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AN EXPONENTIAL DIVISOR FUNCTION OVER GAUSSIAN INTEGERS

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Let $\tau^{(e)}\colon \mathbb{Z} \to \mathbb{Z}$ be a multiplicative function such that $\tau^{(e)}(p^a) = \sum_{d|a} 1$. In the present paper we introduce generalizations of $\tau^{(e)}$ over the ring of Gaussian integers $\mathbb{Z}[i]$. We determine their maximal orders by proving a general result and establish asymptotic formulas for their average orders.

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Нехай $\tau^{(e)}\colon \mathbb{Z} \to \mathbb{Z}$ – така мультиплікативна фукція, що $\tau^{(e)}(p^a) = \sum_{d|a} 1$. У статті означені узагальнення функції $\tau^{(e)}$ на кільце гаусових цілих чисел $\mathbb{Z}[i]$. Як наслідок загального результату визначено максимальні порядки таких функцій. Також побудовано асимптотичні формули для відповідних суматорних функцій.

1. Introduction

In 1972 M.V. Subbarao introduced [8] exponential divisor function $\tau^{(e)}\colon \mathbb{Z} \to \mathbb{Z}$, which is multiplicative and

$$\tau^{(e)}(p^a) = \tau(a),$$

where $\tau\colon \mathbb{Z} \to \mathbb{Z}$ stands for the usual divisor function. Erdös estimated its maximal order and Subbarao proved an asymptotic formula for $\sum_{n\leqslant x} \tau^{(e)}(n)$. Later Wu [11] gave a more precise estimation:

$$\sum_{n\leqslant x} \tau^{(e)}(n) = Ax + Bx^{1/2} + O(x^{\theta_{1,2}+\varepsilon}),$$

where A and B are computable constants, $\theta_{1,2}$ is an exponent in the error term of the estimation $\sum_{ab^2\leqslant x} 1 = \zeta(2)x + \zeta(1/2)x^{1/2} + O(x^{\theta_{1,2}+\varepsilon})$. The best modern result [2] yields the upper bound $\theta_{1,2} \leqslant 1057/4785$.

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In the present paper we generalize the exponential divisor function over the ring of Gaussian integers $\mathbb{Z}[i]$. Namely we introduce multiplicative functions $\tau_*^{(e)}\colon \mathbb{Z} \to \mathbb{Z}$, $\mathfrak{t}^{(e)}$, $\mathfrak{t}_*^{(e)}\colon \mathbb{Z}[i] \to \mathbb{Z}$ such that

$$\tau_*^{(e)}(p^a) = \mathfrak{t}(a), \qquad \mathfrak{t}^{(e)}(\mathfrak{p}^a) = \tau(a), \qquad \mathfrak{t}_*^{(e)}(\mathfrak{p}^a) = \mathfrak{t}(a), \tag{1}$$

where p is prime over $\mathbb{Z}$, $\mathfrak{p}$ is prime over $\mathbb{Z}[i]$, $\mathfrak{t}(a)$ is a number of non-associated in pairs Gaussian integer divisors of a.

The aims of this paper are to determine maximal orders of $\tau_*^{(e)}$, $\mathfrak{t}^{(e)}$, $\mathfrak{t}_*^{(e)}$ and to provide asymptotic formulas for $\sum_{n\leqslant x} \tau_*^{(e)}(n)$, $\sum'_{N(\alpha)\leqslant x} \mathfrak{t}^{(e)}(\alpha)$, $\sum'_{N(\alpha)\leqslant x} \mathfrak{t}_*^{(e)}(\alpha)$. A theorem on the maximal order of multiplicative functions over $\mathbb{Z}[i]$, generalizing [9], is also proved.

2. Notation

Let us denote the ring of Gaussian integers by $\mathbb{Z}[i]$, $N(a+bi) = a^2+b^2$. In asymptotic relations we use $\sim$, $\asymp$, Landau symbols O and o, Vinogradov symbols $\ll$ and $\gg$ in their usual meanings. All asymptotic relations are written for the argument tending to the infinity. Letters $\mathfrak{p}$ and $\mathfrak{q}$ with or without indexes denote Gaussian primes; p and q denote rational primes.

As usual $\zeta(s)$ denotes the Riemann zeta-function, $L(s,\chi)$ is the Dirichlet L-function. Let χ_4 be the single nonprincipal character modulo 4, then $Z(s) = \zeta(s)L(s,\chi_4)$ is the Hecke zeta-function for the ring of Gaussian integers. Real and imaginary components of a complex number s are denoted by $\sigma := \operatorname{Re} s$ and $t := \operatorname{Im} s$, so $s = \sigma + it$. We use abbreviations $\operatorname{llog} x := \log\log x$, $\operatorname{lllog} x := \log\log\log x$.

The notation $\sum'$ means the summation over non-associated elements of $\mathbb{Z}[i]$, and $\prod'$ means the similar relative to multiplication. Notation $a \sim b$ means that a and b are associated, that is $a/b \in \{\pm 1, \pm i\}$. But in asymptotic relations $\sim$ preserves its usual meaning.

The letter γ denotes the Euler–Mascheroni constant. Everywhere $\varepsilon > 0$ is an arbitrarily small number (not always the same). We write $f \star g$ for the notation of the Dirichlet convolution

$$(f \star g)(n) = \sum_{d\mid n} f(d)g(n/d).$$

3. Preliminary lemmas

We need the following auxiliary results.

Lemma 3.1 (Gauss's criterion)**.** *Gaussian integer $\mathfrak{p}$ is prime if and only if one of the following cases holds:*

- $\mathfrak{p} \sim 1+i$,
- $\mathfrak{p} \sim p$, *where* $p \equiv 3 \pmod 4$,
- $N(\mathfrak{p}) = p$, *where* $p \equiv 1 \pmod 4$.

In the last case there are exactly two non-associated $\mathfrak{p}_1$ and $\mathfrak{p}_2$ such that $N(\mathfrak{p}_1) = N(\mathfrak{p}_2) = p$.

Proof. See [1, §34]. □

Lemma 3.2.

$$\sum_{N(\mathfrak{p})\leqslant x}{}' 1 \sim \frac{x}{\log x}, \tag{2}$$

$$\sum_{N(\mathfrak{p})\leqslant x}{}' \log N(\mathfrak{p}) \sim x, \tag{3}$$

Proof. Taking into account Gauss's criterion and the asymptotic law of distribution of primes in the arithmetic progression we get

$$\sum_{N(\mathfrak{p})\leqslant x}{}' 1 \sim \#\{p \mid p \equiv 3 \pmod 4,\ p \leqslant \sqrt{x}\} + 2\#\{p \mid p \equiv 1 \pmod 4,\ p \leqslant x\} \sim$$

$$\sim \frac{\sqrt{x}}{\varphi(4)\log x/2} + 2\frac{x}{\varphi(4)\log x} = \frac{x}{\log x}.$$

A partial summation with use of (2) gives us the second statement of the lemma. □

Lemma 3.3.

$$\max_{n\geqslant 1} \frac{\log \tau(n)}{n} = \frac{\log 2}{2}, \tag{4}$$

$$\max_{n\geqslant 1} \frac{\log \mathfrak{t}(n)}{n} = \frac{\log 3}{2}. \tag{5}$$

Proof. It is well-known that $\tau(n) \leqslant 2\sqrt{n}$. Indeed the set of divisors of n can be divided into pairs $(d, n/d)$ and the least element of a pair is $\leqslant\sqrt{n}$. Similarly the set of non-associated Gaussian divisors of n can be divided into pairs (α, β) such that $\alpha\beta \sim n$, where $N(\alpha) \leqslant n$ or $N(\beta) \leqslant n$, so $\mathfrak{t}(n) \leqslant \pi n/2$.

Consider the functions

$$f(n) = n^{-1}\log(2\sqrt{n}) = n^{-1}\big(\log 2 + (\log n)/2\big),$$
$$g(n) = n^{-1}\log(\pi n/2) = n^{-1}\big(\log \frac{\pi}{2} + (\log n)\big).$$

Both functions are decreasing for $n \geqslant 3$ because $(n^{-1}\log n)' = n^{-2}(1 - \log n)$. Then due to the definition (1)

$$\max_{n\geqslant 1} \frac{\log \tau(n)}{n} = \max\left\{0, \frac{\log 2}{2}, \frac{\log 3}{3}, f(4)\right\} = \frac{\log 2}{2},$$
$$\max_{n\geqslant 1} \frac{\log \mathfrak{t}(n)}{n} = \max\left\{0, \frac{\log 3}{2}, g(3)\right\} = \frac{\log 3}{2}.$$

□

Lemma 3.4. *Let $F : \mathbb{Z} \to \mathbb{C}$ be a multiplicative function such that $F(p^a) = f(a)$, where $f(n) \ll n^\beta$ for some $\beta > 0$. Then*

$$\limsup_{n\to\infty} \frac{\log F(n)\,\mathrm{llog}\, n}{\log n} = \sup_{n\geqslant 1} \frac{\log f(n)}{n}.$$

Proof. See [9]. □

Lemma 3.5. *Let $f(t) \geqslant 0$. If $\int_1^T f(t)\,dt \ll g(T)$, where $g(T) = T^\alpha \log^\beta T$, $\alpha \geqslant 1$, then*

$$I(T) := \int_1^T \frac{f(t)}{t} dt \ll \begin{cases} \log^{\beta+1} T & \text{if } \alpha = 1, \\ T^{\alpha-1} \log^\beta T & \text{if } \alpha > 1. \end{cases}$$

Proof. Let us divide the interval of integration into parts:

$$I(T) \leqslant \sum_{k=0}^{\log_2 T} \int_{T/2^{k+1}}^{T/2^k} \frac{f(t)}{t} dt < \sum_{k=0}^{\log_2 T} \frac{1}{T/2^{k+1}} \int_1^{T/2^k} f(t) dt \ll \sum_{k=0}^{\log_2 T} \frac{g(T/2^k)}{T/2^{k+1}}.$$

Now the lemma's statement follows from elementary estimates. □

Lemma 3.6. *Let $T > 10$ and $|d - 1/2| \ll 1/\log T$. Then we have the following estimates*

$$\int_{d-iT}^{d+iT} |\zeta(s)|^4 \frac{ds}{s} \ll \log^5 T \quad \text{and} \quad \int_{d-iT}^{d+iT} |L(s, \chi_4)|^4 \frac{ds}{s} \ll \log^5 T,$$

for growing T.

Proof. The statement is the result of the application of Lemma 3.5 to the estimates [6, Th. 10.1, p. 75]. □

Lemma 3.7. *Let $\theta > 0$ be a value such that $\zeta(1/2 + it) \ll t^\theta$ as $t \to \infty$, and let $\eta > 0$ be arbitrarily small. Then*

$$\zeta(s) \ll \begin{cases} |t|^{1/2-(1-2\theta)\sigma}, & \sigma \in [0, 1/2], \\ |t|^{2\theta(1-\sigma)}, & \sigma \in [1/2, 1-\eta], \\ |t|^{2\theta(1-\sigma)} \log^{2/3} |t|, & \sigma \in [1-\eta, 1], \\ \log^{2/3} |t|, & \sigma \geqslant 1. \end{cases}$$

The same estimates are valid for $L(s, \chi_4)$ as well.

Proof. The statement follows from Phragmén—Lindelöf principle, exact and approximate functional equations for $\zeta(s)$ and $L(s, \chi_4)$. See [4] and [10] for details. □

The best modern result [3] is that $\theta \leqslant 32/205 + \varepsilon$.

4. Main results

First we give maximal orders of $\tau_*^{(e)}$, $\mathfrak{t}^{(e)}$ and $\mathfrak{t}_*^{(e)}$.

The following theorem generalizes Lemma 3.4 to Gaussian integers; the proof's outline follows the proof of Lemma 3.4 in [9].

Theorem 4.1. *Let $F\colon \mathbb{Z}[i] \to \mathbb{C}$ be a multiplicative function such that $F(\mathfrak{p}^a) = f(a)$, where $f(n) \ll n^\beta$ for some $\beta > 0$. Then*

$$\limsup_{\alpha\to\infty} \frac{\log F(\alpha) \operatorname{llog} N(\alpha)}{\log N(\alpha)} = \sup_{n\geqslant 1} \frac{\log f(n)}{n} := K_f.$$

Proof. Let us fix arbitrarily small $\varepsilon > 0$.

Firstly, let us show that there are infinitely many α such that

$$\frac{\log F(\alpha) \operatorname{llog} N(\alpha)}{\log N(\alpha)} > K_f - \varepsilon.$$

By definition of K_f we can choose l such that $\bigl(\log f(l)\bigr)/l > K_f - \varepsilon/2$. It follows from (3) that for $x \geqslant 2$ the inequality $\sum'_{N(\mathfrak{p})\leqslant x} \log N(\mathfrak{p}) > Ax$ holds, where $0 < A < 1$.

Let $\mathfrak{q}$ be an arbitrarily large Gaussian prime, $N(\mathfrak{q}) \geqslant 2$. Consider

$$r = \sum_{N(\mathfrak{p})\leqslant N(\mathfrak{q})}{}' 1 \quad \text{and} \quad \alpha = \prod_{N(\mathfrak{p})\leqslant N(\mathfrak{q})}{}' \mathfrak{p}^l.$$

Then $F(\alpha) = \bigl(f(l)\bigr)^r$ and we have

$$r \log N(\mathfrak{q}) \geqslant \frac{\log N(\alpha)}{l} = \sum_{N(\mathfrak{p})\leqslant N(\mathfrak{q})}{}' \log N(\mathfrak{p}) > AN(\mathfrak{q}), \tag{6}$$

$$\log F(\alpha) = r \log f(l) \geqslant \frac{\log N(\alpha)}{\log N(\mathfrak{q})} \frac{\log f(l)}{l}. \tag{7}$$

But (6) implies

$$\log A + \log N(\mathfrak{q}) < \log \frac{\log N(\alpha)}{l} \leqslant \operatorname{llog} N(\alpha),$$

so $\log N(\mathfrak{q}) < \operatorname{llog} N(\alpha) - \log A$. Then it follows from (7) that

$$\log F(\alpha) > \frac{\log N(\alpha)}{\operatorname{llog} N(\alpha) - \log A} \frac{\log f(l)}{l}$$

and since $\bigl(\log f(l)\bigr)/l > K_f - \varepsilon/2$ and $A < 1$ we have

$$\frac{\log F(\alpha) \operatorname{llog} N(\alpha)}{\log N(\alpha)} > \frac{\operatorname{llog} N(\alpha)}{\operatorname{llog} N(\alpha) - \log A}(K_f - \varepsilon/2) > K_f - \varepsilon.$$

Second, let us show the existence of $N(\varepsilon)$ such that for all $n \geqslant N(\varepsilon)$ we get

$$\frac{\log F(n) \operatorname{llog} N(\alpha)}{\log N(\alpha)} < (1+\varepsilon)K_f.$$

Let us choose $\delta \in (0, \varepsilon)$ and $\eta \in \bigl(0, \delta/(1+\delta)\bigr)$. Suppose $N(\alpha) \geqslant 3$, and put

$$\omega := \omega(\alpha) = \frac{(1+\delta)K_f}{\operatorname{llog} N(\alpha)}, \qquad \Omega := \Omega(\alpha) = \log^{1-\eta} N(\alpha).$$

By choice of δ and η we have

$$\Omega^\omega = \exp(\omega \log \Omega) = \exp\bigl((1-\eta)(1+\delta)K_f\bigr) > e^{K_f}.$$

Suppose that the canonical expansion of α is

$$\alpha \sim \mathfrak{p}_1^{a_1} \cdots p_r^{a_r} \mathfrak{q}_1^{b_1} \cdots \mathfrak{q}_s^{b_s},$$

where $N(\mathfrak{p}_k) \leqslant \Omega$ and $N(\mathfrak{q}_k) > \Omega$. Then

$$\frac{F(\alpha)}{N^{\omega}(\alpha)} = \prod_{k=1}^{r} \frac{f(a_k)}{N^{\omega a_k}(\mathfrak{p}_k)} \cdot \prod_{k=1}^{s} \frac{f(b_k)}{N^{\omega b_k}(\mathfrak{q}_k)} := \Pi_1 \cdot \Pi_2 \tag{8}$$

Because of $\Omega^{\omega} > e^{K_f}$ and $K_f \geqslant \big(\log f(b_k)\big)/b_k$, we get

$$\frac{f(b_k)}{N^{\omega b_k}(q_k)} < \frac{f(b_k)}{\Omega^{\omega b_k}} < \frac{f(b_k)}{e^{K_f b_k}} \leqslant 1,$$

which implies $\Pi_2 \leqslant 1$. Consider Π_1. From the statement of the theorem we have $f(n) \ll n^{\beta}$, so

$$\frac{f(a_k)}{N^{\omega a_k}(p_k)} \ll \frac{a_k^{\beta}}{(\omega)^{a_k \beta}} \ll \omega^{-\beta}.$$

Then

$$\log \Pi_1 \ll \Omega \log w^{-\beta} \ll log^{1-\eta} N(\alpha) \operatorname{lllog} N(\alpha) = o\left(\frac{\log N(\alpha)}{\operatorname{llog} N(\alpha)}\right)$$

And finally by (8) we get

$$\log F(n) = \omega \log n + \log \Pi_1 + \log \Pi_2 = \frac{(1+\delta) K_f \log n}{\operatorname{llog} n} + \frac{(\varepsilon - \delta) K_f \log n}{\operatorname{llog} n}.$$

□

Theorem 4.2.

$$\begin{aligned}
\limsup_{n\to\infty} \frac{\log \tau_*^{(e)}(n) \operatorname{llog} n}{\log n} &= \frac{\log 3}{2}, \\
\limsup_{\alpha\to\infty} \frac{\log \mathfrak{t}^{(e)}(\alpha) \operatorname{llog} N(\alpha)}{\log N(\alpha)} &= \frac{\log 2}{2}, \\
\limsup_{\alpha\to\infty} \frac{\log \mathfrak{t}_*^{(e)}(\alpha) \operatorname{llog} N(\alpha)}{\log N(\alpha)} &= \frac{\log 3}{2}.
\end{aligned}$$

Proof. The first statement follows from (4) and Lemma 3.4. The second and the third statements follow from (4), (5) and Theorem 4.1. □

A simple corollary of the Theorem 4.2 is that

$$\tau_*^{(e)}(n) \ll n^{\varepsilon}, \qquad \mathfrak{t}^{(e)}(\alpha) \ll N^{\varepsilon}(\alpha), \qquad \mathfrak{t}_*^{(e)}(\alpha) \ll N^{\varepsilon}(\alpha). \tag{9}$$

Now we are ready to provide asymptotic formulas for sums of $\tau_*^{(e)}(n)$, $\mathfrak{t}^{(e)}(\alpha)$, $\mathfrak{t}_*^{(e)}(\alpha)$.
Let us denote

$$\begin{aligned}
G_*(s) &:= \sum_{n} \tau_*^{(e)}(n) n^{-s}, & T_*(x) &:= \sum_{n\leqslant x} \tau_*^{(e)}(n), \\
F(s) &:= {\sum_{\alpha}}' \mathfrak{t}^{(e)}(\alpha) N^{-s}(\alpha), & M(x) &:= {\sum_{N(\alpha)\leqslant x}}' \mathfrak{t}^{(e)}(\alpha), \\
F_*(s) &:= {\sum_{\alpha}}' \mathfrak{t}_*^{(e)}(\alpha) N^{-s}(\alpha), & M_*(x) &:= {\sum_{N(\alpha)\leqslant x}}' \mathfrak{t}_*^{(e)}(\alpha).
\end{aligned}$$

Lemma 4.3.

$$G_*(s) = \frac{\zeta(s)\zeta^2(2s)\zeta(5s)}{\zeta(3s)} K_*(s), \tag{10}$$

$$F(s) = \frac{Z(s)Z(2s)Z(6s)}{Z(5s)Z(7s)} H(s), \tag{11}$$

$$F_*(s) = \frac{Z(s)Z^2(2s)Z(5s)}{Z(3s)} H_*(s), \tag{12}$$

where Dirichlet series $H(s)$ is absolutely convergent for $\operatorname{Re} s > 1/8$ and the Dirichlet series for $H_*(s)$, $K_*(s)$ are absolutely convergent for $\operatorname{Re} s > 1/6$.

Proof. Bell series for $\mathfrak{t}^{(e)}$ have the following representation.

$$\mathfrak{t}^{(e)}_{\mathfrak{p}}(x) = \sum_{k=0}^{\infty} \mathfrak{t}^{(e)}(\mathfrak{p}^k)x^k = 1+x+2x^2+2x^3+3x^4+2x^5+4x^6+O(x^7) = \frac{(1-x^5)\big(1+O(x^7)\big)}{(1-x)(1-x^2)(1-x^6)}.$$

In the case of $\mathfrak{t}^{(e)}_*$ we have

$$\mathfrak{t}^{(e)}_{*\mathfrak{p}}(x) = \sum_{k=0}^{\infty} \mathfrak{t}^{(e)}_*(\mathfrak{p}^k)x^k = 1+x+3x^2+2x^3+5x^4+4x^5+6x^6+O(x^7) = \frac{(1-x^3)\big(1+O(x^6)\big)}{(1-x)(1-x^2)^2(1-x^5)}$$

and the same for $\tau^{(e)}_{*p}$.

Now (10), (11) and (12) follow from the representations of G_*, F, F_*, ζ and Z in the form of infinite products by p or $\mathfrak{p}$:

$$G_*(s) = \prod_p \tau^{(e)}_{*p}(p^{-s}), \qquad \zeta(s) = \prod_p (1-p^{-s})^{-1},$$

$$F(s) = \prod_{\mathfrak{p}} \mathfrak{t}^{(e)}_{\mathfrak{p}}(\mathfrak{p}^{-s}), \qquad F_*(s) = \prod_{\mathfrak{p}} \mathfrak{t}^{(e)}_{*\mathfrak{p}}(\mathfrak{p}^{-s}), \qquad Z(s) = \prod_{\mathfrak{p}} (1-\mathfrak{p}^{-s})^{-1}.$$

□

Theorem 4.4. *$T_*(x) = A_1 x + A_2 x^{1/2}\log x + A_3 x^{1/2} + O(x^{1/3+\varepsilon})$, where A_1, A_2, A_3 are computable constants.*

Proof. Identity (10) implies

$$\tau^{(e)}_* = \tau(1,2,2;\cdot) \star f, \qquad T_*(x) = \sum_{n\leqslant x} T(1,2,2;x/n) f(n), \tag{13}$$

where

$$\tau(1,2,2;n) = \sum_{ab^2c^2=n} 1, \qquad T(1,2,2;x) := \sum_{n\leqslant x} \tau(1,2,2;n) = \sum_{ab^2c^2\leqslant x} 1,$$

and series $\sum_{n=1}^{\infty} f(n)n^{-\sigma}$ is absolutely convergent for $\sigma > 1/3$. Due to [5, (6.4), (6.16)] we have

$$T(1,2,2;x) = \zeta^2(2)x + \frac{1}{2}\zeta(\tfrac12)x^{1/2}\log x + \big((2\gamma-1)\zeta(\tfrac12) + \frac12\zeta'(\tfrac12)\big)x^{1/2} + O(x^{8/25+\varepsilon}). \tag{14}$$

Let us define $C_1 = \zeta^2(2)$, $C_2 = \zeta(1/2)/2$, $C_3 = (2\gamma - 1)\zeta(1/2) + \zeta'(1/2)/2$ and

$$f_1 = \sum_{n=1}^{\infty} \frac{f(n)}{n}, \qquad f_2 = \sum_{n=1}^{\infty} \frac{f(n)}{n^{1/2}}, \qquad f_3 = \sum_{n=1}^{\infty} \frac{f(n)\log n}{n^{1/2}}.$$

One can get the following estimations.

$$\sum_{n>x} \frac{f(n)}{n} = O\left(x^{-2/3+\varepsilon} \sum_{n>x} \frac{f(n)}{n^{1/3+\varepsilon}} \right) = O(x^{-2/3+\varepsilon}), \tag{15}$$

$$\sum_{n>x} \frac{f(n)}{n^{1/2}} = O\left(x^{-1/6+\varepsilon} \sum_{n>x} \frac{f(n)}{n^{1/3+\varepsilon}} \right) = O(x^{-1/6+\varepsilon}), \tag{16}$$

$$\sum_{n>x} \frac{f(n)\log n}{n^{1/2}} = O\left(x^{-1/6+\varepsilon} \sum_{n>x} \frac{f(n)\log n}{n^{1/3+\varepsilon}} \right) = O(x^{-1/6+\varepsilon}). \tag{17}$$

Finally we get by substitution of estimates (14), (15), (16) and (17) into (13)

$$T_*(x) = C_1 x \sum_{n\leqslant x} \frac{f(n)}{n} + C_2 x^{1/2} \log x \sum_{n\leqslant x} \frac{f(n)}{n^{1/2}} - C_2 x^{1/2} \sum_{n\leqslant x} \frac{f(n)\log n}{n^{1/2}} + C_3 x^{1/2} \sum_{n\leqslant x} \frac{f(n)}{n^{1/2}} +$$
$$+ O(x^{8/25+\varepsilon}) = C_1 f_1 x + C_2 f_2 x^{1/2} \log x + (C_3 f_2 - C_2 f_3) x^{1/2} + O(x^{1/3+\varepsilon}).$$

□

Lemma 4.5.

$$\operatorname*{res}_{s=1} F(s)x^s/s = Cx, \qquad \operatorname*{res}_{s=1} F_*(s)x^s/s = C_* x, \tag{18}$$

where

$$C = \frac{\pi}{4} \prod_{\mathfrak{p}} \left(1 + \sum_{a=2}^{\infty} \frac{\tau(a) - \tau(a-1)}{N^a(\mathfrak{p})} \right) \approx 1{,}156\,101, \tag{19}$$

$$C_* = \frac{\pi}{4} \prod_{\mathfrak{p}} \left(1 + \sum_{a=2}^{\infty} \frac{\mathfrak{t}(a) - \mathfrak{t}(a-1)}{N^a(\mathfrak{p})} \right) \approx 1{,}524\,172. \tag{20}$$

Proof. As a consequence of the representation (11) we have

$$\frac{F(s)}{Z(s)} = \prod_{p} \left(1 + \sum_{a=1}^{\infty} \frac{\tau(a)}{N^{as}(\mathfrak{p})} \right) (1 - \mathfrak{p}^{-1}) = \prod_{\mathfrak{p}} \left(1 + \sum_{a=2}^{\infty} \frac{\tau(a) - \tau(a-1)}{N^{as}(\mathfrak{p})} \right),$$

and so function $F(s)/Z(s)$ is regular in the neighbourhood of $s = 1$. At the same time we have

$$\operatorname*{res}_{s=1} Z(s) = L(1, \chi_4) \operatorname*{res}_{s=1} \zeta(s) = \frac{\pi}{4},$$

which implies (19). The proof of (20) is similar.

Numerical values of C and C_* in (19) and (20) were calculated in PARI/GP [7] with the use of the transformation

$$\prod_{\mathfrak{p}} f\big(N(\mathfrak{p})\big) = f(2) \prod_{p=4k+1} f(p)^2 \prod_{p=4k+3} f(p^2)$$

due to Lemma 3.1. □

Theorem 4.6.

$$M(x) = Cx + O(x^{1/2}\log^{13/3} x), \tag{21}$$
$$M_*(x) = C_*x + O(x^{1/2}\log^{17/3} x), \tag{22}$$

where C *and* C_* *were defined in* (19) *and* (20).

Proof. By Perron formula and by (9) for $c = 1 + 1/\log x$, $\log T \asymp \log x$ we have

$$M(x) = \frac{1}{2\pi i}\int_{c-iT}^{c+iT} F(s)x^s s^{-1}\,ds + O\left(\frac{x^{1+\varepsilon}}{T}\right).$$

Suppose $d = 1/2 - 1/\log x$. Let us shift the interval of integration to $[d - iT, d + iT]$. To do this consider an integral about a closed rectangle path with vertexes in $d - iT$, $d + iT$, $c + iT$ and $c - iT$. There are two poles in $s = 1$ and $s = 1/2$ inside the contour. The residue at $s = 1$ was calculated in (18). The residue at $s = 1/2$ is equal to $Dx^{1/2}$, $D = \mathrm{const}$ and will be absorbed by error term (see below).

Identity (11) implies $F(s) = Z(s)Z(2s)H(s)$, where $H(s)$ is regular for $\operatorname{Re} s > 1/3$, so for each $\varepsilon > 0$ it is uniformly bounded for $\operatorname{Re} s > 1/3 + \varepsilon$.

Let us estimate the error term using Lemma 3.6 and Lemma 3.7. The error term absorbs values of integrals about three sides of the integration's rectangle. We take into account $Z(s) = \zeta(s)L(s, \chi_4)$. On the horizontal segments we have

$$\int_{d+iT}^{c+iT} Z(s)Z(2s)\frac{x^s}{s}ds \ll \max_{\sigma\in[d,c]} Z(\sigma + iT)Z(2\sigma + 2iT)x^\sigma T^{-1} \ll$$
$$\ll x^{1/2}T^{2\theta-1}\log^{4/3} T + xT^{-1}\log^{4/3} T,$$

It is well-known that $\zeta(s) \sim (s-1)^{-1}$ in the neighborhood of $s = 1$. So on the vertical segment we have

$$\int_{d}^{d+i} Z(s)Z(2s)\frac{x^s}{s}ds \ll x^{1/2}\int_0^1 \zeta(2d + 2it)dt \ll x^{1/2}\int_0^1 \frac{dt}{|it - 1/\log x|} \ll x^{1/2}\log x,$$

$$\int_{d+i}^{d+iT} Z(s)Z(2s)\frac{x^s}{s}ds \ll$$
$$\ll \left(\left(\int_1^T |\zeta(1/2 + it)|^4\frac{dt}{t}\int_1^T |L(1/2 + it, \chi_4)|^4\frac{dt}{t}\right)^{1/2}\int_1^T |Z(1 + 2it)|^2\frac{dt}{t}\right)^{1/2} \ll$$
$$\ll x^{1/2}(\log^5 T \cdot \log^{8/3+1} T)^{1/2} \ll x^{1/2}\log^{13/3} T.$$

The choice $T = x^{1/2+\varepsilon}$ finishes the proof of (21).

The proof of (22) is similar. □

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