UDC 517.9

OSCILLATION CRITERIA FOR HIGHER ORDER NONLINEAR FUNCTIONAL DIFFERENTIAL EQUATIONS WITH ADVANCED ARGUMENT*

КРИТЕРІЇ ОСЦИЛЯЦІЇ ДЛЯ НЕЛІНІЙНИХ ФУНКЦІОНАЛЬНО-ДИФЕРЕНЦІАЛЬНИХ РІВНЯНЬ ВИЩОГО ПОРЯДКУ З ВИПЕРЕДЖЕНИМ АРГУМЕНТОМ

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In the paper the differential equation

$$u^{(n)}(t) + p(t)\,|u(\sigma(t))|^{\mu(t)}\operatorname{sign} u(\sigma(t)) = 0$$

is considered, where $p \in L_{\rm loc}(R_+; R_+)$, $\mu \in C(R_+; (0, +\infty))$, $\sigma \in C(R_+; R_+)$ *and* $\sigma(t) \geq t$ *for* $t \in R_+$. *We say that the equation is almost linear if the condition* $\lim_{t\to+\infty} \mu(t) = 1$ *is fulfilled, while if* $\limsup_{t\to+\infty} \mu(t) \neq 1$ *or* $\liminf_{t\to+\infty} \mu(t) \neq 1$, *then the equation is an essentially nonlinear differential equation. In case of almost linear differential equations oscillatory properties have been extensively studied. In this paper new sufficient (necessary and sufficient) conditions are established for a general class of essentially nonlinear functional differential equations to have Property* $\mathbf{A}$.

Розглядається диференціальне рівняння

$$u^{(n)}(t) + p(t)|u(\sigma(t))|^{\mu(t)}\operatorname{sign} u(\sigma(t)) = 0,$$

де $p \in L_{\rm loc}(R_+; R_+)$, $\mu \in C(R_+; (0, +\infty))$, $\sigma \in C(R_+; R_+)$ *та* $\sigma(t) \geq t$ *для* $t \in R_+$. *Будемо казати, що рівняння є майже лінійним, якщо виконується умова* $\lim_{t\to+\infty} \mu(t) = 1$, *і суттєво нелінійним, якщо* $\limsup_{t\to+\infty} \mu(t) \neq 1$ *або* $\liminf_{t\to+\infty} \mu(t) \neq 1$. *У випадку майже лінійного диференціального рівняння коливні властивості було широко вивчено. У цій роботі встановлено нові достатні (необхідні та достатні) умови для того, щоб загальний клас суттєво нелінійних функціонально-диференціальних рівнянь задовольнив властивості* $\mathbf{A}$.

1. Introduction. This work deals with investigation of oscillatory properties of solutions of a functional differential equation of the form

$$u^{(n)}(t) + p(t)|u(\sigma(t))|^{\mu(t)}\operatorname{sign} u(\sigma(t)) = 0, \tag{1.1}$$

where

$$\begin{gathered} p \in L_{\rm loc}(R_+; R), \quad \mu \in C(R_+; (0, +\infty)), \\ \sigma \in C(R_+; R_+), \quad \text{and} \quad \sigma(t) \geq t \quad \text{for} \quad t \in R_+. \end{gathered} \tag{1.2}$$

* Was supported by Sh. Rustaveli National Science Foundation (Georgia). Grant No. GNSF/ST09-81-3-101.

It will always be assumed that the condition

$$p(t) \geq 0 \quad \text{for} \quad t \in R_+ \tag{1.3}$$

is fulfilled.

Let $t_0 \in R_+$. A function $u : [t_0, +\infty) \to R$ is said to be a proper solution of (1.1) if it is locally absolutely continuous along with its derivatives up to the order $n-1$ inclusive, $\sup\{|u(s)| : s \geq t\} > 0$ for $t \geq t_0$ and it satisfies (1.1) almost everywhere on $[t_0, +\infty)$.

A proper solution $u : [t_0, +\infty) \to R$ of the equation (1.1) is said to be oscillatory if it has a sequence of zeros tending to $+\infty$. Otherwise the solution u is said to be nonoscillatory.

Definition 1.1. *We say that the equation (1.1) has Property* **A** *if any of its proper solutions is oscillatory when n is even, and is either oscillatory or satisfies*

$$|u^{(i)}(t)| \downarrow 0, \quad \textit{as} \quad t \uparrow +\infty, \quad i = 0, \ldots, n-1, \tag{1.4}$$

when n is odd.

Definition 1.2. *We say that the equation (1.1) is almost linear if the condition*

$$\lim_{t \to +\infty} \mu(t) = 1$$

holds, while if

$$\limsup_{t \to +\infty} \mu(t) \neq 1 \quad \textit{or} \quad \liminf_{t \to +\infty} \mu(t) \neq 1,$$

then we say that the equation is an essentially nonlinear differential equation.

Oscillatory properties of almost linear equations are studied well enough in [1–5]. In the present paper essentially nonlinear differential equations of the type (1.1) are considered with one of the following conditions being satisfied:

$$\mu(t) \leq \lambda \quad (\lambda \in (0,1)) \quad \text{for} \quad t \in R_+, \tag{1.5}$$

or

$$\mu(t) \geq \lambda \quad (\lambda \in (0,1)) \quad \text{for} \quad t \in R_+. \tag{1.6}$$

In the present paper, under conditions (1.5) and (1.6), sufficient (necessary and sufficient) conditions are established for the equation (1.1) to have Property **A**. Of the obtained results, some are specific to generalized equations and do not have analogous for the classical (Emden–Fowler) equations. Analogous results for Emden–Fowler equations are given in the paper [6].

2. Some auxiliary lemmas. In the sequel, $\widetilde{C}_{\text{loc}}([t_0, +\infty))$ will denote the set of all functions $u : [t_0, +\infty) \to R$ absolutely continuous on any finite subinterval of $[t_0, +\infty)$ along with their derivatives of order up to and including $n-1$.

Lemma 2.1 [7]. *Let* $u \in \widetilde{C}_{\rm loc}^{n-1}([t_0,+\infty))$, $u(t) > 0$, $u^{(n)}(t) \le 0$ *for* $t \ge t_0$, *and* $n^{(n)}(t) \not\equiv 0$ *in any neighborhood of* $+\infty$. *Then there exist* $t_1 \ge t_0$ *and* $\ell \in \{0,\ldots,n-1\}$ *such that* $l+n$ *is odd and*

$$\begin{aligned} u^{(i)}(t) > 0 \quad for \quad t \ge t_1, \quad i = 0,\ldots,\ell-1, \\ (-1)^{i+\ell}u^{(i)}(t) > 0 \quad for \quad t \ge t_1, \quad i = \ell,\ldots,n-1. \end{aligned} \tag{2.1$_\ell$}$$

Remark 2.1. If n is odd and $\ell = 0$, then in (2.1_0) it is meant that only the second inequalities are fulfilled.

Lemma 2.2 [7]. *Let* $u \in \widetilde{C}_{\rm loc}^{n-1}([t_0,+\infty))$ *and (2.1$_\ell$) be fulfilled for some* $\ell \in \{0,\ldots,n-1\}$ *with* $l+n$ *odd. Then*

$$\int\limits_{t_0}^{+\infty} t^{n-\ell-1}|u^{(n)}(t)|\,dt < +\infty. \tag{2.2}$$

If, moreover,

$$\int\limits_{t_0}^{+\infty} t^{n-\ell}|u^{(n)}(t)|\,dt = +\infty, \tag{2.3}$$

then there exists $t_* > t_0$ *such that*

$$\frac{u^{(i)}(t)}{t^{\ell-i}}\downarrow, \quad \frac{u^{(i)}(t)}{t^{\ell-i-1}}\uparrow, \quad i = 0,\ldots,\ell-1, \tag{2.4$_i$}$$

$$u(t) \ge \frac{t^{\ell-1}}{\ell!}\,u^{(\ell-1)}(t) \quad for \quad t \ge t_*, \tag{2.5}$$

and

$$u^{(\ell-1)}(t) \ge \frac{t}{(n-\ell)!}\int\limits_t^{+\infty} s^{n-\ell-1}|u^{(n)}(s)|\,ds + \frac{1}{(n-\ell)!}\int\limits_{t_*}^{t} s^{n-\ell}|u^{(n)}(s)|\,ds \quad for \quad t \ge t_*. \tag{2.6}$$

3. Necessary conditions for the existence of solutions of type (2.1_ℓ). The results of this section play an important role in establishing sufficient conditions for the equation (1.1) to have Property **A**.

Definition 3.1. *Let* $t_0 \in R_+$. *By* $\mathbf{U}_{\ell,t_0}$ *we denote the set of all proper solutions of the equation (1.1) satisfying the condition (2.1$_\ell$).*

Theorem 3.1. *Let the conditions (1.2), (1.3) and (1.5) be fulfilled, $\ell \in \{1, \ldots, n-1\}$ with $\ell+n$ odd and let*

$$\int\limits_0^{+\infty} t^{n-\ell}(\sigma(t))^{(\ell-1)\mu(t)} p(t)\,dt = +\infty, \tag{3.1_ℓ}$$

$$\int\limits_0^{+\infty} t^{n-\ell-1}(\sigma(t))^{\ell\mu(t)} p(t)\,dt = +\infty. \tag{3.2_ℓ}$$

If, moreover, $\mathbf{U}_{\ell,t_0} \neq \varnothing$ for some $t_0 \in R_+$, then for any $k \in N$ and $\delta \in [0,\lambda]$, and $\sigma_* \in C([t_0,+\infty))$ such that

$$t \le \sigma_*(t) \le \sigma(t) \quad \textit{for} \quad t \ge t_0 \tag{3.3}$$

we have

$$\int\limits_0^{+\infty} t^{n-\ell-1+\lambda-\delta}(\sigma_*(t))^{\mu(t)-\lambda}(\sigma(t))^{(\ell-1)\mu(t)}(\rho_{\ell,k}(\sigma_*(t)))^{\delta} dt < +\infty, \tag{3.4}$$

where

$$\rho_{\ell,1}(t) = \left(\frac{1-\lambda}{\ell!(n-1)!}\int\limits_0^t\int\limits_s^{+\infty} \xi^{n-\ell-1+\mu(\xi)-\lambda}(\sigma(\xi))^{(\ell-1)\mu(\xi)} p(\xi)d\xi\,ds\right)^{\frac{1}{1-\lambda}}, \tag{3.5_ℓ}$$

$$\rho_{\ell,i}(t) = \frac{1}{\ell!(n-\ell)!}\int\limits_0^t\int\limits_s^{+\infty} \xi^{n-\ell-1}(\sigma(\xi))^{(\ell-1)\mu(\xi)}\,(\rho_{\ell,i-1}(\sigma(\xi)))^{\mu(\xi)}\,p(\xi)\,d\xi\,ds, \quad i = 2, \ldots, k. \tag{3.6_ℓ}$$

Proof. Let $t_0 \in R_+$ and $\mathbf{U}_{\ell,t_0} \neq \varnothing$. By definition of the set $\mathbf{U}_{\ell,t_0}$ (see Definition 3.1), the equation (1.1) has a proper solution $u \in \mathbf{U}_{\ell,t_0}$ satisfying the condition (2.1_ℓ). By (1.1), (2.1_ℓ) and (3.1_ℓ) it is clear that the condition (2.3) holds. Thus by Lemma 2.2 the conditions (2.4_i) – (2.6) are fulfilled and

$$u^{(\ell-1)}(t) \ge \frac{t}{(n-\ell)!}\int\limits_t^{+\infty} s^{n-\ell-1}(u(\sigma(s)))^{\mu(s)} p(s)\,ds +$$
$$+ \frac{1}{(n-\ell)!}\int\limits_{t_*}^{t} s^{n-\ell}(u(\sigma(s)))^{\mu(s)} p(s)\,ds \quad \text{for} \quad t \ge t_*. \tag{3.7}$$

According to (2.5) from (3.7) we get

$$u^{(\ell-1)}(t) \geq \frac{t}{(n-\ell)!} \int\limits_t^{+\infty} s^{n-\ell-1}(u(\sigma(s)))^{\mu(s)} p(s)\, ds -$$

$$- \frac{1}{(n-\ell)!} \int\limits_{t_*}^{t} sd \int\limits_s^{+\infty} \xi^{n-\ell-1}(u(\sigma(\xi)))^{\mu(\xi)} p(\xi)\, d\xi \geq$$

$$\geq \frac{1}{(n-\ell)!} \int\limits_{t_*}^{t} \int\limits_s^{+\infty} \xi^{n-\ell-1} p(\xi)(u(\sigma(\xi)))^{\mu(\xi)}\, d\xi\, ds \geq$$

$$\geq \frac{1}{\ell!(n-\ell)!} \int\limits_{t_*}^{t} \int\limits_s^{+\infty} \xi^{n-\ell-1}(\sigma(\xi))^{(\ell-1)\mu(\xi)}(u^{(\ell-1)}(\sigma(\xi)))^{\mu(\xi)} p(\xi)\, d\xi\, ds. \tag{3.8}$$

Therefore, by (1.2) and $(2.4_{\ell-1})$, from (3.8) we have

$$u^{(\ell-1)}(t) \geq \frac{1}{\ell!(n-\ell)!} \int\limits_{t_*}^{t} \int\limits_s^{+\infty} \xi^{n-\ell-1+\mu(\xi)}(\sigma(\xi))^{(\ell-1)\mu(\xi)} \left(\frac{u^{(\ell-1)}(\xi)}{\xi}\right)^{\mu(\xi)} p\,(\xi)\, d\xi\, ds \ \text{ for } \ t \geq t_*. \tag{3.9}$$

On the other hand, by $(2.4_{\ell-1})$ and (3.2_ℓ) it is obvious that

$$\frac{u^{\ell-1}(t)}{t} \downarrow 0 \quad \text{as} \quad t \uparrow +\infty. \tag{3.10}$$

By (3.10), without loss of generality we can assume that $u^{(\ell-1)}(t)/t \leq 1$ for $t \geq t_*$. Since $0 < \mu(t) \leq \lambda < 1$, from (3.9) we have

$$u^{(\ell-1)}(t) \geq \frac{1}{\ell!(n-\ell)!} \int\limits_{t_*}^{t} \int\limits_s^{+\infty} \xi^{n-\ell-1-\lambda+\mu(\xi)}(\sigma(\xi))^{(\ell-1)\mu(\xi)} p(\xi)(u^{(\ell-1)}(\xi))^{\lambda}\, d\xi\, ds. \tag{3.11}$$

By $(2.4_{\ell-1})$, it is obvious that

$$x'(t) \geq \frac{(u^{(\ell-1)}(t))^{\lambda}}{\ell!(n-\ell)!} \int\limits_t^{+\infty} \xi^{n-\ell-1-\lambda+\mu(\xi)}(\sigma(\xi))^{(\ell-1)\mu(\xi)} p(\xi)\, d\xi, \tag{3.12}$$

where

$$x(t) = \frac{1}{\ell!(n-\ell)!} \int\limits_{t_*}^{t} \int\limits_s^{+\infty} \xi^{n-\ell-1-\lambda+\mu(\xi)}(\sigma(\xi))^{(\ell-1)\mu(\xi)} p(\xi)(u^{(\ell-1)}(\xi))^{\lambda}\, d\xi\, ds. \tag{3.13}$$

Thus, according to (3.11) and (3.13), from (3.12) we get

$$x'(t) \geq \frac{x^{\lambda}(t)}{\ell!(n-\ell)!} \int\limits_{t}^{+\infty} \xi^{n-\ell-1-\lambda+\mu(\xi)} (\sigma(\xi))^{(\ell-1)\mu(\xi)} p(\xi)\, d\xi \quad \text{for} \quad t \geq t_*.$$

Therefore,

$$x(t) \geq \left(\frac{1-\lambda}{\ell!(n-\ell)!} \int\limits_{t_*}^{t} \int\limits_{s}^{+\infty} \xi^{n-\ell-1-\lambda+\mu(\xi)} (\sigma(\xi))^{(\ell-1)\mu(\xi)} p(\xi)\, d\xi\, ds \right)^{\frac{1}{1-\lambda}} \quad \text{for} \quad t \geq t_*.$$

Hence, according to (3.11) and (3.13), we have

$$u^{(\ell-1)}(t) \geq \rho_{t_*,\ell,1}(t) \quad \text{for} \quad t \geq t_*, \tag{3.14}$$

where

$$\rho_{t_*,\ell,1}(t) = \left(\frac{1-\lambda}{\ell!(n-\ell)!} \int\limits_{t_*}^{t} \int\limits_{s}^{+\infty} \xi^{n-\ell-1-\lambda+\mu(\xi)} (\sigma(\xi))^{(\ell-1)\mu(\xi)} p(\xi)\, d\xi\, ds \right)^{\frac{1}{1-\lambda}}. \tag{3.15}$$

Thus by (3.8), (3.13) and (3.14) we get

$$u^{(\ell-1)}(t) \geq \rho_{t_*,\ell,k}(t) \quad \text{for} \quad t \geq t_*, \tag{3.16}$$

where

$$\rho_{t_*,\ell,k}(t) = \frac{1}{\ell!(n-\ell)!} \int\limits_{t_*}^{t} \int\limits_{s}^{+\infty} \xi^{n-\ell-1} (\sigma(\xi))^{(\ell-1)\mu(\xi)} \times$$
$$\times \left(\rho_{t_*,\ell,k-1}(\sigma(\xi))\right)^{\mu(\xi)} p(\xi)\, d\xi\, ds, \quad k = 2, 3, \ldots. \tag{3.17}$$

On the other hand, by (1.2), (2.1_ℓ), (2.5) and (3.3) from (3.7) we have

$$u^{(\ell-1)}(t) \geq \frac{t}{\ell!(n-\ell)!} \int\limits_{t}^{+\infty} s^{n-\ell-1} (\sigma(s))^{(\ell-1)\mu(s)} (u^{(\ell-1)}(\sigma(s)))^{\mu(s)} p(s)\, ds \geq$$
$$\geq \frac{t}{\ell!(n-\ell)!} \int\limits_{t}^{+\infty} s^{n-\ell-1} (\sigma(s))^{(\ell-1)\mu(s)} (u^{(\ell-1)}(\sigma_*(s)))^{\mu(s)} p(s)\, ds =$$
$$= \frac{t}{\ell!(n-\ell)!} \int\limits_{t}^{+\infty} s^{n-\ell-1} (\sigma(s))^{(\ell-1)\mu(s)} (\sigma_*(s))^{\mu(s)} p(s) \left(\frac{u^{(\ell-1)}(\sigma_*(s))}{\sigma_*(s)} \right)^{\mu(s)} ds.$$

Consequently, by $(2.4_{\ell-1})$, (1.5), (3.10) and (3.3), for any $\delta \in [0,\lambda]$

$$u^{(\ell-1)}(t) \geq \frac{t}{\ell!(n-\ell)!} \int\limits_t^{+\infty} s^{n-\ell-1}(\sigma(s))^{(\ell-1)\mu(s)}(\sigma_*(s))^{\mu(s)-\lambda}\times$$

$$\times\, p(s)(u^{(\ell-1)}(\sigma_*(s)))^{\delta}(u^{\ell-1)}(s))^{\lambda-\delta}\, ds.$$

Therefore, according to (3.16), we have

$$u^{(\ell-1)}(t) \geq \frac{t}{\ell!(n-\ell)!} \int\limits_t^{+\infty} s^{n-\ell-1}(\sigma(s))^{(\ell-1)\mu(s)}(\sigma_*(s))^{\mu(s)-\lambda}p(s)\times$$

$$\times\, (\rho_{t_*,\ell,k}(\sigma_*(s))^{\delta}(u^{(\ell-1)}(s))^{\lambda-\delta}\, ds \quad \text{for} \quad t \geq t_*, \quad k = 1,2,\ldots. \tag{3.18}$$

If $\delta = \lambda$, then from (3.18)

$$\int\limits_{t_*}^{+\infty} s^{n-\ell-1}(\sigma(s))^{(\ell-1)\mu(s)}(\sigma_*(s))^{\mu(s)-\lambda}p(s)(\rho_{t_*,\ell,k}(\sigma_*(s)))^{\lambda}\, ds \leq$$

$$\leq \ell!(n-\ell)!\,\frac{u^{(\ell-1)}(t_*)}{t_*} \leq \ell!(n-\ell)!. \tag{3.19}$$

Let $\delta \in [0,\lambda)$. Then from (3.18)

$$(u^{(\ell-1)}(t))^{\lambda-\delta} \geq \frac{t^{\lambda-\delta}}{(\ell!(n-\ell)!)^{\lambda-\delta}} \Bigg(\int\limits_t^{+\infty} s^{n-\ell-1}(\sigma(s))^{(\ell-1)\mu(s)}(\sigma_*(s))^{\mu(s)-\lambda}p(s)\times$$

$$\times\, (\rho_{t_*,\ell,k}(\sigma_*(s))^{\delta}(u^{\ell-1)}(s))^{\lambda-\delta}\, ds\Bigg)^{\lambda-\delta} \quad \text{for} \quad t \geq t_*, \quad k = 1,2,\ldots.$$

Thus we have

$$\frac{\varphi(t)}{\left(\int_t^{+\infty}\varphi(s)\,ds\right)^{\lambda-\delta}} \geq \frac{1}{(\ell!(n-\ell)!)^{\lambda-\delta}}\, t^{n-\ell-1+\lambda-\delta}(\sigma(t))^{(\ell-1)\mu(t)}(\sigma_*(t))^{\mu(t)-\lambda}\times$$

$$\times\, (\rho_{t_*,\ell,k}(\sigma_*(t))^{\delta}p(t) \quad \text{for} \quad t \geq t_*, \quad k = 1,2,\ldots,$$

where

$$\varphi(t) = t^{n-\ell-1}(\sigma(t))^{(\ell-1)\mu(t)}(\sigma_*(t))^{\mu(t)-\lambda}(\rho_{t_*,\ell,k}(\sigma_*(t))^{\delta}(u^{(\ell-1)}(t))^{\lambda-\delta}p(t).$$

From the last inequality we get

$$-\int\limits_{y(t_*)}^{y(t)} \frac{ds}{s^{\lambda-\delta}} \geq \frac{1}{(\ell!(n-\ell)!)^{\lambda-\delta}} \int\limits_{t_*}^{t} s^{n-\ell-1+\lambda-\delta}(\sigma(s))^{(\ell-1)\mu(s)}(\sigma_*(s))^{\mu(s)-\lambda}(\rho_{t_*,\ell,k}(\sigma_*(s)))^{\delta}p(s)\, ds,$$

where

$$y(t) = \int_t^{+\infty} \varphi(s)\, ds. \tag{3.20}$$

Therefore

$$\int_{t_*}^{t} s^{n-\ell-1+\lambda-\delta}(\sigma(s))^{(\ell-1)\mu(s)}(\sigma_*(s))^{\mu(s)-\lambda}(\rho_{t_*,\ell,k}(\sigma_*(s)))^{\delta} p(s)\, ds \leq$$

$$\leq (n-\ell)!)^{\lambda-\delta} \int_0^{y(t_*)} \frac{ds}{s^{\lambda-\delta}}\,. \tag{3.21}$$

By (3.20), without loss of generality we can assume that $y(t_*) \leq 1$. Thus from (3.21) we have

$$\int_{t_*}^{t} s^{n-\ell-1+\lambda-\delta}(\sigma(s))^{(\ell-1)\mu(s)}(\sigma_*(s))^{\mu(s)-\lambda}(\rho_{t_*,\ell,k}(\sigma_*(s)))^{\delta} p(s)\, ds \leq$$

$$\leq (\ell!(n-\ell)!)^{\lambda-\delta} \int_0^{1} \frac{ds}{s^{\lambda-\delta}} = \frac{(\ell!(n-\ell)!)^{\lambda-\delta}}{1-\lambda+\delta} \quad \text{for} \quad t \geq t_*.$$

Passing to limit in the latter inequality, we obtain

$$\int_{t_*}^{+\infty} s^{n-\ell-1+\lambda-\delta}(\sigma(s))^{(\ell-1)\mu(s)}(\sigma_*(s))^{\mu(s)-\lambda}(\rho_{t_*,\ell,k}(\sigma_*(s))^{\delta} p(s)\, ds < +\infty. \tag{3.22}$$

Therefore, since

$$\lim_{t\to+\infty} \frac{\rho_{\ell,k}(t)}{\rho_{t_*,\ell,k}(t)} = 1, \quad k = 1, 2, \dots,$$

by (3.22) and (3.19) it is obvious that for any $\delta \in [0, \lambda]$ and $k \in N$ (3.4) holds, which proves the validity of the theorem.

Analogously we can prove the following theorem.

Theorem 3.2. *Let the conditions (1.2), (1.3), (3.1$_\ell$), (3.2$_\ell$) and (1.6) be fulfilled,* $\ell \in \{1, \dots, n-$ $-1\}$ *with* $\ell + n$ *odd and* $\mathbf{U}_{\ell,t_0} \neq \varnothing$ *for some* $t_0 \in R_+$. *Then for any* $k \in N$ *and* $\delta \in [0, \lambda]$

$$\int_0^{+\infty} t^{n-\ell-1+\delta}(\sigma(t))^{(\ell-1)\mu(t)}(\widetilde{\rho}_{\ell,k}(\sigma(t)))^{\mu(t)-\delta} p(t)\, dt < +\infty, \tag{3.23}$$

where

$$\widetilde{\rho}_{\ell,1}(t) = \left(\frac{1-\lambda}{\ell!(n-\ell)!} \int_0^t \int_s^{+\infty} \xi^{n-\ell-1} (\sigma(\xi))^{(\ell-1)\mu(\xi)} p(\xi)\, d\xi\, ds \right)^{\frac{1}{1-\lambda}}, \tag{3.24$_\ell$}$$

$$\widetilde{\rho}_{\ell,i}(t) = \frac{1}{\ell!(n-\ell)!} \int_0^t \int_s^{+\infty} \xi^{n-\ell-1} (\sigma(\xi))^{(\ell-1)\mu(\xi)} (\widetilde{\rho}_{\ell,i-1}(\sigma(\xi)))^{\mu(\xi)} p(\xi)\, d\xi\, ds, \quad i = 2, \dots, k. \tag{3.25$_\ell$}$$

4. Sufficient conditions for nonexistence of solutions of the type (2.1$_\ell$).

Theorem 4.1. *Let the conditions (1.2), (1.5), (3.1$_\ell$) and (3.2$_\ell$) be fulfilled, where $\ell \in \{1, \dots, n-1\}$ with $\ell + n$ odd, and let there exist $\delta \in [0, \lambda]$, $k \in N$ and $\sigma_* \in C(R_+)$ satisfying the condition (3.3) such that*

$$\int_0^{+\infty} t^{n-\ell-1+\lambda-\delta} (\sigma_*(t))^{\mu(t)-\lambda} (\sigma(t))^{(\ell-1)\mu(t)} (\rho_{\ell,k}(\sigma_*(t)))^{\delta} p(t)\, dt = +\infty \tag{4.1$_\ell$}$$

holds. Then for any $t_0 \in R_+$ we have $\mathbf{U}_{\ell,t_0} = \varnothing$, where $\rho_{\ell,k}$ is defined by (3.5$_\ell$) and (3.6$_\ell$).

Proof. Assume the contrary. Let there exist $t_0 \in R_+$ such that $\mathbf{U}_{\ell,t_0} \neq \varnothing$ (see Definition 3.1). Then the equation (1.1) has a proper solution $u : [t_0, +\infty) \to R$ satisfying the condition (2.1$_\ell$). Since the conditions of Theorem 3.1 are fulfilled, for any $\delta \in [0, \lambda]$, $k \in N$ and $\sigma_* \in C(R_+)$ satisfying the condition (3.3) the condition (3.4) holds, which contradicts (4.1$_\ell$). The obtained contradiction proves the validity of the theorem.

Using Theorem 3.2, analogously we can prove the following theorem.

Theorem 4.2. *Let the conditions (1.2), (1.3), (1.6), (3.1$_\ell$) and (3.2$_\ell$) be fulfilled, where $\ell \in \{1, \dots, n-1\}$ with $\ell + n$ odd, and let there exist $\delta \in [0, \lambda]$ and $k \in N$ such that*

$$\int_0^{+\infty} t^{n-\ell-1+\delta} (\sigma(t))^{(\ell-1)\mu(t)} (\widetilde{\rho}_{\ell,k}(\sigma(t)))^{\mu(t)-\delta} p(t)\, dt = +\infty. \tag{4.2$_\ell$}$$

Then for any $t_0 \in R_+$ we have $\mathbf{U}_{\ell,t_0} = \varnothing$, where $\widetilde{\rho}_{\ell,k}$ is defined by (3.24$_\ell$) and (3.25$_\ell$).

Corollary 4.1. *Let the conditions (1.2), (1.3), (1.5) and (3.2$_\ell$) be fulfilled, where $\ell \in \{1, \dots, n-1\}$ with $\ell + n$ odd, and let for some $\gamma \in (0, 1)$*

$$\liminf_{t \to +\infty} t^{\gamma} \int_t^{+\infty} s^{n-\ell-1+\mu(s)-\lambda} (\sigma(s))^{(\ell-1)\mu(s)} p(s)\, ds > 0. \tag{4.3$_\ell$}$$

If, moreover, there exist $\delta \in [0, \lambda]$ and $\sigma_* \in C(R_+)$ satisfying the condition (3.3) such that

$$\int_0^{+\infty} t^{n-\ell-1+\lambda-\delta}(\sigma_*(t))^{\mu(t)-\lambda+\frac{\delta(1-\gamma)}{1-\lambda}}\,(\sigma(t))^{(\ell-1)\mu(t)}p(t)\,dt = +\infty \tag{4.4$_\ell$}$$

holds, then for any $t_0 \in R_+$ we have $\mathbf{U}_{\ell,t_0} = \varnothing$.

Proof. Clearly the condition (3.1$_\ell$) is fulfilled by virtue of (4.3$_\ell$). On the other hand, according to (3.5$_\ell$) and (4.3$_\ell$), there exist $c > 0$ and $t_1 \in [t_0, +\infty)$ such that

$$\rho_{\ell,1}(t) \geq c\,t^{\frac{1-\gamma}{1-\lambda}} \quad \text{for} \quad t \geq t_1.$$

Therefore from (4.4$_\ell$) it follows (4.1$_\ell$) with $k = 1$. Thus all conditions of Theorems 4.1 are fulfilled, which proves the corollary.

Analogously we can prove the following corollary.

Corollary 4.2. *Let the conditions (1.2), (1.3), (1.5) and (3.2$_\ell$) be fulfilled, where $\ell \in \{1, \ldots, n-1\}$ with $\ell + n$ odd and*

$$\liminf_{t\to+\infty} t\int_t^{+\infty} s^{n-\ell-1+\mu(s)-\lambda}(\sigma(s))^{(\ell-1)\mu(s)}p(s)\,ds > 0. \tag{4.5$_\ell$}$$

If, moreover, there exist $\delta \in [0, \lambda]$ and $\sigma_* \in C(R_+)$ satisfying the condition (3.3) such that

$$\int_0^{+\infty} t^{n-\ell-1+\lambda-\delta}(\sigma_*(t))^{\mu(t)-\lambda}(\sigma(t))^{(\ell-1)\mu(t)}(\ln(1+\sigma_*(t)))^{\frac{\delta}{1-\lambda}}p(t)\,dt = +\infty \tag{4.6$_\ell$}$$

holds, then for any $t_0 \in R_+$ we have $\mathbf{U}_{\ell,t_0} = \varnothing$.

Corollary 4.3. *Let the conditions (1.2), (1.3), (1.6) and (3.2$_\ell$) be fulfilled, where $\ell \in \{1, \ldots, n-1\}$ with $\ell + n$ odd, and let for some $\gamma \in (0, 1)$*

$$\liminf_{t\to+\infty} t^{\gamma}\int_t^{+\infty} s^{n-\ell-1}(\sigma(s))^{(\ell-1)\mu(s)}p(s)\,ds > 0. \tag{4.7$_\ell$}$$

If, moreover, there exists $\delta \in [0, \lambda]$ such that

$$\int_0^{+\infty} t^{n-\ell-1+\delta}(\sigma(t))^{(\ell-1)\mu(t)+\frac{(\mu(t)-\delta)(1-\gamma)}{1-\lambda}}\,p(t)\,dt = +\infty, \tag{4.8$_\ell$}$$

then for any $t_0 \in R_+$ we have $\mathbf{U}_{\ell,t_0} = \varnothing$.

Proof. According to (4.7$_\ell$) and (3.24$_\ell$) there exist $c > 0$ and $t_1 \in [t_0, +\infty)$ such that

$$\widetilde{\rho}_{\ell,1}(t) \geq c\,t^{\frac{1-\gamma}{1-\lambda}} \quad \text{for} \quad t \geq t_1.$$

Therefore, from (4.8_ℓ) it follows (4.2_ℓ) with $k = 1$. Thus all the conditions of Theorems 4.2 hold, which proves the corollary.

Corollary 4.4. *Let the conditions (1.2), (1.3), (1.6) and (3.2_ℓ) be fulfilled, where* $\ell \in \{1, \ldots, n-1\}$ *with* $\ell + n$ *odd, and let*

$$\liminf_{t\to+\infty} t \int\limits_t^{+\infty} s^{n-\ell-1}(\sigma(s))^{(\ell-1)\mu(s)} p(s)\, ds > 0. \tag{4.9_ℓ}$$

If, moreover, for some $\delta \in [0, \lambda]$ *the condition*

$$\int\limits_0^{+\infty} t^{n-\ell-1+\delta}(\sigma(t))^{(\ell-1)\mu(t)}(\ln(1+\sigma(t)))^{\frac{\mu(t)-\delta}{1-\lambda}} p(t)\, dt = +\infty \tag{4.10_ℓ}$$

holds, then for any $t_0 \in R_+$ *we have* $\mathbf{U}_{\ell,t_0} = \varnothing$.

Corollary 4.5. *Let the conditions (1.2), (1.3), (1.6), (3.2_ℓ) and (4.7_ℓ) be fulfilled, where* $\ell \in \{1, \ldots, n-1\}$ *with* $\ell + n$ *odd, and let there exist* $\alpha \in (1, +\infty)$ *such that*

$$\liminf_{t\to+\infty} \frac{\sigma(t)}{t^\alpha} > 0. \tag{4.11}$$

If, moreover, either

$$\alpha\lambda \geq 1 \tag{4.12}$$

or, if $\alpha\lambda < 1$, *for some* $\varepsilon > 0$,

$$\int\limits_0^{+\infty} t^{n-\ell-1+\mu(t)(\frac{\alpha(1-\gamma)}{1-\alpha\lambda}-\varepsilon)}(\sigma(t))^{(\ell-1)\mu(t)} p(t)\, dt = +\infty, \tag{4.13_ℓ}$$

then for any $t_0 \in R_+$ *we have* $\mathbf{U}_{\ell,t_0} = \varnothing$.

Proof. It suffices to show that the condition (4.2_ℓ) is satisfied for $\delta = 0$ and for some $k \in N$. Indeed, according to (4.7_ℓ) and (4.11), there exist $\alpha > 1$, $c > 0$, $\gamma \in (0, 1)$ and $t_1 \in [t_0, +\infty)$ such that

$$t^\gamma \int\limits_t^{+\infty} s^{n-\ell-1}(\sigma(s))^{(\ell-1)\mu(s)} p(s)\, ds \geq c \quad \text{for} \quad t \geq t_1 \tag{4.14}$$

and

$$\sigma(t) \geq c\, t^\alpha \quad \text{for} \quad t \geq t_1. \tag{4.15}$$

Choose $k_0 \in N$ and $c_* \in (1, +\infty)$ such that

$$(1-\gamma)(k_0 - 1) \geq \frac{1}{\lambda} \quad \text{when} \quad \alpha\lambda \geq 1, \tag{4.16}$$

if $\varepsilon > 0$, then

$$1 + \alpha\lambda + \ldots + (\alpha\lambda)^{k_0-2} \geq \frac{1}{1-\alpha\lambda} - \frac{\varepsilon}{\alpha(1-\gamma)} \quad \text{when} \quad \alpha\lambda < 1 \tag{4.17}$$

and for any $k \in \{1, \ldots, k_0\}$

$$c_*^{\lambda^k} \left(\frac{c}{2\ell!(n-\ell)!(1-\gamma)(1+\alpha\lambda+\ldots+(\alpha\lambda)^{k-2}} \right)^{1+\lambda+\ldots+\lambda^{k-2}} \geq 1. \tag{4.18}$$

According to (4.14) and (3.24$_\ell$) it is obvious that $\lim_{t\to+\infty} \widetilde{\rho}_{\ell,1}(t) = +\infty$. Therefore without loss of generality we can assume that $\widetilde{\rho}_{\ell,1}(t) \geq c_*$ for $t \geq t_1$. Thus, by (4.14), from (3.25$_\ell$) we get

$$\widetilde{\rho}_{\ell,2}(t) \geq \frac{1}{\ell!(n-\ell)!} \int\limits_{t_1}^{t} \int\limits_{s}^{+\infty} \xi^{n-\ell-1} (\sigma(\xi))^{(\ell-1)\mu(\xi)} p(\xi)\, d\xi\, ds \geq$$

$$\geq \frac{c_*^{\lambda} c}{\ell!(n-\ell)!} \int\limits_{t_1}^{t} s^{-\gamma}\, ds = \frac{c_*^{\lambda} c}{\ell!(n-\ell)!(1-\gamma)} (t^{1-\gamma} - t_1^{1-\gamma}).$$

Choose $t_2 > t_1$ such that

$$\widetilde{\rho}_{\ell,2}(t) \geq \frac{c_*^{\lambda} c\, t^{1-\gamma}}{2\ell!(n-\ell)!(1-\gamma)} \quad \text{for} \quad t \geq t_2.$$

Then by (1.6), (4.14), (4.15) and (4.18), from (3.25$_\ell$) we have

$$\widetilde{\rho}_{\ell,3}(t) \geq c_*^{\lambda^2} \left(\frac{c}{2\ell!(n-\ell)!(1-\gamma)(1+\alpha\lambda)} \right)^{1+\lambda} t^{(1-\gamma)(1+\alpha\lambda)} \quad \text{for} \quad t \geq t_3,$$

where $t_3 > t_2$ is a sufficiently large number. Therefore, for $k_0 \in N$ there exists $t_{k_0} \in R_+$ such that

$$\widetilde{\rho}_{\ell,k_0}(t) \geq c_*^{\lambda^{k_0-1}} \left(\frac{c}{2\ell!(n-\ell)!(1-\gamma)(1+\alpha\lambda+\ldots+(\alpha\lambda)^{k_0-2})} \right)^{1+\lambda+\ldots+\lambda^{k_0-2}} \times$$

$$\times\, t^{(1-\gamma)(1+\alpha\lambda+\ldots+(\alpha\lambda)^{k_0-2})} \quad \text{for} \quad t \geq t_{k_0}. \tag{4.19}$$

Assume that (4.12) is fulfilled. Then, according to (1.6), (4.14), (4.16) and (4.19) it is obvious that, if $\delta = 0$ for $k = k_0$, (4.2$_\ell$) holds. In the case, where (4.12) holds, the validity of the theorem has been already proved.

Assume now that $\alpha\lambda < 1$ and for some $\varepsilon > 0$ (4.13$_\ell$) is fulfilled. Then, by (4.17) from (4.19) we have

$$(\widetilde{\rho}_{\ell,k_0}(\sigma(t)))^{\mu(t)} \geq c_1 t^{\mu(t)(\frac{\alpha(1-\gamma)}{1-\alpha\lambda}-\varepsilon)} \quad \text{for} \quad t \geq t_{k_0},$$

where $c_1 > 0$. Consequently, according to (4.13$_\ell$), it is obvious that (4.2$_\ell$) holds with if $\delta = 0$ and $k = k_0$.

The theorem is proved.

In a similar manner we can prove the following corollary.

Corollary 4.6. *Let the conditions (1.2), (1.3), (1.6), (3.2$_\ell$) and (4.9$_\ell$) be fulfilled, where $\ell \in$ $\in \{1, \ldots, n-1\}$ with $\ell + n$ odd, and let there exist $\alpha > 0$ such that*

$$\liminf_{t\to+\infty} t^{-\alpha} \ln \sigma(t) > 0. \tag{4.20}$$

Then for any $t_0 \in R_+$ we have $\mathbf{U}_{\ell,t_0} = \varnothing h$.

5. Differential equations with property A.

Theorem 5.1. *Let the conditions (1.2), (1.3), (1.5) be fulfilled and (3.1$_\ell$) and (3.2$_\ell$) hold for any $\ell \in \{1, \ldots, n-1\}$ with $\ell + n$ odd. Let, moreover, there exist $\delta \in [0, \lambda]$, $k \in N$ and $\sigma_* \in C(R_+)$ satisfying the condition (3.3) such that (4.1$_\ell$) holds. If, moreover,*

$$\int_0^{+\infty} t^{n-1} p(t)\, dt = +\infty \tag{5.1}$$

when n is odd, then the equation (1.1) has Property $\mathbf{A}$, *where $\rho_{\ell,k}$ is defined by (3.5$_\ell$) and (3.6$_\ell$).*

Proof. Let the equation (1.1) have a proper nonoscillatory solution $u : [t_0, +\infty) \to (0, +\infty)$ (the case $u(t) < 0$ is similar). Then by (1.2), (1.3) and Lemma 2.1 there exists $\ell \in \{0, 1, \ldots, n-1\}$ such that $\ell + n$ is odd and the condition (2.1$_\ell$) holds. Since the conditions of Theorem 4.1 are fulfilled for any $\ell \in \{1, \ldots, n-1\}$ with $\ell + n$ odd, we have $\ell \notin \{1, \ldots, n-1\}$. Therefore, n is odd and $\ell = 0$. Show that the condition (1.4) holds. If that is not the case, then there exists $c \in (0, 1)$ such that $u(t) \geq c$ for sufficiently large t. According to (2.1$_0$) and (1.5) we have

$$\sum_{i=0}^{n-1} (n-i-1)!\, t_1^i |u^{(i)}(t_1)| \geq \int_{t_1}^{t} s^{n-1} p(s) c^{\mu(s)}\, ds \geq c^\lambda \int_{t_1}^{t} s^{n-1} p(s)\, ds \quad \text{for} \quad t \geq t_1, \tag{5.2}$$

where t_1 is a sufficiently large number. The inequality (5.2) contradicts the condition (5.1). Therefore the equation (1.1) has Property $\mathbf{A}$.

Theorem 5.2. *Let the conditions (1.2), (1.3), (1.6) be fulfilled and for any $\ell \in \{1, \ldots, n-1\}$ with $\ell + n$ odd (3.1$_\ell$), (3.2$_\ell$) and for some $\delta \in [0, \lambda]$ and $k \in N$, (4.2$_\ell$) holds. If, moreover,*

$$\limsup_{t\to+\infty} \mu(t) < +\infty \tag{5.3}$$

and (5.1) holds when n is odd, then the equation (1.1) has Property $\mathbf{A}$, *where $\widetilde{\rho}_{\ell,k}$ is defined by (3.24$_\ell$) and (3.25$_\ell$).*

Proof. The proof of the theorem is analogous to that of Theorem 5.1. We have just to use Theorem 4.2 instead of Theorem 4.1, and change λ by $\mu = \sup\{\mu(t) : t \in R_+\}$ in the inequality (5.2).

Theorem 5.3. *Let the conditions (1.2), (1.3), (1.5), (5.1) and*

$$\liminf_{t\to+\infty} \frac{(\sigma(t))^{\mu(t)}}{t} > 0 \tag{5.4}$$

be fulfilled. If, moreover, there exist $\delta \in [0, \lambda]$, $k \in N$ and $\sigma_* \in C(R_+)$ satisfying the condition (3.3) such that for even n (for odd n) (4.1$_1$) ((4.1$_2$)) holds, then the equation (1.1) has Property $\mathbf{A}$, *where $\rho_{1,k}$ ($\rho_{2,k}$) is defined by (3.5$_1$) and (3.6$_1$) ((3.5$_2$) and (3.6$_2$)).*

Proof. To prove the theorem it suffices to show that the conditions of Theorem 5.1 are fulfilled. Indeed, according to (4.1$_1$) and (5.4) ((4.1$_2$) and (5.4)) it is obvious that (4.1$_\ell$) holds for any $\ell \in \{1, \ldots, n-1\}$ with $\ell + n$ odd. Thus according to (5.1) all the conditions of Theorem 5.1 are fulfilled, which proves the validity of the theorem.

Using Theorem 5.2, the next theorem can be proved similarly.

Theorem 5.4. *Let the conditions (1.2), (1.3), (1.6) and (5.4) hold. If, moreover, there exist $\delta \in [0, \lambda]$ and $k \in N$ such that for even n (for odd n) (4.2$_1$) ((4.2$_2$), (5.1) and (5.3)) holds, then the equation (1.1) has Property* $\mathbf{A}$, *where $\widetilde{\rho}_{1,k}$ ($\widetilde{\rho}_{2,k}$) is defined by (3.24$_1$) and (3.25$_1$) ((3.24$_2$) and (3.25$_2$)).*

Corollary 5.1. *Let the conditions (1.2), (1.3), (1.5) and (5.4) be fulfilled. If, moreover,*

$$\int_0^{+\infty} t^{n-2+\mu(t)} p(t)\, dt = +\infty \tag{5.5}$$

for even n, and

$$\int_0^{+\infty} t^{n-3+\mu(t)} (\sigma(t))^{\mu(t)} p(t)\, dt = +\infty \tag{5.6}$$

for odd n, then the equation (1.1) has Property $\mathbf{A}$.

Proof. It suffices to note that by (1.5), (5,4), (5.5) and (5.6) all the conditions of Theorem 5.3 are fulfilled with $\sigma_*(t) = t$ and $\delta = 0$.

Corollary 5.2. *Let the conditions (1.2), (1.3), (1.5), (5.1) and (5.4) be fulfilled. Let, moreover, for some $k \in N$*

$$\int_0^{+\infty} t^{n-2} (\sigma(t))^{\mu(t)-\lambda} (\rho_{1,k}(\sigma(t)))^{\lambda} p(t)\, dt = +\infty \tag{5.7}$$

hold when n is even, and

$$\int_0^{+\infty} t^{n-3} (\sigma(t))^{2\mu(t)-\lambda} (\rho_{2,k}(\sigma(t)))^{\lambda} p(t) dt = +\infty \tag{5.8}$$

hold when n is odd. Then the equation (1.1) has Property $\mathbf{A}$, *where $\rho_{1,k}$ ($\rho_{2,k}$) is defined by (3.5$_1$) and (3.6$_1$) ((3.5$_2$) and (3.6$_2$)).*

Proof. It suffices to note that by (1.5), (5.4), (5.7) and (5.8) all the conditions of Theorem 5.3 are fulfilled with $\sigma_*(t) = \sigma(t)$ and $\delta = \lambda$.

Corollary 5.3. *Let the conditions (1.2), (1.3), (1.6), (5.1) and (5.4) be fulfilled. Let, moreover, for some $k \in N$*

$$\int_0^{+\infty} t^{n-2}(\widetilde{\rho}_{1,k}(\sigma(t)))^{\mu(t)}p(t)\,dt = +\infty \tag{5.9}$$

hold when n is even, and

$$\int_0^{+\infty} t^{n-3}(\sigma(t))^{\mu(t)}(\widetilde{\rho}_{2,k}(\sigma(t)))^{\mu(t)}p(t)\,dt = +\infty \tag{5.10}$$

hold when n is odd, then the equation (1.1) has Property $\mathbf{A}$, *where $\widetilde{\rho}_{1,k}$ ($\widetilde{\rho}_{2,k}$) is defined by (3.24$_1$) and (3.25$_1$) ((3.24$_2$) and (3.25$_2$)).*

Proof. It suffices to note that by (1.6), (5.4), (5.9) and (5.10) all the conditions of Theorem 5.4 are fulfilled with $\delta = 0$.

Theorem 5.5. *Let the conditions (1.2), (1.3), (3.2$_{n-1}$), (1.5) and*

$$\limsup_{t\to+\infty} \frac{(\sigma(t))^{\mu(t)}}{t} < +\infty \tag{5.11}$$

hold. If, moreover, there exist $\delta \in [0,\lambda]$, $k \in N$ and $\sigma_* \in C(R_+)$ satisfying the condition (3.3) such that (4.1$_{n-1}$) holds, then the equation (1.1) has Property $\mathbf{A}$, *where ρ_{n-1} is defined by (3.5$_{n-1}$) and (3.6$_{n-1}$).*

Proof. By virtue of (1.2), (1.3), (1.5), (3.2$_{n-1}$), (4.1$_{n-1}$) and (5.11), the conditions of Theorem 5.1 are obviously satisfied. Therefore according to that theorem the equation (1.1) has Property $\mathbf{A}$.

The validity of Theorem 5.6 below is proved similarly.

Theorem 5.6. *Let the conditions (1.2), (1.3), (1.6), (3.2$_{n-1}$) and (5.11) be fulfilled. If, moreover, there exist $\delta \in [0,\lambda]$ and $k \in N$ such that (4.2$_{n-1}$) holds, then the equation (1.1) has Property* $\mathbf{A}$, *where $\widetilde{\rho}_{n-1,k}$ is defined by (3.24$_{n-1}$) and (3.25$_{n-1}$).*

Corollary 5.4. *Let the conditions (1.2), (1.3), (1.5) and (5.11) be fulfilled. If, moreover,*

$$\int_0^{+\infty} t^{\lambda}(\sigma(t))^{(n-1)\mu(t)-\lambda}p(t)\,dt = +\infty, \tag{5.12}$$

then the equation (1.1) *has Property* $\mathbf{A}$.

Proof. It suffices to note that by (1.2), (1.3), (1.5), (5.11) and (5.12) all the conditions of Theorem 5.5 are fulfilled with $\delta = 0$ and $\sigma_*(t) = \sigma(t)$.

Corollary 5.5. *Let the conditions (1.2), (1.3), (1.5) and (5.11) be fulfilled. If, moreover,*

$$\int_0^{+\infty} t^{\mu(t)} (\sigma(t))^{(n-2)\mu(t)} p(t)\, dt = +\infty \tag{5.13}$$

holds, then the equation (1.1) has Property $\mathbf{A}$.

Proof. According to Theorem 5.5, it suffices to note that by (5.13) the condition (4.1_{n-1}) holds with $\delta = 0$ and $\sigma_*(t) \equiv t$.

Corollary 5.6. *Let the conditions (1.2), (1.3), (1.5), (3.2_{n-1}) and (5.11) be fulfilled. If, moreover, for some $k \in N$*

$$\int_0^{+\infty} (\sigma(t))^{(n-1)\mu(t)-\lambda} (\rho_{n-1,k}(\sigma(t)))^{\lambda} p(t)\, dt = +\infty \tag{5.14}$$

holds. Then the equation (1.1) has Property $\mathbf{A}$, *where $\rho_{n-1,k}$ is defined by (3.5_{n-1}) and (3.6_{n-1}).*

Proof. Since from (5.14) it follows (4.1_{n-1}) with $\delta = \lambda$ and $\sigma_*(t) \equiv \sigma(t)$, the validity of the corollary follows from Theorem 5.5.

Analogously we can prove the following corollary.

Corollary 5.7. *Let the conditions (1.2), (1.3), (1.6), (5.11) and (3.2_{n-1}) be fulfilled and for some $k \in N$*

$$\int_0^{+\infty} (\sigma(t))^{(n-2)\mu(t)} (\widetilde{\rho}_{n-1,k}(\sigma(t)))^{\mu(t)} p(t) dt = +\infty. \tag{5.15}$$

Then the equation (1.1) has Property $\mathbf{A}$, *where $\widetilde{\rho}_{n-1,k}$ is defined by (3.24_{n-1}) and (3.25_{n-1}).*

Theorem 5.7. *Let the conditions (1.2), (1.3), (1.5), (5.4) hold and for some $\gamma \in (0,1)$ (4.3_1) ((4.3_2) and (5.1)) are fulfilled when n is even (when n is odd). If, moreover, there exist $\delta \in [0, \lambda]$ and $\sigma_* \in C(R_+)$ satisfying (3.3) such that the condition (4.4_1) ((4.4_2)) holds, then the equation (1.1) has Property* $\mathbf{A}$.

Proof. According to (5.4) and (4.3_1) ((4.3_2)) it is obvious that for any $\ell \in \{1, \dots, n-1\}$ with $\ell + n$ odd the condition (4.3_ℓ) holds. Assume that the equation (1.1) has a nonoscillatory solution $u : [t_0, +\infty) \to (0, +\infty)$ satisfying the condition (2.1_ℓ). Then, by Corollary 4.1, $\ell \notin \{1, \dots, n-1\}$. Therefore n is odd and $\ell = 0$. In this case by (5.1) we can show that (1.4) holds. Therefore, the equation (1.1) has Property $\mathbf{A}$.

Corollary 5.8. *Let the conditions (1.2), (1.3), (1.5), (5.1) and (5.4) hold and for some $\gamma \in (0,1)$ (4.3_1) ((4.3_2) and (5.1)) be fulfilled when n is even (when n is odd). If, moreover,*

$$\int_0^{+\infty} t^{n-2-\lambda+\mu(t)+\frac{\lambda(1-\gamma)}{1-\lambda}} p(t)\, dt = +\infty \tag{5.16}$$

holds for even n *and*

$$\int_0^{+\infty} t^{n-3+\lambda+\mu(t)+\frac{\lambda(1-\gamma)}{1-\lambda}} (\sigma(t))^{\mu(t)} p(t)\,dt = +\infty \tag{5.17}$$

hold for odd n, *then the equation (1.1) has Property* $\mathbf{A}$.

Proof. It suffices to note that by (5.16) and (5.17) the conditions (4.4_1) and (4.4_2) are satisfied with $\delta = \lambda$ and $\sigma_*(t) \equiv t$.

Analogously to Theorem 5.7 we can prove the following theorem.

Theorem 5.8. *Let the conditions (1.2), (1.3), (1.5), (5.4) hold and (4.5_1) ((4.5_2) and (5.1)) be fulfilled for even* n (*for odd* n). *If, moreover, there exist* $\delta \in [0,\lambda]$ *and* $\sigma_* \in C(R_+)$ *satisfying (3.3) such that the condition (4.6_1) ((4.6_2)) holds, then the equation (1.1) has Property* $\mathbf{A}$.

Corollary 5.9. *Let the conditions (1.2), (1.3), (1.5), (5.4) hold and (4.5_1) ((4.3_2) and (5.1)) be fulfilled for even* n (*for odd* n). *If, moreover,*

$$\int_0^{+\infty} t^{n-2} (\sigma(t))^{\mu(t)-\lambda} (\ln(1+\sigma(t)))^{\frac{\lambda}{1-\lambda}} p(t)\,dt = +\infty \tag{5.18}$$

for even n *and*

$$\int_0^{+\infty} t^{n-3} (\sigma(t))^{2\mu(t)-\lambda} (\ln(1+\sigma(t)))^{\frac{\lambda}{1-\lambda}} p(t)\,dt = +\infty \tag{5.19}$$

for odd n, *then the equation (1.1) has Property* $\mathbf{A}$.

Proof. It suffices to note that by (5.18) and (5.19) the conditions (4.6_1) and (4.6_2) hold with $\delta = \lambda$ and $\sigma_*(t) = \sigma(t)$.

Theorem 5.9. *Let the conditions (1.2), (1.3), (1.5), (5.11) and (4.3_{n-1}) be fulfilled. If, moreover, there exist* $\delta \in [0,\lambda]$ *and* $\sigma_* \in C(R_+)$ *satisfying (3.3) such that the condition (4.4_{n-1}) holds, then the equation (1.1) has Property* $\mathbf{A}$.

Corollary 5.10. *Let the conditions (1.2), (1.3), (1.5), (5.11) hold, for some* $\gamma \in (0,1)$, *(4.3_{n-1}) be fulfilled and*

$$\int_0^{+\infty} (\sigma(t))^{(n-1)\mu(t)-\lambda+\frac{\lambda(1-\gamma)}{1-\lambda}} p(t)\,dt = +\infty. \tag{5.20}$$

Then the equation (1.1) has Property $\mathbf{A}$.

Proof. It suffices to note that by (5.20) the condition (4.4_{n-1}) holds with $\delta = \lambda$ and $\sigma_*(t) = \sigma(t)$.

Theorem 5.10. *Let the conditions (1.2), (1.3), (1.6), (5.4) hold and for some* $\gamma \in (0,1)$ *(4.7_1) ((4.7_2) and (5.1)) be fulfilled for even* n (*for odd* n). *If, moreover, there exists* $\delta \in [0,\lambda]$ *such that*

(4.8_1) holds when n is even and (4.8_2) and (4.5_1) hold when n is odd, then the equation (1.1) has Property $\mathbf{A}$.

The theorem can be proved similarly to Theorem 5.7.

Corollary 5.11. *Let the conditions (1.2), (1.3), (1.6), (5.4) hold and (4.7_1) ((4.7_2) and (5.1)) be fulfilled for even n (for odd n). If, moreover,*

$$\int\limits_0^{+\infty} t^{n-2}(\sigma(t))^{\frac{\mu(t)(1-\gamma)}{1-\lambda}} p(t)\,dt = +\infty \tag{5.21}$$

when n is even and

$$\int\limits_0^{+\infty} t^{n-3}(\sigma(t))^{\frac{\mu(t)(2-\lambda-\gamma)}{1-\lambda}} p(t)\,dt = +\infty \tag{5.22}$$

when n is odd, then the equation (1.1) has Property $\mathbf{A}$.

Proof. According to Theorem 5.10 it suffices to note that by (5.21) and (5.22) the conditions (4.8_1) and (4.8_2) hold with $\delta = 0$.

Using Theorem 5.6, similarly to Theorem 5.7 one can prove the following theorem.

Theorem 5.11. *Let the conditions (1.2), (1.3), (1.6), (5.11) and (4.7_{n-1}) be fulfilled and for some $\delta \in [0, \lambda]$ (4.8_{n-1}) hold. Then the equation (1.1) has Property* $\mathbf{A}$.

Corollary 5.12. *Let the conditions (1.2), (1.3), (1.6), (5.11) and (4.7_{n-1}) be fulfilled and*

$$\int\limits_0^{+\infty} (\sigma(t))^{\mu(t)\left(n-2+\frac{1-\gamma}{1-\lambda}\right)} p(t)\,dt = +\infty. \tag{5.23}$$

Then the equation (1.1) has Property $\mathbf{A}$.

Proof. According to Theorem 5.11 it suffices to note that by (5.23) the condition (4.8_{n-1}) holds with $\delta = 0$.

Theorem 5.12. *Let the conditions (1.2), (1.3), (1.6), (5.1), (5.4) be fulfilled and (4.7_1) ((4.7_2)) hold for even n (for odd n). If, moreover, there exists $\alpha \in (1, +\infty)$ such that (4.11) holds, then for the equation (1.1) to have Property* $\mathbf{A}$ *it is sufficient that at least one the conditions (4.12) or, if $\alpha\lambda < 1$, (4.13_1) ((4.13_2)) holds for even n (for odd n).*

Proof. According to (4.7_1), (4.13_1) and (5.4) ((5.7_2), (5.4) and (4.13_2)) it is obvious that for any $\ell \in \{1, \dots, n-1\}$ with $\ell + n$ odd (3.2_ℓ) and (3.1_ℓ) hold. Assume that the equation (1.1) has a nonoscillatory solution $u : [t_0, +\infty) \to (0, +\infty)$ satisfying the condition (2.1_ℓ). Then by Corollary 4.5, $\ell \notin \{1, \dots, n-1\}$. Therefore n is odd and $\ell = 0$. In this case by (5.1) it is obvious that (1.4) holds. Therefore the equation (1.1) has Property $\mathbf{A}$.

Using Corollaries 4.5 and 4.6, in a similar manner we can prove Theorems 5.13 and 5.14 below.

Theorem 5.13. *Let the conditions (1.2), (1.3), (1.6), (5.11) and (4.7$_{n-1}$) be fulfilled. If, moreover, there exist $\alpha \in (1, +\infty)$ such that (4.11) holds, then for the equation (1.1) to have Property $\mathbf{A}$ it is sufficient that at least one of the conditions (4.12) or, if $\alpha\lambda < 1$, (4.13$_{n-1}$) holds.*

Theorem 5.14. *Let the conditions (1.2), (1.3), (1.6), (5.4) be fulfilled and (4.9$_1$) ((4.9$_2$) and (5.1)) hold for even n (for odd n). If, moreover, there exists $\alpha > 0$ such that (4.20) holds, then the equation (1.1) has Property $\mathbf{A}$.*

6. Necessary and sufficient conditions.

Theorem 6.1. *Let the conditions (1.2), (1.3) and (1.5) be fulfilled and*

$$\limsup_{t\to+\infty} \frac{\sigma(t)}{t} < +\infty. \tag{6.1}$$

Then the condition

$$\int_0^{+\infty} t^{(n-1)\mu(t)} p(t)\, dt = +\infty \tag{6.2}$$

is necessary and sufficient for the equation (1.1) to have Property $\mathbf{A}$.

Proof. *Necessity.* Assume that the equation (1.1) has Property $\mathbf{A}$ and

$$\int_0^{+\infty} t^{(n-1)\mu(t)} p(t)\, dt < +\infty. \tag{6.3}$$

By (1.5), (6.1) and (6.3)

$$\int_0^{+\infty} (\sigma(t))^{(n-1)\mu(t)} p(t)\, dt < +\infty.$$

Therefore, following Lemma 4.1 [7], there exists $c \neq 0$ such that the equation (1.1) has a proper solution $u : [t_0, \infty) \to R$ satisfying the condition $\lim_{t\to+\infty} u^{(n-1)}(t) = c$. But this contradicts the fact that the equation (1.1) has Property $\mathbf{A}$.

Sufficiency. By (6.1) and (6.2) it is obvious that the condition (5.12) holds. Therefore the sufficiency follows from Corollary 5.4.

From Theorem 6.1, when $\mu(t) \equiv \lambda$ ($\lambda \in (0,1)$) and $\sigma(t) \equiv t$, follows a theorem of Ličko and Švec [8].

Corollary 6.1. *Let the conditions (1.2), (1.3), (1.6) and (6.1) be fulfilled and*

$$\limsup_{t\to+\infty} t^{\mu(t)} < +\infty.$$

Then the condition

$$\int_0^{+\infty} p(t)\, dt = +\infty$$

is necessary and sufficient for the equation (1.1) to have Property **A**.

Remark 6.1. Note that a necessary and sufficient condition of this kind, which does not depend on the order of the equation, is given for the first time.

Theorem 6.2. *Let n be odd, the conditions (1.2), (1.3) and (1.5) be fulfilled and*

$$\liminf_{t\to+\infty} \frac{\sigma(t)}{t^{\frac{2-\mu(t)}{\mu(t)}}} > 0. \tag{6.4}$$

Then the condition (5.1) is necessary and sufficient for the equation (1.1) to have Property **A**.

Proof. *Necessity.* Assume that the equation (1.1) has Property **A** and

$$\int\limits_0^{+\infty} t^{n-1} p(t)\,dt < +\infty. \tag{6.5}$$

According to (6.5), by Lemma 4.1 [7] there exists $c \neq 0$ such that the equation (1.1) has a proper solution $u : [t_0,\infty) \to R$ satisfying the condition $\lim_{t\to+\infty} u(t) = c$. But this contradicts the fact that the equation (1.1) has Property **A**.

Sufficiency. According to (1.5) and (6.4) it is obvious that the condition (5.4) holds. On the other hand, by (5.1) and (6.4) the condition (5.6) holds. Thus, since n is odd, all the conditions of Corollary 5.1 are fulfilled, i.e., the equation (1.1) has Property **A**.

Corollary 6.2. *Let n be odd, the conditions (1.2), (1.3), (1.5) be fulfilled and*

$$\lim_{t\to+\infty} \mu(t) = \lambda \quad (\lambda \in (0,1)), \quad \liminf_{t\to+\infty} t^{\mu(t)-\lambda} > 0, \quad \liminf_{t\to+\infty} \frac{\sigma(t)}{t^{\frac{2-\lambda}{\lambda}}} > 0. \tag{6.6}$$

Then the condition (5.1) is necessary and sufficient for the equation (1.1) to have Property **A**.

Remark 6.2. The condition (6.6) defines a set of the functions σ for which the condition (5.1) is necessary and sufficient. It turns out that the number $\dfrac{2-\lambda}{\lambda}$ is optimal. Indeed, let $\varepsilon > 0$, $\lambda \in (1/(1+\varepsilon), 1)$ and $\gamma \in (1,2)$. Consider the differential equation (1.1) with

$$p(t) = -\gamma(\gamma-1)\dots(\gamma-n+1)t^{-n+\gamma(1-\mu(t)(\frac{2-\lambda}{\lambda}-\varepsilon))},$$

$$\sigma(t) = t^{\frac{2-\lambda}{\lambda}-\varepsilon}, \quad t \geq 1, \quad \lim_{t\to+\infty} \mu(t) = \lambda.$$

It is obvious that the condition (5.1) is fulfilled and

$$\liminf_{t\to+\infty} \frac{\sigma(t)}{t^{\frac{2-\lambda}{\lambda}}} = 0, \quad \text{and} \quad \liminf_{t\to+\infty} \frac{\sigma(t)}{t^{\frac{2-\lambda}{\lambda}-\varepsilon}} > 0.$$

On the other hand, for odd n, $u(t) = t^\gamma$ is a solution of equation (1.1). Therefore, when n is odd, the equation (1.1) does not have Property **A**.

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Received 29.12.11