ON FACTOR GROUPS OF SOME GROUPS

Institute of Mathematics, Ukrainian National Academy of Sciences,
3 Tereshchenkivska Str., Kyiv 01601, Ukraine

e-mail: *chern@imath.kiev.ua*

Let for a prime p, $\mathfrak{X}$ (respectively $\mathfrak{Y}$) be the class of all p-biprimitively finite (respectively periodic p-conjugatively biprimitively finite) groups and $G \in \mathfrak{X}$ (respectively $G \in \mathfrak{Y}$), V be a periodic subgroup of G having an ascending series of normal in G subgroups such that each its factor is an almost layer-finite group or a locally graded group of finite special rank, or a WF-group with $min - q$ on all primes q. We prove that $G/V \in \mathfrak{X}$ (respectively $G/V \in \mathfrak{Y}$). Also some interesting and useful preliminary results are obtained.

Below p and q are primes, $\mathbb{N}$ is the set of all natural numbers, $\mathbb{P}$ is the set of all primes; min (respectively $min - q$) is the minimal condition on (q-) subgroups, $\times$ is the sign of the direct product; $H \ sn \ G$ denotes that H is a subnormal subgroup of the group G. Other notations in the present paper are standard. The empty set is not reckoned finite. Recall that for a class $\mathfrak{X}$ of groups, an almost $\mathfrak{X}$-group is a finite extension of a group belonging to $\mathfrak{X}$.

Remind definitions of some classes of groups: the group G is called p-(conjugatively) biprimitively finite, if for every its finite subgroup K, each subgroup of $N_G(K)/K$, generated by two (conjugate) elements of order

p, is finite; a group is called (conjugatively) biprimitively finite, if for any p, it is p-(conjugatively) biprimitively finite (V.P. Shunkov, 1970–1973). Conjugatively biprimitively finite groups are also called Shunkov groups (V.D. Mazurov, 2003). These classes of groups are very large and contain, for instance, all locally finite groups, all 2-groups. Many deep results of Shunkov's School are connected with groups of these classes (see, for instance, [1]).

Futher, a group is called layer-finite, if the set of all its elements of each order is finite or empty (S.N. Chernikov, see, for instance, [2]). In consequence of Dietzmann's Lemma (see, for instance, [3]) implies that a layer-finite group is locally (finite and normal). A layer-finite group is also called an FL-group [3].

Remind that the FC-centre of the group G is the following its subgroup: $\{g \in G : |G : C_G(g)| < \infty\}$, and the FC-hypercentre of G is the limit of its ascending normal series defined by the rules: $G_0 = 1$, $G_{\alpha+1}/G_\alpha =$ the FC-centre of G/G_α, and $G_\lambda = \cup_{\beta<\lambda} G_\beta$, where α is an ordinal and $\lambda \neq 0$ is a limit ordinal (see, for instance, [3]). The FC-centre (respectively FC-hypercentre) of G contains its centre (respectively hypercentre).

Remind that a group possessing an ascending series with finite and locally nilpotent factors is called an WF-group (B.I. Plotkin, [4]). The class of all WF-groups is large. It contains, for instance, all RN^*-groups, all radical in the sense of B.I. Plotkin groups, all groups possessing an ascending series with FC-factors (at the same time, all hyperfinite groups).

Remind that a group such that any its finitely generated subgroup $\neq 1$ possesses a subgroup of finite index $\neq 1$ is called locally graded (S.N. Chernikov, see, for instance, [2]). The class of all locally graded groups is very large. It contains, for example, all locally finite, locally solvable, residually finite, linear, radical in the sense of B.I. Plotkin groups, all WF-groups, all RN-groups and, at the same time, all groups of all Kurosh-Chernikov classes [5].

The main result of the present paper is the following theorem.

Theorem. *Let G be a p-biprimitively finite (respectively periodic p-conjugatively biprimitively finite) group and V its normal periodic subgroup. Suppose that for V at least one of the following conditions is fulfilled:*

(a) It possesses an ascending series $\mathcal{M}$ of normal in G subgroups, every

factor of which is an almost layer-finite group or a locally graded group of finite special rank, or a WF-group with $min - q$ on all q (i.e., with Artinian q-subgroups on all q).

(b) *It possesses an ascending series with finite factors of normal in G subgroups.*

(c) *It belongs to the FC-hypercentre of G.*

Then G/V is p-biprimitively finite (respectively p-conjugatively biprimitively finite).

The following corollaries are immediate consequences of Theorem.

Corollary 1. *Let G be a p-biprimitively finite (respectively periodic p-conjugatively biprimitively finite) group and V its normal periodic subgroup. If V is almost layer-finite or locally graded of finite special rank, or a WF-group with $min - q$ on all q, then G/V is p-biprimitively finite (respectively p-conjugatively biprimitively finite).*

Corollary 2. *Let G be a biprimitively finite (respectively periodic Shunkov) group and V its normal periodic subgroup. If V is almost layer-finite or locally graded of finite special rank, or a WF-group with $min - q$ on all q, then G/V is biprimitively finite (respectively Shunkov).*

Corollary 3. *Let G be a biprimitively finite (respectively periodic Shunkov) group and V its subgroup possessing an ascending series with finite factors of normal in G subgroups. Then G/V is biprimitively finite (respectively Shunkov).*

Preface the proof of Theorem with the following propositions.

Proposition 1. *Let G be a group having an ascending normal series $G_0 = 1 \subset ... \subset G_\gamma = G$ with finite factors, M a finite set of elements of G, $\Omega_1, ..., \Omega_n$ operator groups of G, generated by finite sets of elements of finite order, such that for any $k = 1, ..., n$ and ordinal $\alpha < \gamma$, $G_\alpha{}^{\Omega_k} = G_\alpha$. Then:*

(i) *All finite subgroups F of G, for which $F^{\Omega_k} = F$, $k = 1, ..., n$, constitute a local system of G.*

(ii) *For any Ω_k, $|\Omega_k : C_{\Omega_k}(M)| < \infty$, $\Omega_k/C_{\Omega_k}(G)$ is residually finite and in the case when Ω_k is a group of automorphisms of G, Ω_k is residually finite.*

Proposition 2. *Let G be a group having a layer-finite subgroup L of finite index. Then G is a locally finite group with Chernikov q-subgroups on all q, its FC-centre F is layer-finite and $|G : F| < \infty$. Further, an arbitrary FC-subgroup B of G (in particular, an arbitrary layer-finite subgroup B of G) belongs to F, if $|G : B| < \infty$ or B sn G.*

Proposition 3. *In each of the following cases the periodic group G possesses an ascending series with finite factors of characteristic subgroups:*

(a) *G is almost layer-finite.*

(b) *G is locally graded of finite special rank.*

(c) *G is a WF-group with $min - q$ on all q (i.e., with Artinian q-subgroups on all q).*

Proposition 4. *Let G be a group, H its subgroup generated by a finite set of elements of finite order and V its normal periodic subgroup. Suppose that for V at lest one of the conditions (a)-(c) of Theorem is fulfilled. Then:*

(i) *All finite subgroup of V, normalized by H, constitute a local system of V.*

(ii) *$H/C_H(V)$ is residually finite.*

The following assertion is an immediate consequence of Proposition 4.

Corollary 4. *Let G be a group, H its subgroup generated by a finite set of elements of finite order and V its normal periodic subgroup. Suppose that V is almost layer-finite or locally graded of finite special rank, or a WF-group with $min - q$ on all q. Then the statements of Proposition 4 are valid.*

Proof of Proposition 1.

Let (i) be false and G be a counterexample with minimal γ. Then for some finite set X of elements of G, there are exists no finite subgroups $F \supseteq X$ such that $F^{\Omega_k} = F$, $k = 1, ..., n$.

For some limit infinite ordinal $\nu \leq \gamma$, $|G : G_\nu| < \infty$. Let T be a transversal to G_ν in G, $\mathfrak{M}_k$ be a finite set of finite cyclic subgroups of Ω_k that generate Ω_k, and $\bigcup_{k=1}^{n} \mathfrak{M}_k = \{\Delta_1, \ldots, \Delta_m\}$; $H_j = \langle X^{\Delta_j}, T^{\Delta_j} \rangle$, $j = 1, \ldots, m$, and $H = \langle H_1, \ldots, H_m \rangle$. Then $H_j^{\Delta_j} = H_j$ and $H = H_j(H \cap G_\nu)$, $j = 1, \ldots, m$.

In view of Corollary [3, p. 35] (for instance), G is locally finite. Therefore because of H is finitely generated, it is finite. Since ν is infinite limit and $H \cap G_\nu$ is finite, for some ordinal $\beta < \nu$, $H \cap G_\nu \subseteq G_\beta$. Put $\zeta = \beta$, if $H \subseteq G_\beta$, and $\zeta = \beta + 1$, if $H \not\subseteq G_\beta$; $L = HG_\beta$ and $L_\alpha = G_\alpha$, $\alpha < \zeta$. Then $X \subseteq L$, $\zeta < \gamma$, and $L_0 = 1 \subset ... \subset L_\zeta = L$ is an ascending normal series of L with finite factors and also for $\alpha < \zeta$, $L_\alpha^{\Omega_k} = L_\alpha$, $k = 1, ..., n$. Further,

$$L^{\Delta_j} = (HG_\beta)^{\Delta_j} = (H_j(H \cap G_\nu)G_\beta)^{\Delta_j} = (H_j G_\beta)^{\Delta_j} =$$

$$= H_j^{\Delta_j} G_\beta^{\Delta_j} = H_j G_\beta = H_j(H \cap G_\nu)G_\beta = HG_\beta = L, j = 1, \ldots, m.$$

Therefore because of Ω_k is generated by some subgroups Δ_j, $L^{\Omega_k} = L$. Thus $L_\alpha^{\Omega_k} = L_\alpha$, $\alpha \leq \zeta$ and $k = 1, \ldots, n$. Further, since $\zeta < \gamma$, L is not a counterexample to (i). Therefore for some finite subgroup F of L, $X \subseteq F$ and $F^{\Omega_k} = F$, $k = 1, \ldots, n$, which is a contradiction. Thus (i) is correct.

Further, in view of (i), for some finite $F \subseteq G$, $M \subseteq F = F^{\Omega_k}$. So $|\Omega_k : C_{\Omega_k}(M)| \leq |\Omega_k : C_{\Omega_k}(F)| < \infty$. Therefore because of $C_{\Omega_k}(G)$ is the intersection of subgroups $C_{\Omega_k}(M)$ by all finite $M \subseteq G$, $\Omega_k/C_{\Omega_k}(G)$ is residually finite. In the last case, $C_{\Omega_k}(G) = 1$ and so Ω_k is residually finite.

The proof is complete.

Proof of Proposition 2.

First, if $|G : B| < \infty$, then for $b \in B$, $|G : C_G(B)| < \infty$. So $B \subseteq F$.

Since $|G : L| < \infty$ and L is an FC-group, $L \subseteq F$. Consequently, $|G : F| < \infty$.

Let X be any finitely generated subgroup of G. Then $|X : X \cap L| < \infty$. In consequence of Schreier's Theorem (see, for instance, [6, p. 228]), $X \cap L$

is finitely generated. Since L is locally finite, $X \cap L$ is finite. So X is finite. Thus G is locally finite.

Let Q be a q-subgroup of G. By Theorem 3.2 [2], $Q \cap L$ is Chernikov. Since $|Q : Q \cap L| < \infty$, obviously Q is Chernikov too.

Let the statement of the present proposition, relating to the case when B sn G, be incorrect, and G be a counter-example to this statement such that the length of some passing through B finite series $G_0 = 1 \subset B = G_1 \subset \ldots \subset G_n = G$ of G is minimal. For the layer-finite subgroup $L \cap G_{n-1}$, $|G_{n-1} : L \cap G_{n-1}| < \infty$. Then the FC-centre K of G_{n-1} contains B. Since obviously K is locally (finite and normal) and satisfies for each q $min - q$, it is layer-finite (Theorem 3.7 [2]). Let $a \in K$. Then for any $g \in G$, $|\langle a \rangle| = |\langle a^g \rangle|$ and $a^g \in K$. Therefore because K is layer-finite, the set $\{a^g : g \in G\}$ is finite. So $a \in F$. Thus $B \subseteq K \subseteq F$, which is a contradiction.

The proof is complete.

Proof of Proposition 3.

(a) In view of Proposition 2, G is locally finite and contains some characteristic layer-finite subgroup F of finite index. For any $k \in \mathbb{N}$, $\langle g \in F : |\langle g \rangle| \leq K \rangle$ is obviously a finite characteristic subgroup of G. Clearly, all distinct subgroups among subgroups: $\langle g \in F : |\langle g \rangle| \leq K \rangle$, $k = 1, 2, \ldots$, F, G, constitute a required series of G.

Let N be the product of all subgroups that possess an ascending series with finite factors of characteristic subgroups of G. Then N has the same series, and G/N has no finite characteristic subgroups $\neq 1$. Consequently: G has the same series iff $G = N$; if $G/N \neq 1$, then G/N is infinite.

Let B be a characteristic Chernikov subgroup of G/N. If $B \neq 1$, then B contains some subgroup H of finite index, which is a direct product of finitely many quasicyclic subgroups. Let p be a prime for which H has an element of order p. Then $\{g \in H : |\langle g \rangle| \leq p\} \neq 1$. But $\{g \in H : |\langle g \rangle| \leq p\}$ is clearly a finite characteristic subgroup of G/N, which is a contradiction. Thus G/N has no characteristic Chernikov subgroups $\neq 1$.

(b) Suppose $G/N \neq 1$. In view of Theorem [7], G and, at the same time, G/N are locally finite and almost hyperabelian. Let K be a normal hyperabelian subgroup of finite index of G/N. Then $K \neq 1$. So K contains some normal abelian subgroup $A \neq 1$. For some p, $O_p(A) \neq 1$. Since

$O_p(A) \trianglelefteq K$ and $O_p(K) \trianglelefteq G/N$, $O_p(G/N) \neq 1$. Since $O_p(G/N)$ is locally finite of finite special rank, it is Chernikov [8], which is a contradiction. Thus $G = N$, so (b) is correct.

(c) Suppose $G/N \neq 1$. In view of Proposition 1.1 [2] and Corollary [3, p. 35] (for instance), G and, at the same time, G/N are locally finite. Therefore for each q all q-subgroups of G are Chernikov (Theorem 1.5 [2]). Then by virtue of Theorem 3.13 [9], for each q all q-subgroups of G/N are Chernikov. Since $O_q(G/N)$ is Chernikov, $O_q(G/N) = 1$ (see above).

Since, obviously, G/N is a WF-group, it has some ascendent subgroup $L \neq 1$ which is locally nilpotent or finite.

In the first case, $L = \times_{q \in \mathbb{P}} L_q$ where L_q are Sylow q-subgroups of L. For some p, $L_p \neq 1$. By Lemma 2.1 [10], $O_p(G/N) \neq 1$, which is a contradiction.

Thus L is finite.

Let M be a subnormal subgroup of L of minimal $\neq 1$ order. Then M is an ascendant finite simple subgroup of G/N. It is non-abelian (see above). By Lemma 2.1 [10], G/N contains some characteristic subgroup $R \neq 1$ which is a direct product of subgroups isomorphic to M. Then R is infinite. So because M is finite, R contains an infinite subgroup which is a direct product of subgroups of order p for some p. But this subgroup is not Chernikov, which is a contradiction.

Thus $G = N$. So (c) is correct.

The proof is complete.

Proof of Proposition 4.

In view of Proposition 3, $\mathcal{M}$ is contained in an ascending series of V with finite factors of normal in G subgroups, i.e. (a) implies (b).

Suppose (c) is fulfilled. Then it is easy to see: V has an ascending series of normal in G subgroups such that every its factor A/B contains some element g, for which $|G/B : C_{G/B}(g)| < \infty$ and $A/B = \langle g^{G/B} \rangle$. In view of Dietzmann's Lemma, $\langle g^{G/B} \rangle$ is finite. So (c) implies (b).

Thus (b) is necessarily fulfilled.

Setting in Proposition 1 $G = V$, $n = 1$ and $\Omega_1 = H$ and applying this proposition, conclude that the present proposition is true.

Proof of the Theorem.

Let L/V be a finite subgroup of G/V, U/V the normalizer of L/V in

G/V and $(T/V)/(L/V)$ a subgroup of $(U/V)/(L/V)$, generated by two (conjugate) elements of order p. It is easy to see that for some $a, b \in N_G(L) \setminus L$, $a^p, b^p \in L$ and $T = \langle a, b \rangle L$ (resp. for some $a \in N_G(L) \setminus L$ and $g \in N_G(L)$, $a^p \in L$ and $T = \langle a, a^g \rangle L$). Note that a, b, g are elements of finite order.

It is easy to see that for L even if one of the conditions (a)-(c) is fulfilled. Therefore by virtue of Proposition 4, for some finite subgroup F of L, $\langle a, b \rangle \subseteq N_G(F)$ and $a^p, b^p \in F$ (respectively $\langle a, g \rangle \subseteq N_G(F)$ and $a^p \in F$ and, at the same time, $a^{pg} \in F$). Make more precise: in Proposition 4, we set $H = \langle a, b \rangle$ or $H = \langle a, g \rangle$ respectively. Since aF and bF (resp. aF and $a^g F$) are (conjugate) elements of order p of $N_G(F)/F$, the subgroup $\langle aF, bF \rangle$ (respectively $\langle aF, a^g F \rangle$) of $N_G(F)/F$ is finite. Therefore $\langle a, b \rangle$ (respectively $\langle a, a^g \rangle$) is finite. At the same time, $(T/V)/(L/V)$ is finite. Thus, the present theorem is true.

The following assertion is an immediate consequence of Proposition 1.

Corollary. *Let G be a Chernikov group, M a finite set of elements of G, $\Omega_1, ..., \Omega_n$ operator groups of G, generated by finite sets of elements of finite order. Then the statements (i) and (ii) of Proposition 1 are valid.*

[1] *Senashov V.I., Shunkov V.P.* Groups with finiteness conditions. — Novosibirsk: Publ. House Siberian Branch Russian Acad. Sci., 2001. — 336 p. (in Russian)

[2] *Chernikov S.N.* Groups with prescribed properties of the system of subgroups. — Moskva: Nauka, 1980. — 384 p. (in Russian)

[3] *Robinson D.J.S.* Finiteness conditions and generalized soluble groups. Pt 1. — Berlin — Heidelberg — New York: Springer, 1972. — 210 p.

[4] *Plotkin B.I.* Radical and semi-simple group // Trudy Moskov. Mat. Obsh. — 1957. — **6**. — P. 299–336 (in Russian).

[5] *Kargapolov M.I., Merzljakov Yu.I.* Fundamentals of the theory of groups. — Moskva: Nauka, 1972. — 240 p. (in Russian)

[6] *Kurosh A.G.* The theory of groups. 3-d ed. — Moskva: Nauka, 1967. — 648 p. (in Russian)

[7] *Chernikov N.S.* One theorem on groups of finite special rank // Ukr.Mat.Ž. — 1990. — **42**, №7. — P. 962–970 (in Russian).

[8] *Myagkova N.N.* On groups of finite rank// Izv. Acad.Nauk SSSR. Ser. Mat. — 1949. — **13**, №6. — P. 495–512 (in Russian).

[9] *Kegel O.H., Wehrfritz B.A.F.* Locally finite groups. — Amsterdam-London: North-Holland Publ. Co, 1973. — 210 p.

[10] *Chernikov S.N.* Finiteness conditions in the general theory of groups // Uspehi. Mat. Nauk. — 1959. — **14**, №5. — P. 45–96 (in Russian).

ПРО ФАКТОР-ГРУПИ ДЕЯКИХ ГРУП

Микола ЧЕРНІКОВ

Інститут математики НАН України,
вул. Терещенківська 3, Київ 01601

e-mail: *chern@imath.kiev.ua*

Нехай для деякого простого p, $\mathfrak{X}$ (відповідно $\mathfrak{Y}$) — клас усіх p-біпримітивно скінченних (відповідно, періодичних p-спряженно біпримітивно скінченних) груп і $G \in \mathfrak{X}$ (відповідно, $G \in \mathfrak{Y}$), V — періодична підгрупа G, яка має зростаючий нормальний ряд інваріантних в G підгруп, такий, що кожен його фактор є або майже шарово скінченною групою або локально скінченною групою скінченного спеціального рангу, або WF-групою із $min - q$ за всіма простими q. Доведено, що $G/V \in \mathfrak{X}$ (відповідно, $G/V \in \mathfrak{Y}$). Отримано також деякі цікаві та корисні попередні результати.