УДК 512.543

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ESSENTIALLY FREE ACTIONS OF SELF-SIMILAR GROUPS

Наведено результати класифікації груп породжених 3-становими автоматами над 2-літерним алфавітом, що діють суттєво вільно на границі бінарного дерева.

We report on the complete classification of groups generated by 3-state automata over 2-letter alphabet that act essentially freely on the boundary of the binary rooted tree.

Dedicated to the sixtieth anniversary of Professor Yaroslav Bigun and to the fiftieth anniversary of the Department of Applied Mathematics of Yuriy Fedkovich Chernivtsi National University

Introduction Groups generated by Mealy type automata represent an important and interesting class of groups with connections to different branches of mathematics, such as dynamical systems (including symbolic dynamics and holomorphic dynamics), computer science, topology and probability. For more details about this class of groups we refer the reader to survey papers [12, 3].

In the whole class of groups generated by automata, there is an important subclass of self-similar groups. These are the groups generated by initial automata that are determined by all states of a non initial automaton. The natural characteristic of such groups, which we will call *complexity*, is the pair (m, n) of two integers, $m \geq 2, n \geq 2$, where m is a number of states and n is a cardinality of the alphabet. There are 6 groups of complexity $(2,2)$ and the most complicated of them is the lamplighter group $\mathcal{L} = (\mathbb{Z}/2\mathbb{Z}) \wr \mathbb{Z}$ [12]. It is shown in [4] and [17] that there is not more than 115 different (up to isomorphism) groups of complexity $(3,2)$, although the number of corresponding automata up to symmetry is 194. Even though the complete characterization of $(3,2)$-groups is not achieved yet, a lot of information about these groups has been obtained. The motivation for this note is twofold: partially it comes from the necessity to understand this class of groups better, and additionally, it represents the venture in the search of new interesting examples of self-similar groups that might potentially serve as answers to questions posed at the end of the paper.

Groups generated by finite automata defined over the m-letter alphabet, in particular self-similar groups, naturally act on the m-regular rooted tree $T = T_m$ (m a cardinality of alphabet) and on its boundary, which topologically is homeomorphic to the Cantor set. This action preserves the uniform Bernoulli measure μ on the boundary. Therefore one can study a topological dynamical system $(G, \partial T)$ or metric dynamical system $(G, \partial T, \mu)$. Ergodicity of the latter is equivalent to the level transitivity of the action of G on T.

The important class of actions are topologically free actions and essentially free actions. These types of actions play especially important role in various studies in dynamical systems, operator algebras, and modern directions of group theory like theory of cost or rank gradient [7, 2]. Self-similar groups acting essentially freely on ∂T can potentially be used to construct new examples of scale-invariant groups [18], and have connection to the class of hereditary just-infinite groups [9].

The opposite to the notion of a free action are totally nonfree actions considered recently in [21, 20, 10]. These are the actions, for which stabilizers of different points are different. Surprisingly many groups generated by finite automata, in particular those of them that are branch or weakly branch) act totally nonfree.

Totally non free actions are also important for the theory of operator algebras and for rapidly developing now theory of invariant random subgroups [21, 1, 5, 6].

The goal of this note is to report on the progress of the project of description of all $(3,2)$-groups acting essentially freely on the boundary of the tree. Or main result is:

Theorem 1. *Among all groups generated by* 3*-state automata over 2-letter alphabet the only groups that act essentially freely on the boundary of the tree T_2 are:*

- *Trivial group;*
- *Group $\mathbb{Z}/2\mathbb{Z}$ of order 2;*
- *Klein group $(\mathbb{Z}/2\mathbb{Z})\times(\mathbb{Z}/2\mathbb{Z})$;*
- $(\mathbb{Z}/2\mathbb{Z})\times(\mathbb{Z}/2\mathbb{Z})\times(\mathbb{Z}/2\mathbb{Z})$*;*
- *Free abelian groups $\mathbb{Z}$ and $\mathbb{Z}^2$;*
- *Infinite dihedral group D_∞;*
- *Baumslag-Solitar groups $BS(1,3)$ and $BS(1,-3)$;*
- *Extension $\big((\mathbb{Z}/2\mathbb{Z})\wr Z\big)\rtimes(\mathbb{Z}/2\mathbb{Z})$ of the lamplighter group by $\mathbb{Z}/2\mathbb{Z}$;*
- *Free group F_3 of rank 3;*
- *Free product $(\mathbb{Z}/2\mathbb{Z})*(\mathbb{Z}/2\mathbb{Z})*(\mathbb{Z}/2\mathbb{Z})$ of three groups of order 2;*
- *Lamplighter group $(\mathbb{Z}/2\mathbb{Z})\wr\mathbb{Z}$;*
- *Extension $\mathbb{Z}^2\rtimes(\mathbb{Z}/2\mathbb{Z})$ of the $\mathbb{Z}^2$ group by $\mathbb{Z}/2\mathbb{Z}$;*
- *Metabelian group $((\frac{1}{2}\mathbb{Z}\left[\frac{1}{3}\right])\rtimes(\mathbb{Z}/2\mathbb{Z}))\rtimes\mathbb{Z}$;*
- *Extension $\big((\mathbb{Z}/2\mathbb{Z})^2\wr\mathbb{Z}\big)\rtimes(\mathbb{Z}/2\mathbb{Z})$ of a rank 2 lamplighter group $(\mathbb{Z}/2\mathbb{Z})^2\wr\mathbb{Z}$ by $\mathbb{Z}/2\mathbb{Z}$.*

The paper is organized as follows. In Section 1 we recall main notions from a theory of groups generated by automata, and discusses various types of free actions. The strategy of proof of the main Theorem is surveyed in Section 2. Finally, we conclude the paper with open questions in Section 3.

Notation and Preliminaries Let X be a finite set of cardinality d and let X^* denote the free monoid generated by X, which consists of finite words over X. This monoid can be naturally endowed with a structure of a rooted d-ary tree T by declaring that v is adjacent to vx for any $v\in X^*$ and $x\in X$. The empty word corresponds to the root of the tree and X^n corresponds to the n-th level of the tree. We will be interested in the groups of automorphisms and semigroups of homomorphisms of X^*. Any such homomorphism can be defined via the notion of initial automaton (see, for example, [12]).

Now we describe shortly the notations used in the classification of $(3,2)$-groups [4]. Every 3-state automaton $\mathcal{A}$ with set of states $S=\{\mathbf{0},\mathbf{1},\mathbf{2}\}$ acting on the 2-letter alphabet $X=\{0,1\}$ is assigned a unique number from 1 to 5832 according to a certain lexicographic order on the set of all automata in this class. Denote by $\mathcal{A}_n$ the automaton numbered by n and by G_n the group generated by $\mathcal{A}_n$.

On the set of all (3,2)-automata one can naturally define an equivalence relation of *minimal symmetry*. Namely, two automata $\mathcal{A}$ and $\mathcal{B}$ are *minimally symmetric* if their minimizations coincide up to symmetry and taking the inverse. At present ([4, 17]), it is known that there are no more than 115 non-isomorphic $(3,2)$-automaton groups out of 194 classes of $(3,2)$-automata that are pairwise not minimally symmetric.

In this note, since we are looking for essentially free actions of groups, we will actually distinguish non minimally symmetric automata generating isomorphic groups. So we will work with 194 classes of not minimally symmetric automata.

There are different ways to define the freeness of a group action. The definition below works in the general context of arbitrary topological (or, respectively, measure) space, but we will work only in the context of actions of self-similar groups on the boundary ∂T of a rooted tree T.

Let G be a countable group acting on a complete metric space Y. Denote by Y_- the set of points with nontrivial stabilizer and by Y_+ the set of points with trivial stabilizer.

Definition 1.

1. *The action* (G,Y) *is* topologically free *if* Y_- *is a meager set (i.e., it can be represented as a countable union of nowhere dense sets).*
2. *Suppose the action* (G,Y) *has a* G*-invariant (not necessarily finite) Borel measure* μ. *The system* (G,Y,μ) *is said to be* essentially free *if* $\mu(Y_-)=0$.

First of all, we note that in our case of groups generated by finite state automata that act spherically transitively on the tree the notions of topological freeness and essential freeness are identical [13, 10].

In order to establish that a group does not act topologically (and essentially) freely on ∂T, one can just find an element $g\in G$ and a vertex $v\in X^*$ fixed by g such that $g|_v$ is identity (because in this case all points in the cylindrical set c_v (consisting of vertices in $T=X^*$ that have v as a prefix), which is open (and has positive measure), will have g in their stabilizers.

Definition 2. *For a vertex* $v\in X^*$ *the set of all* $g\in G$ *that fix* v *and such that* $g|_v$ *is identity forms a subgroup* $\mathrm{triv}_G(v)$ *of* G *called the* trivializer *of* v.

Definition 3. *The action of a group* G *on a rooted tree is called* locally nontrivial *if trivializers of all vertices of the tree are trivial.*

As observed above, if the action is not locally trivial, it cannot be topologically or essentially free. It is not hard to prove the converse in the case of countable group and topological freeness.

Proposition 1 ([11], Proposition 4.2.)**.** *The action of a countable group on the boundary of a tree is topologically free if and only if it is locally nontrivial.*

This observation constitutes one of the main tools to determine that a self-similar group *does not* act essentially freely on the boundary of a tree. Of course, one can simply apply a brute force to find such an element, but in case of self-replicating groups (see, for example, [4]) it can be made almost automatic in many cases by using the the following procedure.

Suppose $G=\langle a_1,a_2,\dots,a_n\rangle$ is a group generated by automaton with states $a_1,a_2,\dots,a_n$. First, we calculate the finite generating set $\{s_j,\ j\in J\}$ of the stabilizer of the first level of the tree $\mathrm{Stab}_G(1)$ in G. This is a subgroup of finite index and a Reidemeister-Schreier procedure can be used for that [14].

Let F_A denote the free group generated by elements $a_1,a_2,...,a_n$. The wreath recursion that defines an automaton induces an embedding

$$F_A\hookrightarrow F_A\wr \mathrm{Sym}(X)$$

defined by

$$F_A\ni g\mapsto (g|_0,g|_1,\dots,g|_{d-1})\lambda(g)\in F_A\wr\mathrm{Sym}(X), \qquad (1)$$

where $g|_i$ denotes the section of g at vertex $i\in X^*$ (see [12]).

With a slight abuse of notation, we will denote by s_j also a word over $A\cup A^{-1}$ in F_A that is mapped to $s_j\in G$ under the canonical epimorphism $F_A\to G$. Then we decompose each $s_j\in F_A$ as a pair $(s_j|_0,s_j|_1)\in F_A\times F_A$ using the wreath recursion embedding (1). The first components $s_j|_0$ of above pairs generate a subgroup H of F_A. After applying the Nielsen reduction to the generators of this subgroup, keeping track of second coordinates, we obtain the generating set of $\langle (s_j|_0,s_j|_1),\ j\in J\rangle < F_A\times F_A$ whose projection onto the first coordinate is Nielsen reduced [14]:

$$\begin{array}{l} t_1=(b_1,w_1),\dots,t_l=(b_m,w_m),\\ t_{m+1}=(1,r_1),\dots,t_{m+l}=(1,r_l), \end{array} \qquad (2)$$

where $\{b_1,\dots,b_m\}$ is a Nielsen reduced generating set for H, $w_i\in F_A$ and $m+l=|J|$. We will call such a representation for $\mathrm{Stab}_G(1)$ the *Mikhailova system* for G. The reason for such name is explained below.

If any of r_i, $i=1,\dots,l$ represents a non-identity element of G, then the corresponding pair $(1,r_i)$ will represent a non-identity element of G that belongs to the trivializer of vertex 1. Thus, the action of G on ∂T_2 would not be essentially free.

Showing that the group actually does act essentially freely is usually much harder. The main tool here is the Proposition 2 below. This proposition is similar to Proposition 1, but it additionally uses self-similarity of a group. We first introduce a notion of a rigid stabilizer.

Definition 4. *Let G be a group acting on the rooted tree X^*.*

- *The* rigid stabilizer of a vertex $v \in X^*$ *in G is a subgroup* $\mathrm{Rist}_G(v)$ *of G that consists of elements that act nontrivially only on the vertices that have v as a prefix.*

- *The* rigid stabilizer of a level n *of X^* in G is a subgroup* $\mathrm{Rist}_G(n)$ *of G that is generated by rigid stabilizers of all the vertices of this level.*

Proposition 2 ([11], Proposition 4.5.)**.** *For a group G generated by finite automaton, acting on a binary tree T_2, the action on ∂T_2 is essentially free if and only if the rigid stabilizer of the first level* $\mathrm{Rist}_G(1)$ *is trivial.*

The problem is that it is harder to show that the rigid stabilizer is trivial, than to find an element witnessing its non-triviality. The main method here is based on finding the presentation of a group. We now go back to Equation (2). In the case when H coincides with F_A we get $m = n$ and this equation is transformed to (after reordering the generators, if necessary):

$$\begin{array}{l} t_1 = (a_1, w_1), \ldots, t_l = (a_n, w_n), \\ t_{n+1} = (1, r_1), \ldots, t_{n+l} = (1, r_l), \end{array} \qquad (3)$$

We can further assume that all r_i's represent identity element in G (otherwise, as stated above, the action of G is not essentially free). Suppose additionally that

$$\langle w_1, w_2, \ldots, w_n \rangle = F_A.$$

Then the map $\phi\colon a_i \to w_i$ extends to an automorphism of F_A. In this case we say that the definition of the group G by a finite automaton belongs to the *diagonal type*. This condition does not depend on how the pairs of elements are reduced by the Nielsen transformations. Note, that the case when ϕ is the identity automorphism one obtains a subgroup of $F_A \times F_A$ that was used by Mikhailova in [15] to to prove that the inclusion problem for direct products of free groups is algorithmically unsolvable.

The following proposition is formulated in [11] and follows immediately from Proposition 2.

Proposition 3 ([11], Proposition 5.1)**.** *Suppose that G is a group generated by finite automaton acting on a binary tree such that its first-level stabilizer can be reduced by the Nielsen transformations to the diagonal type. Let ϕ be the above-constructed automorphism of the free group F_A. Then the action is essentially free if and only if ϕ induces an automorphism of the group G.*

Another useful proposition that allows us to establish essential freeness of the action in the case of groups generated by finite bireversible automata, i.e. invertible automata, whose dual, and dual to the inverse are invertible as well.

Proposition 4 ([19], Corollary 2.10)**.** *A group generated by a bireversible automaton acts topologically and essentially freely on the boundary of the tree.*

Strategy for classification and most interesting new examples.

Our systematic search heavily uses results of [4], in conjunction with computations performed using `AutomGrp` package [16] for `GAP` system [8]. In the first step we compute Mikhailova systems for all automata and filter out those automata, for which Mikhailova system produces a non-identity element in the rigid stabilizer. For the remaining automata we apply a brute force in an attempt to find such elements up to length 5 using the function `FindGroupElement` of `AutomGrp` package. This reduction leaves 57 automata that might generate groups acting essentially freely.

Finally, we investigate these cases separately. Most of the remaining automata generate groups that were either described in [4], or can be reduced to such groups in one or another way. However, two automata, namely A_{2193} (wreath recursion $a = (c,b)\sigma, b = (a,a)\sigma, c = (a,a)$, where σ denotes a nontrivial permutation of the letters in $X = \{0,1\}$) and A_{2372} (wreath recursion $a = (b,b)\sigma, b = (c,a)\sigma, c = (c,a)$) generated groups that have not been studied extensively before. We completely describe the structure and the presentations of these groups in the following two theorems and using the presentations we prove that they act essentially freely on the boundary of the tree.

Theorem 2. *The group $G_{2193} = \langle a,b,c\rangle = \langle a^2, b^{-1}c, b^{-1}a, ac^{-1}a\rangle$ is solvable of derived length 3 and has the following structure:*

$$G \cong \mathcal{L}_2 \rtimes (\mathbb{Z}/2\mathbb{Z}) = \big((\mathbb{Z}/2\mathbb{Z})^2 \wr \mathbb{Z}\big) \rtimes (\mathbb{Z}/2\mathbb{Z}),$$

where the isomorphism is induced by sending the first two generators a^2, $b^{-1}c$ of G to generators of the base group $(\mathbb{Z}/2\mathbb{Z})^2$ in $\mathcal{L}_2$, the generator $b^{-1}a$ to the generator of $\mathbb{Z}$ in $\mathcal{L}_2$, and the generator $ac^{-1}a$ of G to the generator of $\mathbb{Z}/2\mathbb{Z}$ in $\mathcal{L}_2 \rtimes (\mathbb{Z}/2\mathbb{Z})$ acing on $\mathcal{L}_2$.

Moreover, G_{2193} has the following presentation:

$$\begin{aligned} G \cong \langle a,b,c \mid\ & a^4 = (b^{-1}c)^2 = 1,\\ & \left[a^2, (a^2)^{(b^{-1}a)^i}\right] = \left[a^2, (b^{-1}c)^{(b^{-1}a)^i}\right]\\ & = \left[b^{-1}c, (b^{-1}c)^{(b^{-1}a)^i}\right] = 1,\ i\in\mathbb{Z},\\ & (ba^2)^2 = (ca^2)^2 = 1\rangle \end{aligned} \qquad (4)$$

Theorem 3. *The group G_{2372} has the following structure:*

$$\begin{aligned} G = L \rtimes \langle a\rangle &= (K \rtimes \langle v\rangle) \rtimes \langle a\rangle \\ &\cong ((\frac{1}{2}\mathbb{Z}[\frac{1}{3}]) \rtimes (\mathbb{Z}/2\mathbb{Z})) \rtimes \mathbb{Z} \end{aligned}$$

where the action of a on $\frac{1}{2}\mathbb{Z}[\frac{1}{3}]$ corresponds to the multiplication by 3, and the action on v is defined by $v^a = vx_0, v^{a^{-1}} = vx_1^{-1}$.

Moreover, G_{2372} has the following finite presentations

$$\begin{aligned} G_{2372} \cong \langle a,b,c \mid\ & (ac^{-1})^a = (ac^{-1})^3,\\ & (ab^{-1})^2 = 1,\ (ac^{-1})^{ab^{-1}} = ca^{-1},\\ & b^{-1}a = ab^{-1}(ac^{-1})^2\rangle \end{aligned} \qquad (5)$$

Finally, to prove that G_{2193} and G_{2372} act essentially freely on ∂T we use the presentations constructed in the above theorems and Proposition 3.

Open questions

Question 1. *Is there a self-similar group that acts neither essentially freely, nor totally nonfreely on the boundary of a rooted tree?*

Question 2. *Does total non-freeness of action of a self-similar group on ∂T imply weak branchness?*

Question 3. *Classify all $(4,2)$-groups and $(2,3)$-groups that act essentially freely on the boundaries of corresponding rooted trees.*

Question 4. *Are there self-similar groups acting essentially freely on the boundary of rooted tree that are essentially new examples of scale-invariant groups?*

Question 5. *Is there an example of a nonamenable self-replicating group acting essentially freely on the boundary of the tree?*

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