalo B.* On scatteredly continuous maps between topological spaces // Topology Appl. — 2010. — **157.** — P. 108–122.

[6] *Bokalo B.V., Kolos N.M.* On operations on some classes of discontinuous functions // Carpathian Math. Publ. — 2011. — **3**, №2. — P. 36–48.

[7] *Архангельский А.В.* Топологические пространтсва функций. — М.: Изд-во МГУ, 1989. — 222 с.

[8] *Энгелькинг Р.* Общая топология. — М.: Мир, 1986. — 752 с.

[9] *Juhasz I.* Cardinal functions in Topology. — Amsterdam: Math. Centre Tracts **34**, 1971. — 150 p.

[10] *Solecki S.* Decomposing Borel sets and functions and the structure of Baire class 1 functions // J. Amer. Math. Soc. — 1998. — **11**, №3. — P. 521–550.

[11] *Alfsen E.* Compact Convex Sets and Boundary Integrals. — Berlin: Springer-Verlag, 1971, (Ergebnisse der Math., Vol. **57**). — 210 p.

[12] *Takemoto H., Uchiyama A., Zsido L.* The σ-convexity of all bounded convex sets in $\mathbb{R}^n$ and $\mathbb{C}^n$ // Nihonkai Math. J. — 2003. — **14.** — P. 61–64.

[13] *Godefroy G.* Compacts de Rosenthal // Pacific J. Math. — 1980. — **91**, №2. — P.293–306.

[14] *Rosenthal H.P.* Point-wise compact subsets of the first Baire class // Amer. J. Math. — 1977. — **99**, №2. — P.362–378.

[15] *Bourgain J., Fremlin D.H., Talagrand M.* Pointwise compact sets of Baire-measurable functions // Amer. J. Math. — 1978. — **100**, №4. — P.845–886.

ПРО σ-ОПУКЛІ ПІДПРОСТОРИ ПРОСТОРУ РОЗРІДЖЕНО НЕПЕРЕРВНИХ ФУНКЦІЙ

Тарас БАНАХ, Богдан БОКАЛО, Надія КОЛОС

Львівський національний університет імені Івана Франка,
вул. Університетська 1, Львів 79602

e-mail: *tbanakh@yahoo.com, bogdanbokalo@mail.ru, nadiya_kolos@ukr.net*

Доведено, що для довільного топологічного простору X зліченної тісноти кожний σ-опуклий підпростір $\mathcal{F}$ простору $SC_p(X)$ розріджено неперервних дійснозначних функцій на X має сіткову вагу $nw(\mathcal{F}) \le nw(X)$. З цього випливає, що для довільного метризовного сепарабельного простору X кожний компактний опуклий підпростір в $SC_p(X)$ є метризовним, а також, що тихонівські простори X і Y зі зліченною тіснотою мають однакову сіткову вагу $nw(X) = nw(Y)$, якщо лінійні топологічні простори $SC_p(X)$ і $SC_p(Y)$ топологічно ізоморфні. Також доведено, що кожний нуль-вимірний сепарабельний компакт Розенталя вкладається у простір розріджено неперервних функцій $SC_p(\omega^\omega)$ над польським простором ω^ω.