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PARTIAL SOLUTIONS OF A SYSTEM OF EULER EQUATIONS

ЧАСТИННІ РОЗВ'ЯЗКИ СИСТЕМИ РІВНЯНЬ ЕЙЛЕРА

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We consider equations of hydrodynamics with certain additional constraints. Group-theoretical methods are applied to find invariant solutions.

Розглянуто рівняння гідродинаміки з певними додатковими зв'язками. Для знаходження їхніх інваріантних розв'язків використано теоретико-групові методи.

Introduction. The past century in mathematical physics was marked with a large number of research papers on particular solutions to nonlinear differential equations. Besides the fact that exact solutions are almost always interesting themselves, they also have a valuable practical application to verification of various numerical methods of solving of nonlinear differential equations.

There are many examples of explicitly solved problems of fluid mechanics in the literature. All known solutions and multiparametric families of new particular solutions appear to be obtainable by means of group-theoretical methods [1 – 8]. Moreover, these methods are useful for finding particular solutions of nonlinear differential equations that satisfy certain prescribed initial or boundary conditions.

In the present paper we look for invariant solutions of a system of Euler equations that satisfy the Rankine – Hugoniot conditions.

1. Formulation of the problem. To describe the motion of nonviscous compressible liquid we use the system of equations

$$\begin{aligned} &D_t u^k(t,x) + \rho^{-1}\nabla_k p(t,x) = 0, \\ &D_t \rho(t,x) + \rho\nabla_k u^k(t,x) = 0, \end{aligned} \tag{1}$$

where $t \in \mathbb{R}^1,\ x \in \mathbb{R}^n,\ n = 1,\ldots,3,\ u^k(t,x)$ stands for the k-th component of the medium's velocity $k = 1,\ldots,n,\ p$ is the pressure, ρ is the liquid density, and $D_t = \dfrac{\partial}{\partial t} + u^k\nabla_k$ is the total derivative with respect to time with $\nabla_k = \dfrac{\partial}{\partial x_k}$. Repeating indices mean summation, unless otherwise noted.

The main thermodynamical characteristics of the medium $\rho,\ p$ and T are expected to be related by an expression

$$p = \Phi(\rho, T), \tag{2}$$

where Φ is a smooth (piecewise smooth) function. We also assume that the process described by system (1), (2) is either isothermal ($T = \mathrm{const}$) or homothermal ($v_k T = 0,\ k = 1,\ldots,n$). Therefore T does not depend on spatial coordinates and state equation (2) reads as

$$p = F(\rho, t), \tag{3}$$

with another function F.

In order to represent system (1) in a convenient for the following analysis form we introduce the notations

$$u^k_\mu = \frac{\partial u^k}{\partial x_\mu}, \qquad \rho_\mu = \frac{\partial \rho}{\partial x_\mu}, \qquad p_k = \frac{\partial p}{\partial x_k},$$

where $k = 1, \ldots, n,\ \mu = 0, \ldots, n$, and $x_0 = t$. Using these notations we represent system (1) in the form

$$u^k_0 + u^k_j u^j + \rho^{-1} p_k = 0, \qquad \rho_0 + u^j \rho_j + \rho u^j_j = 0. \tag{4}$$

Substituting (3) into the first equation (4) we obtain

$$u^k_0 + u^j u^k_j + \rho^{-1} F_\rho \rho_k = 0, \tag{5}$$

$$\rho_0 + u^j \rho_j + \rho u^j_j = 0, \tag{6}$$

where $F_\rho = \dfrac{\partial F}{\partial \rho}$.

For the symmetry analysis of system (5), (6) we use the infinitesimal Sophus Lie method. Its brief description is following. Let

$$F^\nu(x, u, u_{(1)}) = 0, \quad \nu = 1, \ldots, N, \tag{7}$$

be a system of first order differential equations, where $x = (x_1, \ldots, x_n),\ u = \left(u^1, \ldots, u^m\right)$, and $u_{(1)} = Du$.

We consider a one-parameter local group G of transformations

$$\begin{aligned} x' &= f(x, u; a) = 0 :\ f\big|_{a=0} = x, \\ u' &= g(x, u; a) = 0 :\ g\big|_{a=0} = u \end{aligned} \tag{8}$$

in the space $\mathbb{R}^{n+m}$ of the variables (x, u). Transformations (8) induce a one-parameter group of transformations in the space of the variables $u_{(1)}$,

$${u_{(1)}}' = \Psi(x, u, u_{(1)}; a) = 0 :\ \Psi\big|_{a=0} = u_{(1)}, \tag{9}$$

where $\Psi(x, u, u_{(1)}; a)$ is a function which can be determined once we know f and g. As a result we have a one-parameter group $G_{(1)}$ of transformations in the space $\mathbb{R}^{n+m+nm}$ of the variables $(x, u, u_{(1)})$. Transformations (9) are referred to as the *prolongation* of transformations (8), and the group $G_{(1)}$ is the *first prolongation* of G [1, Chapter 2.3].

Definition 1. *System of equations (7) is said to be* invariant with respect to group G of point transformations *(8) if the manifold determined by equations (7) in the space* $\mathbb{R}^{n+m+nm}$ *is invariant with respect to the first prolongation* $G_{(1)}$ *of group* G.

Let

$$X = \xi^j(x, u) \frac{\partial}{\partial x_j} + \eta^\alpha(x, u) \frac{\partial}{\partial u^\alpha},$$

where

$$\xi^j(x,u) = \left.\frac{\partial f(x,u,a)}{\partial a}\right|_{a=0},$$
$$\eta^\alpha(x,u) = \left.\frac{\partial g(x,u,a)}{\partial a}\right|_{a=0}.$$

The operator X is said to be the *infinitesimal operator* of the one-parameter group G^1 of transformations, and functions ξ^j and η^α are its *coordinates*. The first prolongation of the group G^1 corresponds to an infinitesimal operator of the form

$$X_{(1)} = X + \zeta_i^\alpha \frac{\partial}{\partial u_i^\alpha},$$

where

$$\zeta_i^\alpha = \frac{\partial \eta^\alpha}{\partial x_i} + u_i^\mu \frac{\partial \eta^\alpha}{\partial u^\mu} - u_j^\alpha \left(\frac{\partial \xi^j}{\partial x_i} + u_i^\beta \frac{\partial \xi^j}{\partial u^\beta} \right). \tag{10}$$

One of the prominent results in the group theory of continuous transformations is the fact that the invariance criterion for a differential equation with respect to group G^1 is stated in terms of the correspondent infinitesimal symmetry operator, cf. [2].

Proposition 1. *System of equations (7) is invariant with respect to group G^1 if and only if*

$$X_{(1)} F^\nu(x,u,u_{(1)})\big|_{F=0} = 0, \quad \nu = 1,\ldots,N. \tag{11}$$

Condition (11) is equivalent to a system of first order linear differential equations in $x,\ u$ and $u_{(1)}$ named the *system of determining equations*.

Thus the problem of finding the maximal local group of point transformations that are admissible for system (7) is to determine the coordinates of the infinitesimal operators that generate its one-parameter subgroups.

In the case of system (5), (6) the infinitesimal symmetry operator is expected to be of the form

$$Z = \xi^\mu(x,u,\rho)\frac{\partial}{\partial x_\mu} + \eta^k(x,u,\rho)\frac{\partial}{\partial u^k} + \Lambda(x,u,\rho)\frac{\partial}{\partial \rho}, \tag{12}$$

where $\mu = 0,\ldots,n,\ k = 1,\ldots,n.$

Acting by operator (12) on equations (5), (6) we obtain a rather cumbersome system of first-order linear differential equations. Eliminating the variables u_0^k and ρ_0 by virtue of their expressions from (5) and (6), we transform it to another system of equations where the quantities $x_\alpha,\ u^k,\ u_j^k$ and ρ_j will be treated as independent variables from this point. As the coordinates of the infinitesimal operator do not depend on u_j^k and ρ_j, the two equations obtained from (5) and (6) by means of criterion (11) can be split with respect to these variables. As a result we have the system of differential equations

$$\begin{gathered}
\eta^k u^l + \xi_k^l = 0, \quad \eta^k u^l + \xi_l^k = 0, \quad k \neq l, \\
\eta^j + u^j \xi_0^0 - \xi_0^j - \sum_{i=1}^n \xi_i^j u^i = 0, \quad \Lambda_\rho + \rho^{-1}\Lambda + \xi_k^k - \xi_0^0 - \eta^k u^k = 0, \\
\Lambda_0 + \sum_{l=1}^n \left(u^l \Lambda_l + \rho u_l^l \right) = 0,
\end{gathered} \tag{13}$$

$$2F_\rho\left(\xi_0^0-\xi_k^k\right)+F_{\rho\rho}A+F_{0\rho}\xi^0=0, \tag{14}$$

where $\xi^0=\xi^0(x_0),\ \xi^k=\xi^k(x),\ \eta^k=\eta^k(x,u),\ \Lambda=\Lambda(x,u,\rho)$. In all the formulae (13), (14) there is no summation over repeating indices.

Note that the arbitrary function F appears only in (14). This equation is refereed to as a *classifying condition*.

2. Symmetry of system (5), (6). It is easy to check by direct calculations that system (13) has the solution

$$\xi^0=\varkappa x_0^2+\lambda x_0+\alpha,\quad \Lambda=\left(c-\frac{n}{2}\,\dot\xi^0(x_0)\right)\rho,$$
$$\xi^k=\left(\frac12\,\dot\xi^0(x_0)+\delta\right)x_k+\mu^k x_0+\sum_{l=1}^n a_l^k x_l+\nu^k,$$
$$\eta^k=\varkappa x_k+\mu^k+\sum_{l=1}^n a_l^k u^l+\left(\delta-\frac12\,\dot\xi^0(x_0)\right)u^k,$$

where $\dot\xi^0(x_0)=d\xi^0(x_0)/dx_0,\ a_l^k=-a_k^l,\ c,\ \alpha,\ \delta,\ \varkappa,\ \lambda,\ \mu^k$ and ν^k are arbitrary parameters. Substituting this solution into (14) we have

$$\left(\frac{n}{2}\dot\xi^0(x_0)-c\right)\varphi_\rho-\xi^0\varphi_0=\left(\dot\xi^0(x_0)-2\delta\right)\varphi, \tag{15}$$

where $\varphi(\rho,t)=F_\rho(\rho,t)$. Note that the parameters $a_l^k,\ \mu^k$ and ν^k (and α in the case when $\varphi_0=0$) are not involved in system (15). Therefore, for arbitrary function $F(\rho,l)$ system (13), (14) admits the solution

$$\begin{gathered}\xi^0=0,\quad \xi^k=\sum_{j=1}^n a_j^k x_j+\mu^k x_0+\nu^k,\\ \eta^k=\sum_{j=1}^n a_j^k u^j+\mu^k,\quad a_j^k=-a_k^j.\end{gathered} \tag{16}$$

In the case when $F_\rho=\varphi(\rho)$ the same solution with $\xi^0=\alpha=\mathrm{const}$ is also possible.

The functions $\xi^0,\ \xi^k$ and $\eta^k,\ k=1,\ldots,n,$ defined by (16) correspond to the differential operators

$$\begin{gathered}P_k=\frac{\partial}{\partial x_k},\quad G_k=x_0\,\frac{\partial}{\partial x_k}+\frac{\partial}{\partial u^k},\\ J_{kr}=x_k\,\frac{\partial}{\partial x_r}-x_r\,\frac{\partial}{\partial x_k}+u^k\,\frac{\partial}{\partial u^r}-u^r\,\frac{\partial}{\partial u^k}.\end{gathered} \tag{17}$$

It is easy to check that the vector space $\langle P_k,G_k,J_{kr}\rangle,\ k=1,\ldots,n,\ r=1,\ldots,n,$ is closed under the Lie bracket

$$X,Y\longrightarrow[X,Y]=XY-YX$$

and therefore the space of these operators possesses the structure of a Lie algebra. This is a general property [2], namely, the set of infinitesimal operators that generate the one-parameter groups of transformations admissible for a differential equation (or a system) necessarily form a Lie algebra. Operators (17) with $P_0=\dfrac{\partial}{\partial x_0}$ (the case $\alpha=\xi^0\neq0$) form the Lie algebra of the Galilean group. Therefore the following statement holds.

Theorem 2.1. *For arbitrary function $F_\rho = \varphi(\rho, t)$ system of equations (5), (6) admits $\dfrac{n(n+3)}{2}$-parameter group of transformations with the Lie algebra generated by the operators (17). In the case when F_ρ does not depend on x_0 explicitly system (5), (6) admits the Galilean group $G(n)$.*

Thereby, we have found the symmetry of system (5), (6) under arbitrary functional relationship $p = F(\rho, t)$. However, for some values of F symmetry of this system appears to be essentially wider. In order to list all the cases of symmetry extensions, it is necessary to get the set of solutions to equation (15) under various constraints on the parameters involved in this equation.

As a result of solving equation (15) we have found 12 cases of symmetry extension for the system in question. The corresponding functions $\varphi_\nu =$ and the set of infinitesimal symmetry operators admitted by system (5), (6) are presented in Table 1.

Observe that for all state equations that admit an extension of the symmetry (except the first one, where $\varphi = \varphi_1 = M\rho^{2/n}$) an arbitrary one-parameter invariance group of Euler equations is generated by an operator of the form

$$Z = (\alpha + \lambda x_0)\frac{\partial}{\partial x_0} + Ax_k\,\frac{\partial}{\partial x_k} + Bu^k\,\frac{\partial}{\partial u^k} + L\rho\,\frac{\partial}{\partial \rho}. \tag{18}$$

The operator (18) with the constraint $\alpha = 0$ is referred to as the *generator of scale transformations*. The solutions of system (5), (6) that are invariant with respect to this operator are called *self-similar*, or *automodel solutions*.

Theorem 2.2. *The symmetry extension of system (5), (6) is possible in 12 cases presented in Table 1. The maximal invariance group for this system is the $\dfrac{n(n+3)}{2} + 4$-parameter projective group. This group is admissible for system (5), (6) if and only if $F_\rho = c\rho^{2/n}$.*

Remark 2.1. Observe that the one-dimensional case is special. Namely, the two first equations in system (13) appear only when $n > 1$. As it is demonstrated in [9], for the state equation of the form $p = \dfrac{M}{3}\rho^3$, which describes an ideal polytropic gas, system (5), (6) under $n = 1$ admits an infinite group. Due to this fact the general solution was obtained for system (5), (6) in this case [9].

3. Invariant solutions of system (5), (6) and Rankine – Hugoniot conditions. In this section we find solutions of system (5), (6) in the case $n = 1$ that are compatible with the Rankine – Hugoniot conditions.

In this case each operator that generates a one-parameter group of admissible transformations for system (5), (6) can be presented as

$$\begin{aligned} Z = \left(\alpha + \lambda t + \varkappa t^2\right)\frac{\partial}{\partial t} + (\mu t + \nu + Ax + \varkappa xt)\,\frac{\partial}{\partial x} + \\ + (\varkappa x + \mu + Bu - \varkappa tu)\,\frac{\partial}{\partial u} + (L - \varkappa t)\rho\,\frac{\partial}{\partial \rho}, \end{aligned} \tag{19}$$

where $t = x_0,\ x = x_1,\ \alpha,\ \delta,\ \varkappa,\ \lambda,\ \mu$ and ν are arbitrary constant parameters, $B = \delta - \lambda/2$, $A = B + \lambda$, and L is a function of these parameters.

Following the well known technique [1, 2] we find the solutions of (5), (6) that are invariant with respect to a one-parameter group of transformations with infinitesimal symmetry operator of the form (19) by means of transition to invariant variables which can be expressed via solutions of equation

$$ZJ(t, x, u, \rho) = 0. \tag{20}$$

Table 1. List of inequivalent cases for the state equations and the corresponding operators

$\varphi = F_\rho$	Z_ν	Notes
$\varphi_1 = M\rho^{2/n}$	$Z_1 = \alpha P_0 + \lambda L_1 + \delta L_2 + n\left(\delta - \dfrac{\lambda}{2}\right) L_3 + \varkappa L_0,$ $L_0 = x_0^2\,\dfrac{\partial}{\partial x_0} + x_0 x_k\,\dfrac{\partial}{\partial x_k} +$ $+\left(x_k - x_0 u^k\right)\dfrac{\partial}{\partial u^k} + (-1)^n x_0 \rho\,\dfrac{\partial}{\partial \rho},$ $P_0 = \dfrac{\partial}{\partial x_0},$ $L_1 = x_0\,\dfrac{\partial}{\partial x_0} + \dfrac{1}{2}\,x_k\,\dfrac{\partial}{\partial x_k} - \dfrac{1}{2}\,u_k\,\dfrac{\partial}{\partial u^k},$ $L_2 = x_k\,\dfrac{\partial}{\partial x_k} + u_k\,\dfrac{\partial}{\partial u^k}, \quad L_3 = \rho\dfrac{\partial}{\partial \rho}$	
$\varphi_2 = M\rho^{\varkappa}$	$Z_2 = \alpha P_0 + \lambda L_1 + \delta L_2 + \dfrac{2}{\varkappa}\left(\delta - \dfrac{\lambda}{2}\right) L_3$	$\varkappa \neq 0$
$\varphi_3 = M x_0^\sigma \rho^{\varkappa}$	$Z_3 = \lambda L_1 + \delta L_2 + \left[\dfrac{2}{\varkappa}\left(\delta - \dfrac{\lambda}{2}\right) - \dfrac{\sigma}{\varkappa}\lambda\right] L_3$	$\varkappa \neq 0$
$\varphi_4 = \rho^{2/n} G(\gamma),$ $\gamma = \rho^{2/n} x_0^\sigma$	$Z_4 = \lambda L_1 + \dfrac{\kappa}{n}\lambda L_2 + n\lambda\left(\dfrac{\kappa}{n} - \dfrac{1}{2}\right) L_3$	$\sigma = 1 - \dfrac{2\kappa}{n}$
$\varphi_5 = M x_0^\sigma$	$Z_5 = \lambda L_1 + \dfrac{\sigma + 1}{2}\,\lambda L_2 + \left(\mu - \dfrac{n}{2}\,\lambda\right) L_3$	
$\varphi_6 = x_0^{-1} G(\rho)$	$Z_6 = \lambda L_1$	
$\varphi_7 = \Phi\left(\rho^{2/n} x_0^\sigma\right)$	$Z_7 = \lambda L_1 + \dfrac{\lambda}{2}\,L_2 + n\,\dfrac{\sigma}{2}\,\lambda L_3$	$\sigma = 1 - \dfrac{2\kappa}{n}$
$\varphi_8 = \Phi\left(\rho^{2/n} e^{-\sigma x_0}\right)$	$Z_8 = \alpha P_0 + n\,\dfrac{\sigma}{2}\,\alpha L_3$	
$\varphi_9 = e^{\sigma x_0}\Phi(\rho)$	$Z_9 = \alpha P_0 + \dfrac{\sigma}{2}\,\alpha L_2$	
$\varphi_{10} = x_0^\sigma \Phi(\rho)$	$Z_{10} = \lambda\left[L_1 + \dfrac{\sigma + 1}{2} L_2\right]$	
$\varphi_{11} = \Phi(\rho)$	$Z_{11} = \alpha P_0 + \lambda\left[L_1 + \dfrac{1}{2}\,L_2\right]$	
$\varphi_{12} = \rho^\kappa \Phi(x_0)$	$Z_{12} = \delta\left(L_3 + \dfrac{\kappa}{2}\,L_2\right)$	

In order to list the cases, when invariant solutions are applicable to description of point explosion in a medium with the state equation $p = F(\rho, t)$ it is necessary to analyze the invariance of the manifold, determined by boundary conditions with respect to transformations generated by the operator (19).

The role of “boundary conditions” in the case of point explosion is played by the Rankine – Hugoniot conditions [10]

$$\rho_2(u_2 - D) + \rho_1 D = 0, \qquad \rho_2(u_2 - D)^2 + p_2 = \rho_1 D^2 + p_1 \tag{21}$$

which represent the discontinuity of main characteristics at a material medium of the shock wave. In formula (21) the quantities with the index 2 describe the values of these functions behind the shock wave front, and those with the index 1 before it. The medium is expected to be motionless, $u_1 = 0,$ D is the velocity of the shock wave front and $p_1,$ ρ_1 are constants, $\rho_1 > 0.$

It is obvious that in the one-dimensional case the motion of the shock wave front in the point explosion problem can be determined by a relation $x_{\text{front}} = g(t)$ with a certain function $g.$ Therefore, the manifold M defined by the boundary conditions (21) is determined by the system

$$x - g(t) = 0,$$
$$\rho[u - \dot{g}(t)] + \rho_1 \dot{g}(t) = 0, \tag{22}$$
$$\rho[u - \dot{g}(t)]^2 + p(\rho, t) - \rho_1 \dot{g}^2(t) - p_1 = 0, \tag{23}$$

where $\rho_1,$ p_1 are constants that are equal to initial values of the density and the pressure in the medium, correspondingly, $g(t)$ is unknown function, and $\dot{g}(t) = dg(t)/dt.$

Note that infinitesimal operator of the form (19) with coefficients involving quadratic terms as admissible if and only if $p = \dfrac{M}{3}\rho^3.$ In this case system (5), (6) has a general solution, therefore we can set $\varkappa = 0$ in (19).

Applying the infinitesimal invariance criterion (11) to the manifold M we obtain the system

$$\mu - L\dot{g} = 0, \tag{24}$$
$$\nu + \mu t + Ag(t) - (\alpha + \lambda t)\dot{g}(t) = 0, \tag{25}$$
$$\frac{\rho_1^2}{\rho}\, \dot{g}^2(t)(L + 2B) + \rho L p_\rho + (\alpha + \lambda t) p_t - 2\rho_1(\mu + B\dot{g})\dot{g} = 0. \tag{26}$$

To satisfy the condition (24) in the case when $L \neq 0$ it is necessary that

$$g(t) = St + R,$$

where S and R are certain constants, $S \neq 0.$ Formula (25) implies that $L = -B.$ Analyzing the functions $\varphi = F_\rho$ and the corresponding operators Z_ν (see Table 1) we conclude that the case $L \neq 0$ is possible only for a state equation of the form

$$p = c - \frac{M}{\rho}, \qquad M = (S\rho_1)^2,$$

which corresponds to the function $\varphi = M\rho^\sigma$ with $\sigma = -2.$

Since $L = 0$ for the other cases conditions (24) – (26) can be represented as

$$\mu = L = 0, \tag{27}$$
$$\nu + Ag - (\alpha + \lambda t)\dot{g} = 0, \tag{28}$$
$$2B\frac{\rho_1^2}{\rho}\dot{g}^2 - 2\rho_1 \dot{g}^2 B + (\alpha + \lambda t) p_t = 0. \tag{29}$$

It is necessary to analyze condition (29) now. Note that the operators listed in Table 1 can be partitioned into two groups according to the criterion whether L is a multiple of B. Thereby, the first group consists of $Z_1,\ Z_2,\ Z_4,\ Z_{11}$ and Z_{12}. For the operator Z_4 the restrictions (27) imply that the corresponding function φ_4 does not depend on t and, therefore, it coincides with φ_{11}. For the operator Z_{12} the restriction $L = 0$ makes the operator vanish.

By virtue of (27) the functions $\varphi_1,\ \varphi_2$ and φ_{11} correspond to the same infinitesimal symmetry operator

$$Z_{II} = (\alpha + \lambda t)\,\frac{\partial}{\partial t} + (\nu + \lambda x)\,\frac{\partial}{\partial x}. \tag{30}$$

Therefore we can consider these three cases together. Denote the function that corresponds to operator (30) by $\Phi_{II}(\rho) = \varphi_1(\rho) = \varphi_2(\rho) = \varphi_{11}(\rho)$. It is clear that $p_{II} = \Phi_{II}(\rho) + H(t)$ with a certain function $H(t)$. For operator (30) equation (29) is equivalent to the condition

$$(\alpha + \lambda t)(p_{II})_t = 0,$$

which leads to $H = c = \text{const}$.

Draw our attention to other cases. If $L = 0$ then the functions φ_7 and φ_8 coincide with φ_{11}, and the functions $\varphi_3,\ \varphi_5,\ \varphi_6$ and φ_{10} can be represented as

$$\varphi_{III} = t^{\sigma}\dot{\Phi}(\rho)$$

due to the fact that the infinitesimal symmetry operator for all these cases is the same, namely

$$Z_{III} = \lambda t\,\frac{\partial}{\partial t} + (\nu + Ax)\frac{\partial}{\partial x} + Bu\,\frac{\partial}{\partial u},$$

where $A = \dfrac{\lambda}{2}\,(\sigma + 2),\ B = \dfrac{\lambda\sigma}{2},\ \sigma \neq 0$.

Formula (29) enables one to recover p_{III},

$$p_{III} = t^{\sigma}\Phi(\rho) + H(t).$$

Observe that the derivative of p_{III} with respect to t can be expressed as

$$(p_{III})_t = \frac{\sigma}{t}\,(p_{III} - H) + \frac{dH}{dt}.$$

So, the condition (29) is equivalent to the equation

$$\sigma p_1 - \sigma H + t\,\frac{dH}{dt} = 0.$$

Hence $H(t) = c_1 t^{\sigma} + c_2$ and

$$p_{III} = t^{\sigma}\Phi(\rho) + c, \qquad c = p_1.$$

The last case to be considered is $F_\rho = \varphi_9$. Then

$$p_{IV} = e^{2\kappa t}\Phi(\rho) + H(t), \qquad Z_{IV} = \alpha\left(\frac{\partial}{\partial t} + \kappa x\,\frac{\partial}{\partial x} + \kappa u\,\frac{\partial}{\partial u}\right),$$

and hence $B = \kappa\alpha$. Expressing $(p_{IV})_t$ in terms of $p,\ H$ and $\dfrac{dH}{dt}$ making use of formulae (22) and (23) we find that

$$p_{IV} = e^{2\kappa t}\Phi(\rho) + p_1.$$

Hereby, all the functional relationships $p = F(\rho, t)$ for which the corresponding invariance solutions are compatible with the Rankine – Hugoniot conditions are listed. In what follows we determine the function $g(t)$ for each of these cases and verify that $g = \mathrm{const}$ i.e., that a shock wave really propagates in a medium. Solving equation (28) we obtain

$$g_{II}(t) = \begin{cases} c_2 + \dfrac{\nu}{\alpha} t, & \text{if} \quad \lambda = 0, \\ c_2(\alpha + \lambda t) - \dfrac{\nu}{\alpha}, & \text{if} \quad \lambda \neq 0 \end{cases} \tag{31}$$

in the case $Z = Z_{II},\ p = \Phi(\rho) + p_1$,

$$g_{III}(t) = \begin{cases} \dfrac{\nu}{\lambda} \ln t + c_3, & \text{if} \quad \varkappa = \dfrac{\sigma}{2} + 1 = 0, \\ c_3 t^{\varkappa} - \dfrac{\nu}{\lambda\varkappa}, & \text{if} \quad \varkappa \neq 0 \end{cases} \tag{32}$$

for the case $Z = Z_{III},\ p = t^{\sigma}\Phi(\rho) + p_1$, and

$$g_{IV}(t) = c_4 e^{\kappa t} - \frac{\nu}{\kappa\alpha} \tag{33}$$

for $Z = Z_{IV},\ p = e^{2\kappa t}\Phi(\rho) + p_1$. In expressions (31) – (33) $c_2,\ c_3$ and c_4 are arbitrary constants.

Therefore, if we restrict the consideration to the symmetry operators that do not contain any quadratic terms in their coefficients, the following theorem holds.

Theorem 3.1. *The four classes of invariant solutions to system (5), (6) compatible with the Rankine – Hugoniot conditions under $n = 1$ are:*

(a) solutions that are invariant with respect to the one-parameter subgroup generated by the operator

$$Z_I = (\alpha + \lambda t)\frac{\partial}{\partial t} + (\nu - BSt + Ax)\frac{\partial}{\partial x} + Bu\frac{\partial}{\partial u} + L\rho\frac{\partial}{\partial \rho},$$

where $L = -B,\ \alpha,\ \lambda,\ \nu,\ A,\ B$ and S are constants, $B \neq 0$ and $S \neq 0$, if

$$p = c - \frac{(S\rho_1)^2}{\rho};$$

(b) solutions invariant with respect to the one-parameter subgroup generated by the operator

$$Z_{II} = (\alpha + \lambda t)\frac{\partial}{\partial t} + (\nu + \lambda x)\frac{\partial}{\partial x},$$

if

$$p = \Phi(\rho) + p_1;$$

(c) solutions invariant with respect to the one-parameter subgroup generated by the operator

$$Z_{III} = \lambda t\,\frac{\partial}{\partial t} + (\nu + Ax)\,\frac{\partial}{\partial x} + Bu\,\frac{\partial}{\partial u},$$

where $A = \lambda\left(\frac{\sigma}{2} + 1\right),\ B = \frac{\lambda\sigma}{2}$, *if*

$$p = t^{\sigma}\Phi(\rho);$$

(d) solutions invariant with respect to the one-parameter subgroup generated by the operator

$$Z_{IV} = \alpha\,\frac{\partial}{\partial t} + (\nu + \kappa\alpha x)\,\frac{\partial}{\partial x} + \kappa\alpha u\,\frac{\partial}{\partial u},$$

if

$$p = e^{2\kappa t}\Phi(\rho) + p_1.$$

Hereby, the cases when the boundary-value problem (5), (6), (21) admits invariant solutions are exhaustively described.

4. Partial solutions. Invariant variables depend on the parameters $\alpha,\ \lambda,\ \nu$ and σ involved in the coordinates of an infinitesimal symmetry operator, so the solution of the system depends on relations existing between these parameters. For the case $Z = Z_1$ there are three possibilities:

(a) if $A \neq 0,\ B \neq 0$ and $\lambda \neq 0$ then solving equation (20) we have

$$\begin{gathered} \omega = \omega_1 = (x - St - R)\tau^{-A/\lambda}, \qquad \tau = \alpha + \lambda t, \\ u = S + U(\omega)\tau^{B/\lambda}, \qquad \rho = R(\omega)\tau^{-B/\lambda}. \end{gathered} \tag{34}$$

Substituting expressions (34) into (5), (6) we obtain the equations for the point explosion problem in the invariant variables,

$$\begin{gathered} \left(U - \frac{A}{\lambda}\,\omega\right)\frac{dU}{d\omega} + \frac{M}{R^3}\,\frac{dR}{d\omega} + \frac{B}{\lambda}U = 0, \\ R\,\frac{dU}{d\omega} + \left(U - \frac{A}{\lambda}\,\omega\right)\frac{dR}{d\omega} - \frac{B}{\lambda}\,R = 0 \end{gathered} \tag{35}$$

with the boundary conditions of the form

$$\begin{gathered} R\left(\omega^*\right)U\left(\omega^*\right) + \rho_1 S = 0, \qquad c - p_1 - \rho_1 S^2 = 0, \\ R\left(\omega^*\right)\left[U\left(\omega^*\right)\right]^2 - M\left[R\left(\omega^*\right)\right]^{-1} = 0 \end{gathered} \tag{36}$$

at the shock wave front, where $\omega^* = 0$;

(b) if $\lambda = 0$, then

$$\omega = \omega_2 = e^{-\frac{\delta}{\alpha}t}(x - St - R), \qquad u = S + U(\omega)e^{\frac{\delta}{\alpha}t}, \qquad \rho = R(\omega)e^{-\frac{\delta}{\alpha}t}.$$

In the new variables system (5), (6) rewrites as

$$\begin{gathered} \left(U - \frac{\delta}{\alpha}\omega\right)\frac{dU}{d\omega} + \frac{M}{R^3}\,\frac{dR}{d\omega} + \frac{\delta}{\alpha}\,U = 0, \\ R\frac{dU}{d\omega} + \left(U - \frac{\delta}{\alpha}\omega\right)\frac{dR}{d\omega} - \frac{\delta}{\alpha}\,R = 0; \end{gathered} \tag{37}$$

(c) if $A = 0,\ \lambda = -2\delta$ then

$$\omega = \omega_3 = St - x, \qquad u = S + U(\omega)\tau^{-1}, \qquad \rho = R(\omega)\tau, \qquad \tau = \alpha + \lambda t,$$

and in the invariant variables system (5), (6) takes the form

$$\begin{aligned} &U\frac{dU}{d\omega} + \frac{M}{R^3}\frac{dR}{d\omega} + \lambda U = 0, \\ &R\frac{dU}{d\omega} + U\frac{dR}{d\omega} - \lambda R = 0. \end{aligned} \tag{38}$$

The boundary conditions for the cases (b) and (c) are expressed by formula (36) with $\omega^* = \omega_2^* = 0$ and $\omega^* = \omega_3^* = -R$.

Consider the case $Z = Z_{II}$. Two possibilities appear:

(a) if $\lambda = 0$ then

$$\omega = \omega_4 = \nu t - \alpha x, \qquad u = U(\omega_4), \qquad \rho = R(\omega_4).$$

Rewrite system (5), (6) in the new variables:

$$\begin{aligned} &\left(U - \frac{\nu}{\alpha}\right)\frac{dU}{d\omega} + \frac{1}{R}\frac{d\Phi(R)}{dR}\frac{dR}{d\omega} = 0, \\ &R\frac{dU}{d\omega} + \left(U - \frac{\nu}{\alpha}\omega\right)\frac{dR}{d\omega} = 0. \end{aligned}$$

The boundary conditions (20) after the substitution take the form

$$\begin{aligned} &\left[R\left(U - \frac{\nu}{\alpha}\right) + \rho_1\frac{\nu}{\alpha}\right]\Big|_{\omega=\omega_4^*} = 0, \\ &\left[R\left(U - \frac{\nu}{\alpha}\right)^2 + \Phi(R)\right]\Bigg|_{\omega=\omega_4^*} = c_1\left(\frac{\nu}{\alpha}\right)^2, \end{aligned}$$

where $\omega_4^* = -c_1\alpha$;

(b) if $\lambda \neq 0$ then

$$\omega = \omega_5 = \frac{x + \nu/\lambda}{t + \alpha/\lambda}, \qquad u = U(\omega), \qquad \rho = R(\omega).$$

In the new variables system (5), (6), (21) takes the form

$$\begin{aligned} &(U - \omega)\frac{dU}{d\omega} + \frac{1}{R}\frac{d\Phi(R)}{dR}\frac{dR}{d\omega} = 0, \\ &R\frac{dU}{d\omega} + (U - \omega)\frac{dR}{d\omega} = 0, \\ &[R(U - c_1\lambda) + \lambda c_1\rho_1]|_{\omega=\omega_5^*} = 0, \\ &\left[R(U - c_1\lambda)^2 + \Phi(R) - \rho_1(\lambda c_1)^2\right]\Big|_{\omega=\omega_5^*} = 0, \end{aligned}$$

where $\omega_5^* = c_1\lambda$.

Consider now the case $Z = Z_{III}$.

(a) let $\varkappa = \dfrac{\sigma}{2} + 1 = 0$. Then

$$\omega = \omega_6 = x - \frac{\nu}{\lambda} \ln t, \qquad u = U(\omega) t^{-1}, \qquad \rho = R(\omega)$$

and system (5), (6), (21) in the invariant variables takes the form

$$\begin{gathered}
\left(U - \frac{\nu}{\lambda}\right) \frac{dU}{d\omega} + \frac{1}{R} \frac{d\Phi(R)}{dR} \frac{dR}{d\omega} = U, \\
R \frac{dU}{d\omega} + \left(U - \frac{\nu}{\lambda}\right) \frac{dR}{d\omega} = 0, \\
\left[R \left(U - \frac{\nu}{\lambda}\right) + \rho_1 \frac{\nu}{\lambda}\right]\Big|_{\omega=\omega_6^*} = 0, \\
\left[R \left(U - \frac{\nu}{\lambda}\right)^2 + \Phi(R) - \rho_1 \left(\frac{\nu}{\lambda}\right)^2\right]\Big|_{\omega=\omega_6^*} = 0,
\end{gathered}$$

where $\omega_6^* = c_3$;

(b) if $\varkappa \neq 0$ then

$$\omega = \omega_7 = \frac{x + \dfrac{\nu}{\lambda \varkappa}}{t^{\varkappa}}, \qquad u = t^{\varkappa - 1} U(\omega), \qquad \rho = R(\omega), \tag{39}$$

after the substitution of (39) equations (5), (6) take the form

$$\begin{gathered}
(U - \varkappa \omega_7) \frac{dU}{d\omega} + \frac{1}{R} \frac{d\Phi(R)}{dR} \frac{dR}{d\omega} = (1 - \varkappa) U, \\
R \frac{dU}{d\omega} + (U - \varkappa \omega) \frac{dR}{d\omega} = 0,
\end{gathered}$$

and boundary conditions (21) become

$$\begin{gathered}
[R(U - \varkappa c_3) + \rho_1 \varkappa c_3]|_{\omega=\omega_7^*} = 0, \\
\left[R(U - \varkappa c_3)^2 + \Phi(R) - \rho_1 (\varkappa c_3)^2\right]\big|_{\omega=\omega_7^*} = 0,
\end{gathered}$$

where $\omega_7^* = c_3$.

Lastly, for the case $Z = Z_{IV}$ we have

$$\omega = \omega_8 = \left(x + \frac{\nu}{\alpha \kappa}\right) e^{-\kappa t}, \qquad u = U(\omega) e^{\kappa t}, \qquad \rho = R(\omega),$$

system (5), (6) in the invariant variables takes the form

$$\begin{gathered}
(U - \kappa \omega) \frac{dU}{d\omega} + \frac{1}{R} \frac{d\Phi(R)}{dR} \frac{dR}{d\omega} + kU = 0, \\
R \frac{dU}{d\omega} + (U - \kappa \omega) \frac{dR}{d\omega} = 0,
\end{gathered}$$

and the conditions at the shock wave front have the form

$$[R(U-\kappa c_4)+\rho_1\kappa c_4]\big|_{\omega=\omega_8^*}=0,$$
$$\big[R(U-\kappa c_4)^2+\Phi(R)-\rho_1(\kappa c_4)^2\big]\big|_{\omega=\omega_8^*}=0,$$

where $\omega_8^*=c_4$.

It is clear that all the systems of ordinary differential equations presented in this section can be presented in the form

$$\begin{aligned}&(U-a-b\omega)\,\frac{dU}{d\omega}+\Psi(R)\,\frac{dR}{d\omega}=hU,\\&R\,\frac{dU}{d\omega}+(U-a-b\omega)\,\frac{dR}{d\omega}=lR,\end{aligned}\tag{40}$$

where $a,\ b,\ h$ and l are some constants and $\Psi(R)=R^{-1}\,\dfrac{d\Phi}{dR}$.

Suppose that in formula (40) $b=0,\ h=0$ and $l=0$. The solution of the system will be nonconstant only if

$$U=a\pm\sqrt{R\Psi}.$$

Substituting this expression into formula (40) we have

$$\left[3\Psi(R)+R\,\frac{d\Psi}{dR}\right]\frac{dR}{d\omega}=0.$$

Thus, system (40) in this case has nonconstant solutions if and only if $\Psi=cR^{-3}$. These solutions are

$$R=f(\omega),\qquad U=a+c_5f^{-1},$$

where f is an arbitrary function, c_5 is a constant.

Let $b=0,\ l=0$ and $b=1$ in formula (40). It is easy to see that under these conditions the system has the first integral $U=a+c_6R^{-1}$. Substituting U into the first equation of system (40) after simple algebraic transformations we have

$$\int\frac{\left[R^3\Psi(R)-c_6^2\right]R}{aR+c_6}\,dR=\omega-\omega_0.$$

Consider system (40) under the constraints $a=0,\ h=0$ and $l=0$. The transformation $z=b\omega$ converts it to

$$\begin{aligned}&(\tilde U-z)\,\frac{d\tilde U}{dz}+\tilde\Psi(\tilde R)\,\frac{d\tilde R}{dz}=0,\\&\tilde R\,\frac{d\tilde U}{dz}+(\tilde U-z)\,\frac{d\tilde R}{dz}=0,\end{aligned}\tag{41}$$

where $\tilde U,\ \tilde R,\ \tilde\Psi$ are the old functions expressed in terms of the variable z. For the function $\tilde R$ to be nonconstant it is necessary that $\tilde U=\varepsilon[\tilde R\tilde\Psi(\tilde R)]^{1/2}+z$, where $\varepsilon=\pm1$. Substituting $\tilde U$ into equation (41) we have

$$\int\frac{3\tilde\Psi(\tilde R)+\tilde R d\tilde\Psi/d\tilde R}{2\varepsilon\sqrt{\tilde R\tilde\Psi}}d\tilde R=\omega-\omega_0.$$

In the case when $a = 0,\ l = 0$ and $h = b$ system (40) has the first integral

$$\frac{U}{2}\,(U - 2b\omega) + \chi(R) = c_7, \qquad \Psi(R) = \frac{d\chi(R)}{dR}.$$

Setting $\chi(R) = MR^{\kappa+1}$ we have

$$R = T[2c_7 - U(U - 2b\omega)]^{\frac{1}{\kappa+1}}, \qquad \text{where} \quad T = (2M)^{-\frac{1}{\kappa+1}}.$$

Substituting the value of R into the second equation of system (40) we have

$$\left[2c_7 - U(U - 2b\omega)^{\frac{1}{\kappa+1}}\right] \frac{dU}{d\omega} + \\ + \frac{b\omega - U}{\kappa+1}\left[2c_7 - U(U - 2b\omega)^{-\frac{\kappa}{\kappa+1}}\right]\left[\frac{dU}{d\omega}(U - 2b\omega) + U\left(\frac{dU}{d\omega} - 2b\right)\right] = 0.$$

If $c_7 = 0$ then the change $U = V_\omega$ brings this expression to

$$\left(V + \omega\frac{dV}{d\omega}\right) L - \frac{2b}{\kappa+1} V(V - b) = 0, \tag{42}$$

where $L = \dfrac{2(V-b)^2}{\kappa+1} + V(V - 2b)$. Equation (42) can be integrated in quadratures,

$$\int \frac{L}{\dfrac{2b}{\kappa+1}V(V-b) - VL}\, dV = \ln\left(\frac{\omega}{\omega_0}\right).$$

The ways of solving system (40) under $p = t^\sigma(A\rho^{\kappa+1} + B)$ and arbitrary values of b and h are analyzed in [11]. Particularly, it is demonstrated therein that this system can be always transformed to a single ordinary differential equation.

If we set $h = -l$ and $a = 0$ in system (40), see formulae (35), (37) and (38), then it is possible to specify three cases when the system has the first integral:

(a) if $a = 0$ and $b = l = -h$, then

$$R(U - b\omega) = c_8;$$

(b) if $a = 0$ and $b = h = -l$, then

$$\frac{U^2}{2} - b\omega U + \chi(R) = c_9, \qquad \frac{d}{dR}\chi(R) = \Psi(R);$$

(c) if $a = 0,\ b = 0$ and $h = -l$, then

$$RU^2 + \chi(R) = c_{10}, \qquad \frac{d}{dR}\chi(R) = \Psi(R).$$

In each of these cases the original problem can be reduced to solving of a single differential equation.

5. Conclusion. In this paper group analysis of a system of Euler equations with a state equation of the medium is carried out. The group classification provided for the state equations is of great practical importance, because there is no unified analytical expression that satisfactorily describes the relationship of thermodynamic parameters of liquid throughout the domain where these parameters vary. In many cases the state equations listed in Table 1 coincide with functional relationships known as state equations for liquid in limited ranges of values of thermodynamic parameters.

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