



## SOME PROPERTIES OF BOUNDARY VALUE PROBLEMS FOR BESSEL'S EQUATION

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Some properties of systems of the Bessel functions of negative order less than  $-1$  generated by one boundary value problem are studied.

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Досліджено деякі властивості систем функцій Бесселя з від'ємним індексом меншим за  $-1$ , які виникають в одній крайовій задачі.

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Let  $\nu > 0$  be a non-integer number,  $\overline{0; \nu} = \mathbb{Z} \cap [0; \nu]$ ,  $p \in \mathbb{R}$ , and  $L_2((0; 1); x^p dx)$  be the Hilbert space of functions  $f : (0; 1) \rightarrow \mathbb{C}$  such that the function  $t^{p/2} f(t)$  belongs to the space  $L_2(0; 1)$ ; the inner product and the norm in  $L_2((0; 1); x^p dx)$  are given respectively by  $\langle f_1; f_2 \rangle = \int_0^1 t^p f_1(t) \overline{f_2(t)} dt$  and  $\|f\| = \sqrt{\int_0^1 t^p |f(t)|^2 dt}$ . Further, let  $J_\nu$  be Bessel's function of order  $\nu$ . We study the approximation properties of the system  $\{J_{-\nu}(\rho_j x)\}$  for some sequence  $(\rho_j)$  tending to infinity. Such a system arises, for example, when considering the boundary value problem

$$-f'' + \frac{\nu^2 - 1/4}{x^2} f = \lambda f, \quad (1)$$

$$f(1) = 0, \quad (2)$$

$$\exists c_k \in \mathbb{C} : f(x) = \sum_{k \in \overline{0; \nu}} c_k x^{-\nu+2k+1/2} + o(x^{\nu+1/2}), \quad x \rightarrow 0+. \quad (3)$$

Indeed, one easily derives the following result:

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**Proposition 1.** *The boundary value problem (1)–(3) has a countable set of eigenvalues  $\{\lambda_j : j \in \mathbb{N}\}$ , and  $\lambda_j = \rho_j^2$ , where  $\rho_j$  are the zeros of the Bessel function  $J_{-\nu}$ ; moreover,  $v_j(x) = \rho_j^\nu \sqrt{\pi x/2} J_{-\nu}(\rho_j x)$  are the corresponding eigenfunctions.*

*Proof.* For  $\lambda \neq 0$ , equation (1) has two linearly independent solutions given by

$$u(x) = \rho^{-\nu-1/2} \sqrt{\pi \rho x/2} J_\nu(\rho x), \quad v(x) = \chi_\nu \rho^{\nu-1/2} \sqrt{\pi \rho x/2} J_{-\nu}(\rho x),$$

where  $\rho = \sqrt{\lambda}$  and  $\chi_\nu = -1/\sin(\nu\pi)$ . Using the asymptotic behaviour of the Bessel functions, we find that

$$u(x) = \sqrt{\pi/2} \frac{x^{\nu+1/2}}{2^\nu \Gamma(\nu+1)} + o(x^{\nu+1/2}), \quad x \rightarrow 0+,$$

$$v(x) = \chi_\nu \sqrt{\pi/2} \sum_{k \in \mathbb{N}; \nu} \frac{(-1)^k \rho^{2k} x^{-\nu+2k+1/2}}{2^{-\nu+2k} k! \Gamma(-\nu+k+1)} + O(x^{-\nu+2[\nu]+5/2}), \quad x \rightarrow 0+.$$

Therefore the general solution  $\alpha u(x) + \beta v(x)$  of (1) satisfies the condition (3) if and only if  $\alpha = 0$ . Recalling (2), we conclude that the eigenfunctions of the boundary value problem (1)–(3) are  $v_j(x) = \rho_j^\nu \sqrt{\pi x/2} J_{-\nu}(\rho_j x)$ , and the corresponding eigenvalues are  $\{\lambda_j : j \in \mathbb{N}\}$ , where  $\lambda_j = \rho_j^2$  and  $\rho_j$  are the zeros of the Bessel function  $J_{-\nu}$ . The proposition is proved.  $\square$

The function  $J_{-3/2}$  has ([1]) an infinite set  $\{\rho_j : j \in \mathbb{Z} \setminus \{0\}\}$  of zeros; moreover, the zeros  $\rho_1$  and  $\rho_{-1} = \overline{\rho_1} = -\rho_1$  are purely imaginary,  $\rho_j$  with  $j \in \mathbb{N} \setminus \{1\}$  are positive, and  $\rho_{-j} := -\rho_j$ ,  $j \in \mathbb{N} \setminus \{1\}$ , are negative. In [2]–[4] the following results were proved.

**Theorem A.** *Let  $\rho_j$  be the zeros of the function  $J_{-3/2}$ . Then the system  $\{\rho_j \sqrt{\rho_j x} J_{-3/2}(\rho_j x) : j \in \mathbb{N} \setminus \{1\}\}$  is complete in the space  $L_2((0; 1); x^2 dx)$ , has in this space a biorthogonal system  $\{\gamma_k : k \in \mathbb{N} \setminus \{1\}\}$ ,*

$$\overline{\gamma_k}(x) := \frac{\pi(\rho_k \sqrt{\rho_k x} J_{-3/2}(\rho_k x) - \rho_1 \sqrt{\rho_1 x} J_{-3/2}(\rho_1 x))}{x^2 \rho_k^2 \cos^2 \rho_k},$$

*and is not a basis of this space.*

**Theorem B.** *For  $\nu = 3/2$  the boundary value problem (1)–(3) has a countable set of eigenvalues  $\{\lambda_j : j \in \mathbb{N}\}$ ; moreover,  $\lambda_j = \rho_j^2$ , where  $\rho_j$  are the zeros of the Bessel function  $J_{-3/2}$ . In particular, all eigenvalues are real,  $\lambda_1$  is negative, and  $\lambda_j$  are positive for  $j > 1$ . The corresponding eigenfunctions  $\rho_j \sqrt{\rho_j x} J_{-3/2}(\rho_j x)$  enjoy the properties described in Theorem A.*

Similar results can be obtained for  $\nu = 5/2$ . Their extension to arbitrary  $\nu$  leads to some difficulties. In this connection we will prove here the following proposition.

**Theorem 1.** *Let  $z \in \mathbb{C}$  and let  $\rho_j$  be the zeros of the function  $J_{-\nu}$  with  $\nu = m + 1/2$ ,  $m \in \mathbb{N}$ . Then*

$$\int_0^1 \cos(zt) \frac{J_{-\nu}(\rho_j t)}{(\rho_j t)^{-\nu}} dt = (-1)^{m+1} \sqrt{\frac{\pi}{2}} \rho_j^{1+\nu} J_{-\nu+1}(\rho_j) \left( \frac{1}{z} \frac{d}{dz} \right)^m \frac{J_{-\nu}(z)}{(z^2 - \rho_j^2) z^{-\nu}},$$

*where  $(z^{-1} d/dz)^m$  is the  $m$ -th power of the differential operator  $z^{-1} d/dz$ .*

*Proof.* The functions  $f(t) = J_{\pm\nu}(zt)$  are the solutions of the equation  $f'' + f'/t - \nu^2 f/t^2 = -z^2 f$ . Hence

$$\begin{aligned} \frac{d}{dt} \left( t \frac{dJ_{-\nu}(zt)}{dt} \right) + \left( z^2 t - \frac{\nu^2}{t} \right) J_{-\nu}(zt) &= 0, \\ \frac{d}{dt} \left( t \frac{dJ_{-\nu}(\rho_j t)}{dt} \right) + \left( \rho_j^2 t - \frac{\nu^2}{t} \right) J_{-\nu}(\rho_j t) &= 0. \end{aligned}$$

Multiply the first equality by  $J_{-\nu}(\rho_j t)$ , the second by  $J_{-\nu}(zt)$  and subtract. Then

$$\frac{d}{dt} \left( t \left( \frac{dJ_{-\nu}(zt)}{dt} J_{-\nu}(\rho_j t) - \frac{dJ_{-\nu}(\rho_j t)}{dt} J_{-\nu}(zt) \right) \right) = -(z^2 - \rho_j^2) t J_{-\nu}(zt) J_{-\nu}(\rho_j t)$$

or, in turn,

$$\rho_j^{-\nu} t^{-2\nu+1} \frac{J_{-\nu}(zt)}{(zt)^{-\nu}} \frac{J_{-\nu}(\rho_j t)}{(\rho_j t)^{-\nu}} = \frac{d}{dt} \left( t \frac{-(J_{-\nu}(zt))'_t J_{-\nu}(\rho_j t) + (J_{-\nu}(\rho_j t))'_t J_{-\nu}(zt)}{(z^2 - \rho_j^2) z^{-\nu}} \right),$$

and

$$\begin{aligned} \rho_j^{-\nu} t^{-2\nu+1} \frac{J_{-\nu}(\rho_j t)}{(\rho_j t)^{-\nu}} \left( \frac{1}{z} \frac{\partial}{\partial z} \right)^m \frac{J_{-\nu}(zt)}{(zt)^{-\nu}} &= \\ &= \left( \frac{1}{z} \frac{\partial}{\partial z} \right)^m \frac{\partial}{\partial t} \left( t \frac{-(J_{-\nu}(zt))'_t J_{-\nu}(\rho_j t) + (J_{-\nu}(\rho_j t))'_t J_{-\nu}(zt)}{(z^2 - \rho_j^2) z^{-\nu}} \right). \end{aligned}$$

Using the equality [5, p.56]

$$\frac{d}{dz} \frac{J_{-\nu}(z)}{z^{-\nu}} = -z^\nu J_{-\nu+1}(z),$$

we obtain

$$\left( \frac{1}{z} \frac{\partial}{\partial z} \right)^m \frac{J_{-\nu}(zt)}{(zt)^{-\nu}} = (-1)^m t^{2m} \frac{J_{-\nu+m}(zt)}{(zt)^{-\nu+m}}.$$

Therefore

$$\begin{aligned} (-1)^m \rho_j^{-\nu} \frac{J_{-\nu}(\rho_j t)}{(\rho_j t)^{-\nu}} t^{-2\nu+1+2m} \frac{J_{-\nu+m}(zt)}{(zt)^{-\nu+m}} &= \\ &= \left( \frac{1}{z} \frac{\partial}{\partial z} \right)^m \frac{\partial}{\partial t} \left( t \frac{-(J_{-\nu}(zt))'_t J_{-\nu}(\rho_j t) + (J_{-\nu}(\rho_j t))'_t J_{-\nu}(zt)}{(z^2 - \rho_j^2) z^{-\nu}} \right). \end{aligned}$$

Since  $-\nu + m = -1/2$  and  $t^{1/2} J_{-1/2}(t) = \sqrt{2/\pi} \cos t$ , we have

$$\begin{aligned} (-1)^m \rho_j^{-\nu} \sqrt{\frac{2}{\pi}} \frac{J_{-\nu}(\rho_j t)}{(\rho_j t)^{-\nu}} \cos(zt) &= \\ &= \left( \frac{1}{z} \frac{\partial}{\partial z} \right)^m \frac{\partial}{\partial t} \left( t \frac{-(J_{-\nu}(zt))'_t J_{-\nu}(\rho_j t) + (J_{-\nu}(\rho_j t))'_t J_{-\nu}(zt)}{(z^2 - \rho_j^2) z^{-\nu}} \right) = \\ &= \frac{\partial}{\partial t} \left( \frac{1}{z} \frac{\partial}{\partial z} \right)^m \left( t \frac{-(J_{-\nu}(zt))'_t J_{-\nu}(\rho_j t) + (J_{-\nu}(\rho_j t))'_t J_{-\nu}(zt)}{(z^2 - \rho_j^2) z^{-\nu}} \right). \end{aligned}$$

Hence

$$\begin{aligned}
 & (-1)^m \rho_j^{-\nu} \sqrt{\frac{2}{\pi}} \int_0^1 \cos(zt) \frac{J_{-\nu}(\rho_j t)}{(\rho_j t)^{-\nu}} dt = \\
 & = \int_0^1 \frac{\partial}{\partial t} \left( \frac{1}{z} \frac{\partial}{\partial z} \right)^m \left( t \frac{-(J_{-\nu}(zt))'_t J_{-\nu}(\rho_j t) + (J_{-\nu}(\rho_j t))'_t J_{-\nu}(zt)}{(z^2 - \rho_j^2) z^{-\nu}} \right) dt = \\
 & = \left( \frac{1}{z} \frac{\partial}{\partial z} \right)^m \left. t \frac{-(J_{-\nu}(zt))'_t J_{-\nu}(\rho_j t) + (J_{-\nu}(\rho_j t))'_t J_{-\nu}(zt)}{(z^2 - \rho_j^2) z^{-\nu}} \right|_0^1 = \\
 & = \rho_j J'_{-\nu}(\rho_j) \left( \frac{1}{z} \frac{d}{dz} \right)^m \frac{J_{-\nu}(z)}{(z^2 - \rho_j^2) z^{-\nu}}.
 \end{aligned}$$

But ([5, p.56])  $tJ'_{-\nu}(t) + \nu J_{-\nu}(t) = -tJ_{-\nu+1}(t)$ . Then  $J'_{-\nu}(\rho_j) = -J_{-\nu+1}(\rho_j)$ , and we obtain the desired result. The theorem is proved.  $\square$

Theorem 1 for  $\nu = 3/2$  was used to prove completeness of the corresponding systems of eigenfunction in Theorem B.

## REFERENCES

1. A. Hurwitz, *Ueber die Nullstellen der Bessel'schen Function*, Math. Ann. **33** (1889), 246–266 (in German).
2. B.V. Vynnyts'kyi, O.V. Shavala, *Boundness of the solutions of a second-order linear differential equation and a boundary value problem for Bessel's equation*, Matematychni Studii **30**:1 (2008), 31–41 (in Ukrainian).
3. B.V. Vynnyts'kyi, O.V. Shavala, Abstracts, International Conference Analysis and Topology (26 May – 7 June 2008), Lviv, (2008), p. 54–55.
4. O.V. Shavala, *Some properties of linear differential equations of the second order with meromorphic coefficients*, Cand. Sci. (Phys.-Math.) Dissertation, Drohobych, 2008, 127 p. (in Ukrainian)
5. G.N. Watson, *A treatise on the theory of Bessel functions, Part I*, Publishing Foreign Literature, Moskow (1949), 798p. (in Russian); Translated from: At the University Press, Cambridge (1944), 804p.

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