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STOCHASTIC INTEGRAL AND STOCHASTIC DERIVATIVE CONNECTED WITH A LÉVY PROCESS

The extended stochastic integral with respect to a Lévy process and the corresponding Hida stochastic derivative have many applications in the stochastic analysis, in particular, in the theory of stochastic differential and integral equations. One of main lacks of these operators, if one considers them as linear operators on the space of square integrable random variables (L^2), consists in their unboundedness. Moreover, the domain of the extended stochastic integral (and of the conjugated to this integral operator – the stochastic derivative) depends on an interval of integration. As a result, there are problems with some applications of the above-mentioned operators. Therefore an important problem is a modification of definitions of the extended stochastic integral and the Hida stochastic derivative that gives us the possibility to define these operators as linear bounded (i.e., continuous) ones. In this paper, using the theory of Hilbert equipments, we introduce and study the extended stochastic integral with respect to a Lévy process and the Hida stochastic derivative as linear continuous operators on a suitable rigging of (L^2). This gives a possibility to extend an area of applications of these operators.

Introduction

Denote $R_+ := [0,+\infty)$. Let $L = (L_t)_{t\in R_+}$ be a Lévy process, i.e., a random process on R_+ with stationary independent increments and such that $L_0 = 0$ (e.g., [1]). In particular cases, when L is a Wiener or Poisson process, any square integrable random variable can be decomposed in a series of repeated stochastic integrals from nonrandom functions with respect to L. This property of L, known as the *chaotic representation property* (CRP) (e.g., [2]), plays a very important role in the stochastic analysis. In particular, it can be used in order to construct extended stochastic integrals, see, e.g., [3]. Unfortunately, for a general Lévy process the CRP does not hold (e.g., [4]).

There are different generalizations of the CRP for Lévy processes. In particular, under the Itô's approach [5] one decomposes a Lévy process L in the sum of a Gaussian process and a stochastic integral with respect to a Poisson random measure and then uses the CRP for both terms in order to obtain a generalized CRP for L. The Nualart-Schoutens's approach [6] consists in decomposition of a square integrable random variable in a series of repeated stochastic integrals from nonrandom functions with respect to so-called *orthogonalized centered power jump processes*, these processes are constructed with using of a cádlág version of L (i.e., of a random process, which is stochastically equivalent to L and has right continuous with finite left limits trajectories). The Lytvynov's approach [7] is based on orthogonalization of continuous monomials in the space of square integrable random variables.

The interconnection between above-mentioned generalizations of the CRP is described in, e.g., [7–10].

In order to construct an extended (Skorohod) stochastic integral with respect to L, one can take any generalization of the CRP described above, see [10] for details. But if we consider this integral as an operator on the space of square integrable random variables, as in [10], then this operator is unbounded and, moreover, its domain depends on the interval of integration. The conjugate operator of the extended stochastic integral (the Hida stochastic derivative) is, of course, also an unbounded linear operator on the space of square integrable random variables. The unboundedness of these operators lead to some problems with their applications. Therefore an important problem is a modification of definitions of the extended stochastic integral and the Hida stochastic derivative that gives us the possibility to define these operators as linear *bounded* (i.e., *continuous*) ones. In order to do this, we offer to define the stochastic integral and derivative on a suitable rigging of the space of square integrable random variables.

Problem definition

The aim of this paper is to define the extended (Skorohod) stochastic integral with respect to a Lévy process and the corresponding Hida stochastic derivative as linear *continuous* operators on a suitable rigging of the space of square integrable random variables; and to describe some properties of these operators.

Preliminaries

In this paper we deal with a real-valued locally square integrable Lévy process L without Gaussian part and drift. As is well known, the characteristic function of L is

$$E[e^{iuL_t}] = \exp[t\int_R (e^{iux} - 1 - iux)\nu(dx)], \quad (1)$$

where ν is the Lévy measure of L, E denotes the expectation. We assume that ν *is a Radon measure whose support contains an infinite number of points,* $\nu(\{0\}) = 0$, *there exists* $\varepsilon > 0$ *such that* $\int_R x^2 e^{\varepsilon|x|}\nu(dx) < \infty$, *and* $\int_R x^2\nu(dx) = 1$.

Let us define a measure of the white noise of L. Let D denote the set of all real-valued infinite-differentiable functions on R_+ with compact supports. As is well known, D can be endowed by the projective limit topology generated by some Sobolev spaces (see, e.g., [11]}). Let D' be the set of linear continuous functionals on D. For $\omega \in D'$ and $\varphi \in D$ denote $\omega(\varphi) = \langle\omega, \varphi\rangle$; note that one can understand $\langle\cdot,\cdot\rangle$ as the dual pairing generated by the scalar product in the space $L^2(R_+)$ of square integrable with respect to the Lebesgue measure real-valued functions on R_+. The notation $\langle\cdot,\cdot\rangle$ will be preserved for dual pairings in tensor powers of spaces.

A probability measure μ on $(D', C(D'))$, where C denotes the cylindrical σ-algebra, with the Fourier transform

$$\int_{D'} e^{i\langle\omega,\varphi\rangle}\mu(d\omega) =$$

$$= \exp[\int_{R_+\times R}(e^{i\varphi(u)x} - 1 - i\varphi(u)x)du\nu(dx)], \quad (2)$$

$$\varphi \in D,$$

is called the *Lévy white noise measure.*

We reckon that the σ-algebra $C(D')$ is augmented with respect to μ. Denote $(L^2) := L^2(D', C(D'), \mu)$ the space of real-valued square integrable with respect to μ functions on D'; let also $H := L^2(R_+)$. Let $f \in H$ and a sequence $(\varphi_k \in D)_{k\in N}$ converges to f in H as $k \to \infty$. One can show ([7, 12, 10]) that $\langle\circ, f\rangle := (L^2) - \lim_{k\to\infty}\langle\circ, \varphi_k\rangle$ is well-definite as an element of (L^2). Let us consider $\langle\circ, 1_{[0,t)}\rangle \in (L^2)$, $t \in R_+$ (here and below 1_A denotes the indicator of a set A). It follows from (1) and (2) that $(\langle\circ, 1_{[0,t)}\rangle)_{t\in R_+}$ can be identified with a Lévy process L.

Finally, consider the Lytvynov's generalization of the CRP (see [7] for details). Denote by $\hat{\otimes}$ the symmetric tensor product. Let P_n ($n \in Z_+ := N \cup \{0\}$) be the set of continuous polynomials on D' of power $\le n$ (elements of P_n have a form $\varphi(x) = \sum_{k=0}^{m}\langle x^{\otimes k}, \varphi^{(k)}\rangle$, $\varphi^{(k)} \in D^{\hat{\otimes} k}$, $x \in D'$, $m \le n$). Denote by $\bar{P}_n$ the closure of P_n in (L^2). Let for $n \in N$ $\mathbf{P}_n$ be defined from the condition $\bar{P}_n = \mathbf{P}_n \oplus \bar{P}_{n-1}$, $\mathbf{P}_0 := \bar{P}_0$. Let $f^{(n)} \in D^{\hat{\otimes} n}$, $n \in Z_+$. Denote by $:\langle\circ^{\otimes n}, f^{(n)}\rangle:$ the orthogonal projection of a monomial $\langle\circ^{\otimes n}, f^{(n)}\rangle$ onto $\mathbf{P}_n$. (Note that for $n > 1$ $:\langle\circ^{\otimes n}, f^{(n)}\rangle:$ is not a continuous polynomial, generally speaking ([7]).) Let us define scalar products $\langle\cdot,\cdot\rangle_{\text{ext}}$ on $D^{\hat{\otimes} n}$, $n \in Z_+$, by setting for $f^{(n)}, g^{(n)} \in D^{\hat{\otimes} n}$

$$\langle f^{(n)}, g^{(n)}\rangle_{\text{ext}} :=$$

$$:= \frac{1}{n!}\int_{D'} :\langle\omega^{\otimes n}, f^{(n)}\rangle::\langle\omega^{\otimes n}, g^{(n)}\rangle: \mu(d\omega),$$

and let $|\cdot|_{\text{ext}}$ be the corresponding norms. Denote by $H_{\text{ext}}^{(n)}$, $n \in Z_+$, the closures of $D^{\hat{\otimes} n}$ with respect to the norms $|\cdot|_{\text{ext}}$.

Theorem 1 ([7]). Let $F \in (L^2)$. Then there exists a unique sequence of kernels $f^{(n)} \in H_{\text{ext}}^{(n)}$, $n \in Z_+$, such that

$$F = \sum_{n=0}^{\infty} :\langle\circ^{\otimes n}, f^{(n)}\rangle: \quad (3)$$

and

$$E|F|^2 = \|F\|_{(L^2)}^2 =$$

$$= \int_{D'}|F(\omega)|^2\mu(d\omega) = \sum_{n=0}^{\infty} n!|f^{(n)}|_{\text{ext}}^2.$$

One can show ([7,10]) that $H_{\text{ext}}^{(1)} = H$ and for $n \in N\backslash\{1\}$ one can identify $H^{\hat{\otimes} n}$ with the proper subspace of $H_{\text{ext}}^{(n)}$ that consists of "vanishing on diagonals” elements. In this sense the *space* $H_{\text{ext}}^{(n)}$ *is an extension of* $H^{\hat{\otimes} n}$ (this explains why we used the subscript ext in the designations $H_{\text{ext}}^{(n)}$, $\langle\cdot,\cdot\rangle_{\text{ext}}$ and $|\cdot|_{\text{ext}}$).

An extended stochastic integral

Let $F \in (L^2)\otimes H$. By (3) F can be uniquely presented in the form

$$F(\cdot)=\sum_{n=0}^{\infty} :\langle \circ^{\otimes n}, f_{\cdot}^{(n)}\rangle:,\quad f_{\cdot}^{(n)} \in H_{\text{ext}}^{(n)}\otimes H. \quad (4)$$

If the kernels from (4) $f_{\cdot}^{(n)} \in H^{\hat{\otimes} n}\otimes H \subset H_{\text{ext}}^{(n)}\otimes H$ then one can define the extended stochastic integral from F with respect to L by analogy with the Gaussian case [3], i.e.,

$$\int_{R_+} F(u)\hat{d}L_u := \sum_{n=0}^{\infty} :\langle \circ^{\otimes n+1}, \hat{f}^{(n)}\rangle:, \quad (5)$$

where $\hat{f}^{(n)} \in H^{\hat{\otimes} n+1}$ are the projections of $f_{\cdot}^{(n)}$ onto $H^{\hat{\otimes} n+1}\subset H_{\text{ext}}^{(n+1)}$, if series (5) converges in (L^2). In a general case one can not project elements of $H_{\text{ext}}^{(n)}\otimes H$ onto $H_{\text{ext}}^{(n+1)}$; nevertheless, by $f_{\cdot}^{(n)} \in H_{\text{ext}}^{(n)}\otimes H$ one can construct kernels $\hat{f}^{(n)} \in H_{\text{ext}}^{(n+1)}$ that can be used in order to define the extended stochastic integral by (5). Namely, let $f_{\cdot}^{(n)} \in H_{\text{ext}}^{(n)}\otimes H$, $n\in N$. We select a representative (a function) $\dot{f}_{\cdot}^{(n)} \in f_{\cdot}^{(n)}$ such that $\dot{f}_u^{(n)}(u_1,...,u_n)=0$ if for some $k\in\{1,...,n\}$ $u=u_k$. Let $\tilde{f}^{(n)}$ be the symmetrization of $\dot{f}_u^{(n)}$ by $n+1$ variables. Define $\hat{f}^{(n)} \in H_{\text{ext}}^{(n+1)}$ as the equivalence class in $H_{\text{ext}}^{(n+1)}$ generated by $\tilde{f}^{(n)}$. It is proved in [10] that $\hat{f}^{(n)}$ is well-definite and

$$|\hat{f}^{(n)}|_{\text{ext}} \le |f_{\cdot}^{(n)}|_{H_{\text{ext}}^{(n)}\otimes H}. \quad (6)$$

For $F\in (L^2)\otimes H$ we define an extended stochastic integral $\int_{R_+} F(u)\hat{d}L_u \in (L^2)$ by formula (5), where $\hat{f}^{(n)} \in H_{\text{ext}}^{(n+1)}$ are constructed by the kernels $f_{\cdot}^{(n)} \in H_{\text{ext}}^{(n)}\otimes H$ from decomposition (4) as above, $\hat{f}^{(0)} := f_{\cdot}^{(0)}$. The domain of this integral, i.e., of the operator $\int_{R_+} \circ(u)\hat{d}L_u : (L^2)\otimes H \to (L^2)$, consists of $F\in (L^2)\otimes H$ such that $\|\int_{R_+} F(u)\hat{d}L_u\|_{(L^2)}^2 = \sum_{n=0}^{\infty}(n+1)!\,|\hat{f}^{(n)}|_{\text{ext}}^2 < \infty$.

For $t_1,t_2\in[0,+\infty]$, $t_1<t_2$, set

$$\int_{t_1}^{t_2} F(u)\hat{d}L_u := \int_{R_+} F(u)1_{[t_1,t_2)}(u)\hat{d}L_u. \quad (7)$$

Theorem 2 ([10]). Let $F\in (L^2)\otimes H$ be integrable by Itô. Then for any $t_1,t_2\in[0,+\infty]$, $t_1<t_2$, F is integrable on $[t_1,t_2)$ in the extended sense and $\int_{t_1}^{t_2} F(u)\hat{d}L_u = \int_{t_1}^{t_2} F(u)\,dL_u$, where the last integral is the Itô one.

One of main lacks of the extended stochastic integral consists in its unboundedness and, moreover, in dependence of its domain on an interval of integration. This essentially constricts an area of possible applications. A possible solution of this problem – to define the stochastic integral as an operator acting from $(L^2)\otimes H$ to a suitable space that must be wider than (L^2). Let us describe a possible construction of such a space. Define $(L^2)_1 \subset (L^2)$ as a Hilbert space of functions $F: D'\to R$ of form (3) for which $\|F\|_{(L^2)_1}^2 = \sum_{n=0}^{\infty} n!2^n |f^{(n)}|_{\text{ext}}^2 < \infty$. Let $(L^2)_{-1}$ be the Hilbert space of generalized functions that is dual of $(L^2)_1$ with respect to (L^2), i.e., $(L^2)_{-1}$ is the negative space of the chain $(L^2)_1\subset (L^2)\subset (L^2)_{-1}$. This space consists of formal series of form (3) with $\|F\|_{(L^2)_{-1}}^2 = \sum_{n=0}^{\infty} n!2^{-n}|f^{(n)}|_{\text{ext}}^2 < \infty$.

For $F\in (L^2)\otimes H$, $t_1,t_2\in[0,+\infty]$, $t_1<t_2$, we define the extended stochastic integrals $\int_{R_+} F(u)\hat{d}L_u, \int_{t_1}^{t_2} F(u)\hat{d}L_u \in (L^2)_{-1}$ by formulas (5) and (7) respectively. Note that one can understand $\int_{R_+} F(u)\hat{d}L_u$ as a particular case of $\int_{t_1}^{t_2} F(u)\hat{d}L_u$ with $t_1=0$ and $t_2=+\infty$. Further, since $(L^2)_1\otimes H\subset (L^2)\otimes H$, one can consider the linear operator $\int_{t_1}^{t_2} \circ(u)\hat{d}L_u : (L^2)_1\otimes H \to (L^2)$.

Theorem 3. The operators $\int_{t_1}^{t_2} \circ(u)\hat{d}L_u : (L^2)\otimes H\to (L^2)_{-1}$ and $\int_{t_1}^{t_2} \circ(u)\hat{d}L_u : (L^2)_1\otimes H \to (L^2)$ are linear continuous ones.

One can obtain this result by direct calculation of norms of the integrals with use of estimate (6).

A Hida stochastic derivative

Let $g^{(n)} \in H^{(n)}_{\text{ext}}$, $n \in N$. It is shown in [10] that $g^{(n)}$ can be considered as an element $g^{(n)}(\cdot)$ of $H^{(n-1)}_{\text{ext}} \otimes H$ (but $H^{(n)}_{\text{ext}} \not\subset H^{(n-1)}_{\text{ext}} \otimes H$ because different elements of $H^{(n)}_{\text{ext}}$ can coincide as elements of $H^{(n-1)}_{\text{ext}} \otimes H$), and

$$|g^{(n)}(\cdot)|_{H^{(n-1)}_{\text{ext}} \otimes H} \le |g^{(n)}|_{\text{ext}}. \tag{8}$$

Let $t_1, t_2 \in [0, +\infty]$, $t_1 < t_2$. For $G \in (L^2)$ we define a Hida stochastic derivative $1_{[t_1,t_2)}(\cdot)\partial_{\cdot} G \in (L^2) \otimes H$ by setting

$$1_{[t_1,t_2)}(\cdot)\partial_{\cdot} G :=$$

$$:= \sum_{n=0}^{\infty} (n+1) : \langle \circ^{\otimes n}, g^{(n+1)}(\cdot) 1_{[t_1,t_2)}(\cdot) \rangle :,$$

where $g^{(n+1)} \in H^{(n+1)}_{\text{ext}}$ are the kernels from decomposition (3) for G, in point as elements of $H^{(n)}_{\text{ext}} \otimes H$. The domain of this derivative, i.e., of the operator $1_{[t_1,t_2)}(\cdot)\partial_{\cdot} : (L^2) \to (L^2) \otimes H$, consists of $G \in (L^2)$ such that

$$\| 1_{[t_1,t_2)}(\cdot)\partial_{\cdot} G \|^2_{(L^2)\otimes H} =$$

$$= \sum_{n=0}^{\infty} (n+1)!(n+1) |g^{(n+1)}(\cdot) 1_{[t_1,t_2)}(\cdot)|^2_{H^{(n)}_{\text{ext}} \otimes H} < \infty.$$

It is shown in [10] that the Hida stochastic derivative $1_{[t_1,t_2)}(\cdot)\partial_{\cdot} : (L^2) \to (L^2) \otimes H$ and the extended stochastic integral $\int_{t_1}^{t_2} \circ(u) \widehat{d} L_u : (L^2) \otimes H \to (L^2)$ are conjugated one to another and, in particular, are closed operators.

As in the case of the extended stochastic integral, the Hida stochastic derivative will be a linear *continuous* operator if we suitably extend a “target” space. More exactly, the next statement is fulfilled.

Theorem 4. The operators $1_{[t_1,t_2)}(\cdot)\partial_{\cdot} : (L^2) \to (L^2)_{-1} \otimes H$ and $1_{[t_1,t_2)}(\cdot)\partial_{\cdot} : (L^2)_1 \to (L^2) \otimes H$ are linear continuous ones.

One can obtain this result by direct calculation of norms of the derivatives with use of estimate (8).

Finally, the next statement can be proved by analogy with [10].

Theorem 5. The extended stochastic integral $\int_{t_1}^{t_2} \circ(u) \widehat{d} L_u : (L^2) \otimes H \to (L^2)_{-1}$ and the Hida stochastic derivative $1_{[t_1,t_2)}(\cdot)\partial_{\cdot} : (L^2)_1 \to (L^2) \otimes H$ are conjugated one to another.

It is easy to verify that this statement holds true if we consider $\int_{t_1}^{t_2} \circ(u) \widehat{d} L_u : (L^2)_1 \otimes H \to (L^2)$ and correspondingly $1_{[t_1,t_2)}(\cdot)\partial_{\cdot} : (L^2) \to (L^2)_{-1} \otimes H$.

Conclusions

In this paper the extended stochastic integral with respect to a Lévy process and the corresponding Hida stochastic derivative are defined as linear *continuous* operators on a rigging of the space (L^2) of square integrable random variables, and the interconnection between these operators is established. This gives a possibility to extend an area of applications of the mentioned stochastic integral and derivative. In the forthcoming papers we’ll consider these operators on more general riggings of (L^2), study the Wick calculus and its interconnection with the stochastic integration, etc.

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