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STRONG LAW OF LARGE NUMBERS FOR RANDOM VARIABLES WITH SUPERADDITIVE MOMENT FUNCTION

In this paper, we study random variables with moment function of superadditive structure. We do not impose any assumptions on the structure of dependence of these random variables. We prove the strong law of large numbers for such random variables under regularly varying normalization by the method developed by Fazekas and Klesov. In this proof we use different properties of superadditive and ragularly varying functions. The key role in the proof is played by the possibility of approximating the nondifferentiable slowly varying function by differentiable slowly varying function. This result can be applied for obtaining strong law of large numbers for independent, orthogonal and stationary dependent random variables, submartingales. It can be used for obtaining an analogical result in the case of random fields.

Introduction

Strong law of large numbers for different types of random variables are studied very widely in probability theory and mathematical statistics. Such results have wide application in economics, biology, chemistry and other sciences. There are two basic approaches to proving the strong law of large numbers.

The first is to prove the desired result for a subsequence and then reduce the problem for the whole sequence to that for the subsequence. In so doing, a maximal inequality for cumulative sums is usually needed for the second step. Note that maximal inequalities make up a well-developed branch of probability theory and many inequalities are known for different classes of random variables.

The second approach is to use directly a maximal inequality for normed sums. Inequalities of this kind are said to be of Hajek–Renyi type. Inequalities of this type are not easy to obtain and the first approach prevails. In the paper of Fazekas and Klesov [1] there is shown that a Hajek–Renyi type inequality is, in fact, a consequence of an appropriate maximal inequality for cumulative sums and the latter automatically implies the strong law of large numbers.

We study strong law of large numbers for random variables with superadditive moment function under regularly varying normalization. The asymptotic behavior of sums of random variables whose r-th moment function is superadditive for some $r > 0$ is considered. The method of the proof is based on a general result of Fazekas and Klesov [1]. This result can be applied for obtaining strong law of large numbers for independent, orthogonal and stationary dependent random variables, submartingales.

Problem definition

Definition. A sequence of random variables $\{X_n, n \ge 1\}$ is said to have the r-th $(r > 0)$ moment function of superadditive structure if there exists a a nonnegative function $g(i, j)$ such that for every $b \ge 0$ and $1 \le k < b + l$

$$g(b,k) + g(b+k,l) \le g(b,k+l)$$

and for some $\alpha > 1$

$$E|S_{b,n}|^r \le g^{\alpha}(b,n),$$

where $S_{b,n} = \sum_{v=b+1}^{b+n} X_v$.

Under the superadditive property Moricz [2] proved that there exists a constant $A_{r,\alpha}$ depending only on r and α such that

$$E[\max_{k \le n} |S_k|]^r \le A_{r,\alpha} g_n^{\alpha},$$

where $g_n = g(0,n)$. Under these assumptions Klesov and Fazekas [1] proved the following strong law of large numbers.

Theorem 1. Assume that a sequence of random variables $\{X_n, n \ge 1\}$ has an r-th moment function of superadditive structure with $r > 0$, $\alpha > 1$. Let $q > 0$, $b_n = n^{1/q}$. If

$$\sum_{n=1}^{\infty} \frac{g_n^{\alpha}}{n b_n^r} < \infty, \tag{1}$$

then

$$\lim_{n\to\infty} \frac{S_n}{b_n} = 0 \text{ a.s.} \tag{2}$$

Here and in what follows “a.s.” means “almost surely”.

The goal of the paper is to prove the strong law of large numbers for random variables with r-th moment function of superadditive structure with $r>0$ under regularly varying normalization. Namely, we prove a theorem analogous to Theorem 1 putting $b_n = n^{1/q}L(n)$, where $L(n)$ is a slowly varying function, $q>0$, $n \ge 1$.

Preliminaries

To prove our result we need the following theorem and corollary of [1].

Theorem 2. Let $b_1, b_2, \ldots$ be a nondecreasing unbounded sequence of positive numbers. Let $\alpha_1, \alpha_2, \ldots$ be nonnegative numbers. Let r be a fixed positive number. Assume that for every $n \ge 1$

$$E[\max_{1\le l\le n} |S_l|]^r \le \sum_{l=1}^{n} \alpha_l. \tag{3}$$

If $\sum_{l=1}^{\infty}(\alpha_l / b_l^r) < \infty$, then

$$\lim_{n\to\infty} \frac{S_n}{b_n} = 0 \text{ a.s.} \tag{4}$$

Corollary. Let $b_1, b_2, \ldots$ be a nondecreasing sequence of positive numbers. Let $\alpha_1, \alpha_2, \ldots$ be nonnegative numbers; $\Lambda_k = \alpha_1 + \alpha_2 + \ldots + \alpha_k$, $k \ge 1$. Let r be a fixed positive number. Assume that, for every $n \ge 1$, (3) holds. If

$$\sum_{l=1}^{\infty} \Lambda_l \left(\frac{1}{b_l^r} - \frac{1}{b_{l+1}^r} \right) < \infty \tag{5}$$

and

$$\frac{\Lambda_n}{b_n^r} \text{ is bounded,} \tag{6}$$

then (4) holds.

Main result

Theorem 3. Assume that a sequence of random variables $\{X_n, n \ge 1\}$ has r-th superadditive moment function with $r>0$, $\alpha>1$. Let $q>0$, $b_n = n^{1/q}L(n)$, where $L(n)$ is a slowly varying function. If

$$\sum_{n=1}^{\infty} \frac{g_n^{\alpha}}{n b_n^r} < \infty, \tag{7}$$

then

$$\lim_{n\to\infty} \frac{S_n}{b_n} = 0 \text{ a.s.}$$

P r o o f. To prove this theorem we have to show that $b_n = n^{1/q}L(n)$ is an unbounded nondecreasing sequence and (7) implies (5) and (6). The fact that b_n is unbounded follows from the property of slowly varying function (see [3]): for any $\gamma>0$ with $x\to\infty$: $x^{\gamma}L(x)\to\infty$, where $L(x)$ is a slowly varying function. The theorem on the representation of a slowly varying function (see [4]) implies that there exist $B>0$ that, for any $u \ge B$,

$$L(u) = \exp\left(\eta(u) + \int_B^u \frac{\varepsilon(u)}{u} du \right),$$

where η is a bounded and measurable on $[B,\infty]$ function such that $\eta(u)\to c$ $(|c|<\infty)$, $\varepsilon(u)$ is bounded on $[B,\infty]$ and such that $\varepsilon(u)\to 0$, $u\to\infty$.

First we consider the case $\eta=0$, $b_n = n^{\frac{1}{q}}L(n)$ increases monotonically and L is a differentiable function. We prove that in this case (7) implies (5). Put in (5) $\Lambda_n = g_n^{\alpha}$, $n\ge 1$ and $\Lambda_0 = 0$. Then

$$b^r(x) = x^{\frac{r}{q}} L^r(x) = x^{\frac{r}{q}} \exp\left(\int_B^x \frac{\varepsilon(u)}{u} du \right).$$

The mean value theorem in this case implies $\dfrac{b^r(n+1)-b^r(n)}{n+1-n} : (b^r)'(\xi)$, where $\xi \in [n, n+1]$. Taking first derivative we get

$$(b^r(x))' = \frac{r}{q} x^{\frac{r}{q}-1} \exp\left(\int_B^x \frac{\varepsilon(u)}{u} du \right) +$$

$$+ x^{\frac{r}{q}} \exp\left(\int_B^x \frac{\varepsilon(u)}{u} du \right) \frac{\varepsilon(x)}{x} =$$

$$= x^{\frac{r}{q}-1} \exp\left(\int_B^x \frac{\varepsilon(u)}{u} du \right) \left(\frac{r}{q} + \varepsilon(x) \right) \le$$

$$\le \left(\frac{r}{q} + 1 \right) x^{\frac{r}{q}-1} \exp\left(\int_B^x \frac{\varepsilon(u)}{u} du \right).$$

Then from (5) we have

$$\sum_{n=1}^{\infty}\Lambda_n\left(\frac{1}{b_n^r}-\frac{1}{b_{n+1}^r}\right)=\sum_{n=1}^{\infty}\frac{\Lambda_n}{b_n^r b_{n+1}^r}[b_{n+1}^r-b_n^r]\le$$

$$\le\sum_{n=1}^{\infty}\frac{\Lambda_n\left(\frac{r}{q}+1\right)(n+1)^{\frac{r}{q}-1}\exp\left(\int_B^{n+1}\frac{\varepsilon(u)}{u}du\right)}{n^{\frac{r}{q}}\exp\left(\int_B^{n}\frac{\varepsilon(u)}{u}du\right)(n+1)^{\frac{r}{q}}\exp\left(\int_B^{n+1}\frac{\varepsilon(u)}{u}du\right)}=$$

$$=\sum_{n=1}^{\infty}\frac{\Lambda_n\left(\frac{r}{q}+1\right)}{b_n^r(n+1)}\le\left(\frac{r}{q}+1\right)\sum_{n=1}^{\infty}\frac{\Lambda_n}{nb_n^r}<\infty.$$

Now we prove that (7) implies (6). Inequality (7) implies

$$\sum_{n=1}^{\infty}\frac{\Lambda_n}{nb_n^r}=\sum_{n=1}^{\infty}\frac{\Lambda_n}{n^{1+\frac{r}{q}}L^r(n)}=\sum_{m=0}^{\infty}\sum_{n=2^m}^{2^{m+1}}\frac{\Lambda_n}{n^{1+\frac{r}{q}}L^r(n)}\ge$$

$$\ge\frac{1}{2}\sum_{m=0}^{\infty}\frac{\Lambda_{2^m}}{2^{(m+1)\frac{r}{q}}L^r(2^{m+1})}.$$

Thus the last series converges. Now we prover (6). For any $m\ge 0$,

$$0\le\max_{2^m\le n<2^{m+1}}\frac{\Lambda_n}{b_n^r}=\max_{2^m\le n<2^{m+1}}\frac{\Lambda_n}{n^{\frac{r}{q}}L^r(n)}\le$$

$$\le\frac{\Lambda_{2^{m+1}}}{2^{\frac{mr}{q}}L^r(2^m)}=2^{\frac{r}{q}}\frac{\Lambda_{2^{m+1}}}{2^{(m+1)\frac{r}{q}}L^r(2^m)\frac{L^r(2^{m+1})}{L^r(2^{m+1})}}.$$

The definition of slowly varying function implies $\frac{L^r(2^m)}{L^r(2^{m+1})}\to 1$. So we get

$$2^{\frac{r}{q}}\frac{\Lambda_{2^{m+1}}}{2^{(m+1)\frac{r}{q}}L^r(2^{m+1})}\to 0,$$

as function g_n^{α} is nondecreasing and the convergence of the series $\sum_{m=0}^{\infty}\frac{\Lambda_{2^m}}{2^{(m+1)\frac{r}{q}}L^r(2^{m+1})}$ implies the convegence of $\sum_{m=0}^{\infty}\frac{\Lambda_{2^{m+1}}}{2^{(m+1)\frac{r}{q}}L^r(2^{m+1})}$. Thus theorem follows from Corollary in the case $b_n=n^{\frac{1}{q}}L(n)$ increases monotoniccally and L is a differentiable function with $\eta(n)=0$. Now we consider the case where η differs from 0, but b_n still increases monotonically and L is a differentiable function. In this case we can estimate b_n as follows

$$b_n^r=x^{\frac{r}{q}}\exp r\left(\eta(n)+\int_B^n\frac{\varepsilon(u)}{u}du\right)=$$

$$=x^{\frac{r}{q}}\exp(r\eta(n))\exp\left(r\int_B^n\frac{\varepsilon(u)}{u}du\right)\le$$

$$\le x^{\frac{r}{q}}e^{rc}\exp\left(r\int_B^n\frac{\varepsilon(u)}{u}du\right)=$$

$$=Cx^{\frac{r}{q}}\exp\left(r\int_B^n\frac{\varepsilon(u)}{u}du\right)=C\bar{b}_n^{\,r},$$

where $C=e^r c$ and $\bar{b}_n=\exp\left(\int_B^n\frac{\varepsilon(u)}{u}du\right)$. Hence (7) implies (5) and (6). Thus we have the strong law of large numbers for $b_n=x^{\frac{1}{q}}L(n)$, where $L(n)$ increases monotonically. Now we consider the general case. It is proved in [3] that, for any slowly varying function L, there exists a slowly varying function $\hat{L}$ such that

a) $n^{\frac{1}{q}}\hat{L}(n)$ is nondecreasing;

b) $\hat{L}\in C^{\infty}$;

c) $\frac{\hat{L}(u)}{L(u)}\to 1,\ u\to\infty$.

Put $\hat{b}_n=n^{\frac{1}{q}}\hat{L}(n)$. Then

a) $\hat{b}_n$ is nondecreasing;

b) $\hat{b}_n\in C^{\infty}$;

c) $\frac{\hat{b}_n}{b_n}\to 1,\ n\to\infty$.

The strong law of large numbers is proved for the sequence $\hat{b}_n$. So we have

$$\frac{S_n}{b_n}=\frac{S_n}{b_n\frac{\hat{b}_n}{\hat{b}_n}}=\frac{S_n}{\hat{b}_n\frac{b_n}{\hat{b}_n}}\to 0,\ n\to\infty,$$

which proves our theorem in the general case.

Conclusions

We studied the strong law of large numbers for random variables with moment function of superadditive structure. A similar result is considered in [1], [2], [4]. Our goal was to study the strong law of large numbers under special normalization, namely, under regularly varying normalization of sums of random variables.

The main result of the paper is Theorem 3 concerning the strong law of large numbers for random variables with r-th moment function of superadditive structure for some $r > 0$ under regularly varying normalization of sums of random variables. The proof of this theorem is based on results of Fazekas and Klesov [1].

Theorem 3 can be applied for obtaining strong law of large numbers for independent, orthogonal and stationary dependent random variables, submartingales. It can be applied for obtaining of analogical result in the case of random fields.

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