УДК 519.21

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PARABOLIC EQUATION OF MATHEMATICAL PHYSICS WITH RANDOM INITIAL CONDITIONS FROM ORLICZ SPACE

A new method is proposed in this paper to construct solutions of boundary-value problems for parabolic equation with random initial conditions. We assume that the initial conditions are stochastic processes belonging to the Orlicz space of random variables (in particular case, processes with zero mean).

В даній роботі запропоновано новий метод для побудови розв'язків крайових задач для параболічних рівнянь з випадковими початковими умовами. Будемо вважати, що початкові умови є випадковими процесами з просторів Орліча ($E\xi = 0$).

In this paper construct solutions of boundary-value problems for parabolic equation with random initial conditions. We assume that the initial conditions are stochastic processes belonging to the Orlicz space of random variables (in particular, $E\xi = 0$). Physical interpretation of which is: find the law of temperature change in a homogeneous rod length l, with insulated lateral surface, when initial temperature of the rod points $\xi(x)$ — is a random process belonging to the Orlicz space $L_U(\Omega)$, left end of the rod is insulated, and the right is heat exchange with the environment of zero temperature under the law Newton. Conditions for justification of the Fourier method for parabolic equations with random initial conditions from Orlicz spaces of random variables are obtained. Bounds for the distribution of the supremum of solutions of such equations are found. Similar problems for hyperbolic equations are considered in [6], for parabolic equations when boundary conditions the simple are considered in [5]. A survey of the corresponding results can be found in [1–4].

1. Stochastic processes belonging to an Orlicz space.

Definition 1. *([2]) A continuous even convex function $U(x), x \in \boldsymbol{R}$ is called a C-function if $U(x)$ is monotonically increasing function for $x > 0$ and $U(0) = 0$.*

Let $\{\Omega, \Im, P\}$ be a standard probability space.

Definition 2. *([2]) We say that C-function $U = \{U(x), x \in \boldsymbol{R}\}$ satisfies g-condition if there exist constants $z_0 \geq 0$, $K > 0$ and $A > 0$ such that the inequality*

$$U(x)U(y) \leq AU(Kxy).$$

holds for all $x \geq z_0$ and all $y \geq z_0$.

Definition 3. *([4]) The space $L_U(\Omega)$ of random variables $\xi(\omega) = \xi, \omega \in \Omega$, is called the Orlicz space generating by a C-function $U(x)$ if, for any $\xi \in L_U(\Omega)$, there exist a constant $r_\xi > 0$ such that*

$$EU\left(\frac{\xi}{r_\xi}\right) < \infty.$$

The Orlicz space $L_u(\Omega)$ is a Banach space with respect to the norm

$$\|\xi\|_{L_u} = inf\left\{r > 0 : Eu\left(\frac{\xi}{r_\xi}\right) \leq 1\right\}.$$

Definition 4. *([1]) Let $X = \{X(t), t \in T\}$ be a stochastic process. We say that X belongs to the Orlicz space $L_u(\Omega)$ if, for all $t \in T$, the random variable $X(t)$ belongs to the $L_u(\Omega)$.*

Definition 5. *([1]) Let $U(x)$ be a C-function. A family Δ of centered random variables ξ, $(E\xi = 0)$, $\xi \in \Delta$ from the Orlicz space $L_u(\Omega)$ is called a strictly Orlicz family if there exist a constant C_Δ such that*

$$\left\| \sum_{i \in I} \lambda_i \xi_i \right\|_{L_u} \le C_\Delta \left(E\left(\sum_{i \in I} \lambda_i \xi_i \right)^2 \right)^{1/2}.$$

for all finite collection of random variables $\xi_i \in \Delta$, $i \in I$ and for all $\lambda_i \in \mathbf{R}^1$.

Definition 6. *([1]) A stochastic process $X = \{X(t), t \in T\}$, $(X \in L_u(\Omega))$, is called a strictly Orlicz process if the collection of the random variables $X = \{X(t), t \in T\}$ is a strictly Orlicz family. Two stochastic processes $X = X(t), t \in T$ and $Y = Y(t), t \in T$ are called jointly strictly Orlicz processes if the collection of the random variables $\{X(t), Y(t), t \in T\}$ is a strictly Orlicz family.*

Theorem 1. *([4]) Let $X_i = \{X_i(t), t \in T, i \in I\}$ - a family of jointly strictly Orlicz processes. If there exist the integral what converge on mean square*

$$\xi_{ki} = \int_T \varphi_k(t)\, x_i(t)\, d\mu(t),$$

then the family of random variables $\Delta_\xi = \left\{\xi_{ki}, i \in I, k = \overline{1,\infty}\right\}$ is a strictly Orlicz family.

Next theorem is a particular case of the theorem in [6].

Theorem 2. *Let in $\mathbf{R}^2$:*

$$d(t,s) = \max_{i=1,2} |t_i - s_i|,$$

$T = \{0 \le t_i \le T_i, i = 1,2\}$, $X_n = \{X_n(t), t \in T\}$, $n = 1,2,\dots$ - a sequance of stochastic processes belonging to the Orlicz space, and the function U satisfies g-condition. Let

1) *$X_n(t)$ - separable processes;*
2) *$X_n(t) \to X(t)$ when $n \to \infty$, $t \in T$ in probability;*
3) *$\sup\limits_{d(t,s)\le h} \sup\limits_{n=\overline{1,\infty}} \|X_n(t) - X_n(s)\| \le \sigma(h)$, where $\sigma = \{\sigma(h), h > 0\}$ continuous increasing function, such that $\sigma(h) \to 0$ as $h \to 0$;*
4) *for some $\epsilon > 0$*

$$\int_0^\epsilon U^{(-1)}\left(\left(\frac{T_1}{2\sigma^{(-1)}(u)} + 1 \right) \left(\frac{T_2}{2\sigma^{(-1)}(u)} + 1 \right) \right) du < \infty,$$

where $\sigma^{(-1)}(u)$ - inverse function of $\sigma(u)$.

Then processes $X_n(t)$ are converge in probability to $X(t)$ in space $C(T)$.

Lemma 1. *([5]) Let $Y_\lambda(u), \lambda > 0, u \in T$, $T \in (0, \infty)$, such that*

1) $\sup\limits_{u\in T} |Y_\lambda(u)| \le B$,

2) $|Y_\lambda(u) - Y_\lambda(v)| \le C\lambda\, |u - v|$ *for all* $u, v \in T$.

Let $\varphi(\lambda), \lambda > 0$ be a continuous increasing function, $\varphi(\lambda) > 0$ for all $\lambda > 0$, such that function $\frac{\lambda}{\varphi(\lambda)}$ increasing for $\lambda > v_0$, for some constant $v_0 \ge 0$. Then

$$|Y_\lambda(u) - Y_\lambda(v)| \le \max(C, 2B)\frac{\varphi(\lambda + v_0)}{\varphi\left(\frac{1}{|u-v|} + v_0\right)}.$$

for all $\lambda \ge 0$ and $v > 0$

Theorem 3. *[2] Let $X = \{X(t), t \in T\}, T = [a, b], -\infty < a < b < +\infty$ — separable stochastic processes belonging to the space $L_p(\Omega), p \ge 1$. Let there exist a function $\sigma = \{\sigma(h), 0 \le h \le b - a\}$, such that $\sigma(h)$ — continuous, monotonically increasing, $\sigma(0) = 0$ and*

$$\sup_{\substack{|t-s|\le h\\ t,s\in[a,b]}} \left(E\,|X(t) - X(s)|^p\right)^{\frac{1}{p}} \le \sigma(h).$$

Let for some $0 < \epsilon < b - a$ converge integral

$$\int_0^\epsilon \left(\sigma^{(-1)}(u)\right)^{-\frac{1}{p}} du < \infty.$$

Then for random variable $\sup\limits_{t\in[a,b]} |X(t)| \in L_p(\Omega)$ there is inequality

$$\left\|\sup_{t\in[a,b]} |X(t)|\right\|_p = \left(E\left(\sup_{t\in[a,b]} |X(t)|\right)^p\right)^{\frac{1}{p}} \le$$

$$\le \left(E\,|X(t_0)|^p\right)^{\frac{1}{p}} + \frac{1}{\theta(1-\theta)} \cdot \int_0^{\omega_0\theta} \left(\frac{b-a}{2\sigma^{(-1)}(u)} + 1\right)^{\frac{1}{p}} du =$$

$$D_p(t_0) \le \left(E\,|X(t_0)|^p\right)^{\frac{1}{p}} + \frac{\alpha_p}{\theta(1-\theta)} \cdot \int_0^{\omega_0\theta} \left(\sigma^{(-1)}(u)\right)^{-\frac{1}{p}} du = \tilde{D}_p(t_0),$$

where t_0 — any point from $[a, b]$; $0 < \theta < 1$; $\omega_0 = \sigma\left(\sup\limits_{t\in[a,b]} |t - t_0|\right)$, $\alpha_p = \left[\frac{b-a}{2} + \sup\limits_{t\in[a,b]} |t - t_0|\right]$.

For any $\epsilon > 0$

$$P\left\{\sup_{t\in[a,b]}|X(t)| > \epsilon\right\} \le \frac{[D_p(t_0)]^p}{\epsilon^p} \le \frac{\left[\tilde{D}_p(t_0)\right]^p}{\epsilon^p}$$

and process $X(t)$ *continuous with probability one.*

Definition 7. *([1]) We say that a* C*-function* U *is subordinate to a* C*-function* V *and denote* $U \prec V$ *if there exist two numbers* $x_0 \ge 0$ *and* $c > 0$ *such that* $U(x) \le V(cx)$ *for all* x *such that* $|x| > x_0$*. We say that two* C*-functons* $U(x)$ *and* $V(x)$ *are equivalent if* $U(x) \prec V(x)$ *and* $V(x) \prec U(x)$*.*

2. Main results. Consider a boundary value problem for a parabolic equation with two independent variables $0 \le x \le l$ and $t \ge 0$, physical interpretation of which is: find the law of temperature change in a homogeneous rod length l, with insulated lateral surface, when initial temperature of the rod points $\xi(x)$ — is a random process belonging to the Orlicz space $L_U(\Omega)$, left end of the rod is insulated, and the right is heat exchange with the environment of zero temperature under the law Newton. Namely

$$Z_t\left(t,x\right) = a^2 Z_{xx}\left(t,x\right), O < x < l, 0 < t < T, \tag{1}$$

$$Z\left(0,x\right) = \xi\left(x\right), O \le x \le l, \tag{2}$$

$$Z_x\left(t,0\right) = 0, Z_x\left(t,l\right) + Z\left(t,l\right) = 0, 0 \le t < T, \tag{3}$$

Initial condition $(\xi\left(x\right), x \in [0,l])$ is a strictly Orlicz random process. Using the method of Fourier solution is sought in the form [7]:

$$Z\left(t,x\right) = \sum_{k=1}^{\infty} A_k e^{-\left(\frac{a\mu_k}{l}\right)^2 t} X_k\left(x\right), \tag{4}$$

where $\mu_k = \sqrt{\lambda_k} l, k \in N$ solution of equation $\operatorname{ctg}\sqrt{\lambda_k} l = \sqrt{\lambda_k}$, namely eigenvalues which corresponds eigenfunctons $\hat{X}_k\left(x\right) = -\frac{\mu_k}{l}\sin\frac{\mu_k}{l}x + \cos\frac{\mu_k}{l}x$, of the Sturm-Liouville problem

$$X''\left(x\right) - \lambda X\left(x\right) = 0,$$

$$X'\left(0\right) = 0,$$

$$X'\left(l\right) + X\left(l\right) = 0.$$

Denote

$$X_k\left(x\right) = \sqrt{\frac{2}{l}}\hat{X}_k\left(x\right).$$

Then the coefficients

$$A_k = \frac{2}{l}\int_0^l \xi\left(x\right) X_k\left(x\right) dx.$$

Consider the series $S_{ms}\left(t,x\right)$ — when $m = 0$, then $s = 0, 1, 2$; then $m = 1$, then $s = 0$ — derivative $Z\left(t,x\right)$ once in t, once and twice in x (m-th derivative in t, s-th in x), namely:

$$S_{ms}\left(t,x\right) = \sum_{k=1}^{\infty} A_k e^{-\left(\frac{a\mu_k}{l}\right)^2 t}\left(\frac{a\mu_k}{l}\right)^{2m} X_k^{(s)}\left(x\right) \tag{5}$$

Moreover, the series $S_{ms}(t,x)$ strictly Orlicz processes (follows from [3]) in the domain $D=[0,T]\times[0,l]$. Denote

$$S_{msN}(t,x)=\sum_{k=1}^{N}A_k e^{-\left(\frac{a\mu_k}{l}\right)^2 t}\left(\frac{a\mu_k}{l}\right)^{2m}X_k^{(s)}(x). \tag{6}$$

Lemma 2. *For $\forall\epsilon>0$ series $S_{ms}(t,x)$ converge with probability one uniformly in the domain*

$$D_\epsilon=[0\le x\le l]\times[\epsilon;+\infty).$$

Proof. Have

$$X_n'(x)=\sqrt{\frac{2}{l}}\left(-\frac{\mu_n}{l}\sin\frac{\mu_n}{l}x+\cos\frac{\mu_n}{l}x\right)'=\sqrt{\frac{2}{l}}\left(-\left(\frac{\mu_n}{l}\right)^2\cos\frac{\mu_n}{l}x-\frac{\mu_n}{l}\sin\frac{\mu_n}{l}x\right),$$

$$X_n''(x)=\sqrt{\frac{2}{l}}\left(-\left(\frac{\mu_n}{l}\right)^2\cos\frac{\mu_n}{l}x-\frac{\mu_n}{l}\sin\frac{\mu_n}{l}x\right)'=\sqrt{\frac{2}{l}}\left(\left(\frac{\mu_n}{l}\right)^3\sin\frac{\mu_n}{l}x-\left(\frac{\mu_n}{l}\right)^2\cos\frac{\mu_n}{l}x\right).$$

Hence

$$|X_n'(x)|\le\sqrt{\frac{2}{l}}\frac{\mu_n}{l}\left(\frac{\mu_n}{l}+1\right),$$

$$|X_n''(x)|\le\sqrt{\frac{2}{l}}\left(\frac{\mu_n}{l}\right)^2\left(\frac{\mu_n}{l}+1\right).$$

So

$$\sup_{\substack{t>\epsilon\\0\le x\le l}}S_{ms}(t,x)\le\sum_{k=1}^{\infty}|A_k|\sup_{t>\epsilon}e^{-\left(\frac{a\mu_k}{l}\right)^2 t}\left(\frac{a\mu_k}{l}\right)^{2m}\left|X_k^{(s)}(x)\right|\le$$

$$\le c\sum_{k=1}^{\infty}|A_k|\,e^{-\left(\frac{a\mu_k}{l}\right)^2\epsilon}\sqrt{\frac{2}{l}}\frac{\mu_k}{l}\left(\frac{\mu_k}{l}+1\right),$$

where $c=\max\left\{\frac{\mu_k}{l},1\right\}$.

Last series converge with probability one when converge series

$$\sum_{k=1}^{\infty}E|A_k|\,e^{-\left(\frac{a\mu_k}{l}\right)^2\epsilon}\frac{\mu_k}{l}\left(\frac{\mu_k}{l}+1\right). \tag{7}$$

Since

$$E|A_k|\le\left(EA_k^2\right)^{1/2}\le a,$$

where a — positive constant. There

$$\sum_{k=1}^{\infty}E|A_k|\,e^{-\left(\frac{a\mu_k}{l}\right)^2\epsilon}\frac{\mu_k}{l}\left(\frac{\mu_k}{l}+1\right)\le a\sum_{k=1}^{\infty}e^{-\left(\frac{a\mu_k}{l}\right)^2\epsilon}\frac{\mu_k}{l}\left(\frac{\mu_k}{l}+1\right). \tag{8}$$

As for [7]:

$$\sqrt{\lambda_n}=dn+o\left(\frac{1}{n}\right),$$

then $l\sqrt{\lambda_n}=\mu_n=ldn+o\left(\frac{1}{n}\right)$ (where d — positive constant).

Herefrom follows, that for sufficiently large n:

$$\mu_n \leq ld_1 n, \mu_n \geq ld_2 n, 0 < d_2 < d_1.$$

Denote series

$$\sum_{k=1}^{\infty} e^{-a^2 d_2^2 k^2} k(k+1). \tag{9}$$

As converge series (9), then converge (7), namely converges uniformly with probability one series $S_{ms}(t,x)$.

Lemma 3. *For so as uniformly in probability in the domain D $Z(t,x)$ converge to the $\xi(x)$ by $t \to 0$ enought so as for every $0 \leq x \leq l$*

$$E\left|Z(t,x) - \xi(x)\right| \to 0, t \to 0,$$

$\xi(x)$ continuous process at $[0;l]$ with probability one and exist monotonically increament function $\sigma = (\sigma(h), 0 < h \leq l)$ such that $\sigma(h) \to 0$ when $h \to 0$ and

$$\sup_{0<t<t_0} \sup_{\substack{|x-y|\leq h \\ x,y\in[0;l]}} \left(E\left|Z(t,x) - Z(t,y)\right|^2\right)^{1/2} \leq \sigma(h), \tag{10}$$

where $0 < t_0 < T$ and for $\forall\epsilon > 0$

$$\int_0^{\epsilon} U^{(-1)}\left(\frac{l}{2\sigma^{(-1)}(h)} + 1\right) dh < \infty. \tag{11}$$

Proof. The validity of this theorem is follows from the fact that $Z(t,x)$ strictly Orlicz process, such that

$$\|Z(t,x) - Z(t,y)\| \leq C_\Delta \left(E\left|Z(t,x) - Z(t,y)\right|^2\right)^{1/2},$$

and $Z(t,x) \to \xi(x)$ in probability (follows from theorem 2).

Lemma 4. *Considered problem (1)-(3). Let $\xi = \{\xi(x), x \in [0;l]\}$ from (2) be a strictly Orlicz stochastic process belonging to the Orlicz space $L_U(\Omega)$ of random variables, where $U(x)$ is a C-function, such that the function $V(x) = x^2$ subordinate to $U(x)$, and for $U(x)$ and condition g holds for $U(x)$. Assume that the stochastic process ξ- is separable and mean square continuous, $E\xi(x) = 0$. Let function $\varphi = (\varphi(\lambda), \lambda > 0)$ — continuous, monotonically increasing and $\Psi(\lambda) = \frac{\lambda}{\varphi(\lambda)}$ monotonocally increasing when $\lambda \geq v_0$, where v_0 is a constant such that $\forall\epsilon > 0$ and $c > 0$:*

$$\sup_{|x-y|\leq h} \left(E\left(\xi(x) - \xi(y)\right)^2\right)^{1/2} \leq \frac{c}{\varphi\left(\frac{1}{h} + v_0\right)} \tag{12}$$

when

$$\int_0^{\epsilon} U^{(-1)}\left(\frac{l}{2}\left(\varphi^{(-1)}\left(\frac{c}{v}\right) - v_0\right) + 1\right) dv < \infty, \tag{13}$$

and converge sum

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \left|EA_iA_j\right| \varphi\left(\left(\frac{a\mu_i}{l}\right)^2 + v_0\right) \varphi\left(\left(\frac{a\mu_j}{l}\right)^2 + v_0\right). \tag{14}$$

Then uniformly in probability $Z(t,x) \to \xi(x)$ in the domain $D = [0;l]$ when $t \to 0$.

Proof. In order to prove this lemma need to satisfies conditions from lemma 3. From conditions of these lemma it follows that $E\left|Z(t,x)-\xi(x)\right|^2 \to 0$ when $t \to 0$. Since $\xi(x)=\sum\limits_{k=1}^{\infty} A_k X_k(x)$, have:

$$E\left|Z(t,x)-\xi(x)\right|^2 = E\left[\sum_{k=1}^{\infty} A_k e^{-\left(\frac{a\mu_k}{l}\right)^2 t} X_k(x) - \sum_{k=1}^{\infty} A_k X_k(x)\right]^2 =$$

$$= E\left[\sum_{k=1}^{\infty} A_k X_k(x)\left(e^{-\left(\frac{a\mu_k}{l}\right)^2 t}-1\right)\right]^2 =$$

$$= \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} EA_iA_jX_i(x)X_j(x)\left(e^{-\left(\frac{a\mu_i}{l}\right)^2 t}-1\right)\left(e^{-\left(\frac{a\mu_j}{l}\right)^2 t}-1\right)$$

In lemma 1 put $Y_\lambda(t)=e^{-\left(\frac{a\mu_k}{l}\right)^2 t}$, $B=1$, $C=1$.

It obvious that $\left|X_k(x)\right|\le 1$.

Then

$$E\left|Z(t,x)-\xi(x)\right|^2 \le$$

$$\le \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\left|EA_iA_jX_i(x)X_j(x)\right|\varphi\left(\left(\frac{a\mu_i}{l}\right)^2+v_0\right)\varphi\left(\left(\frac{a\mu_j}{l}\right)^2+v_0\right)\varphi^{-2}\left(\frac{1}{t}+v_0\right)\le \quad (15)$$

$$\le \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\left|EA_iA_j\right|\varphi\left(\left(\frac{a\mu_i}{l}\right)^2+v_0\right)\varphi\left(\left(\frac{a\mu_j}{l}\right)^2+v_0\right)\varphi^{-2}\left(\frac{1}{t}+v_0\right)\to 0,$$

when $t\to 0$. We find a function $\sigma(h)$, that is satisfies condition:

$$\sup_{0<t<t_0}\ \sup_{\substack{|x-x_1|\le h\\ x,x_1\in[0;l]}}\left(E\left|Z(t,x)-Z(t,x_1)\right|^2\right)^{1/2}\le\sigma(h). \quad (16)$$

It is obvious that

$$E\left(Z(t,x)-Z(t,x_1)\right)^2 = E\left[\sum_{k=1}^{\infty}\left(A_kX_k(x)e^{-\left(\frac{a\mu_k}{l}\right)^2 t}-A_kX_k(x_1)e^{-\left(\frac{a\mu_k}{l}\right)^2 t}\right)\right] =$$

$$= E\left[\sum_{k=1}^{\infty} A_k e^{-\left(\frac{a\mu_k}{l}\right)^2 t}\left(X_k(x)-X_k(x_1)\right)\right] =$$

$$= \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} e^{-\left(\frac{a\mu_i}{l}\right)^2 t}e^{-\left(\frac{a\mu_j}{l}\right)^2 t}EA_iA_j\left(X_i(x)-X_i(x_1)\right)\left(X_j(x)-X_j(x_1)\right)\le$$

$$\le \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} e^{-\left(\frac{a\mu_i}{l}\right)^2 t}e^{-\left(\frac{a\mu_j}{l}\right)^2 t}\left|EA_iA_j\right|\left|X_i(x)-X_i(x_1)\right|\left|X_j(x)-X_j(x_1)\right|\le$$

$$\le \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\left|EA_iA_j\right|\left|X_i(x)-X_i(x_1)\right|\left|X_j(x)-X_j(x_1)\right|.$$

Let estimate the absolute of difference eigenfunctions

$$|X_k(x) - X_k(x_1)| = \sqrt{\frac{2}{l}} \left| -\frac{\mu_k}{l} \sin\frac{\mu_k}{l}x + \cos\frac{\mu_k}{l}x + \frac{\mu_k}{l}\sin\frac{\mu_k}{l}x_1 - \cos\frac{\mu_k}{l}x_1 \right| \leq$$

$$\leq \sqrt{\frac{2}{l}}\frac{\mu_k}{l}\left|\sin\frac{\mu_k}{l}x - \sin\frac{\mu_k}{l}x_1\right| + \sqrt{\frac{2}{l}}\left|\cos\frac{\mu_k}{l}x - \cos\frac{\mu_k}{l}x_1\right| =$$

$$= \sqrt{\frac{2}{l}}\frac{2\mu_k}{l}\left|\sin\frac{\mu_k}{2l}(x-x_1)\cos\frac{\mu_k}{2l}(x+x_1)\right| + 2\sqrt{\frac{2}{l}}\left|\sin\frac{\mu_k}{2l}(x-x_1)\sin\frac{\mu_k}{2l}(x+x_1)\right| \leq$$

$$\sqrt{\frac{2}{l}}\frac{\mu_k^2}{l^2}|x-x_1| + \sqrt{\frac{2}{l}}\frac{\mu_k}{l}|x-x_1| = \sqrt{\frac{2}{l}}\mu_k\left(\frac{\mu_k}{l^2} + \frac{1}{l}\right)|x-x_1|.$$

We received that

$$|X_k(x) - X_k(x_1)| \leq \sqrt{\frac{2}{l}}\mu_k\left(\frac{\mu_k}{l^2} + \frac{1}{l}\right)|x-x_1|. \tag{17}$$

Denote

$$D = \sqrt{\frac{2}{l}}\left(\frac{\mu_k}{l} + 1\right). \tag{18}$$

From used above conditions have

$$|X_k(x) - X_k(x_1)| \leq$$

$$\leq \max(2, D)\,\varphi\left(\left(\frac{a\mu_k}{l}\right)^2 + v_0\right)\varphi^{-1}\left(\frac{1}{|x-x_1|} + v_0\right), \tag{19}$$

that

$$E\left(Z(t,x) - Z(t,y)\right)^2 \leq$$

$$\leq \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}|EA_iA_j|\,4D^2\varphi\left(\left(\frac{a\mu_i}{l}\right)^2 + v_0\right)\varphi\left(\left(\frac{a\mu_j}{l}\right)^2 + v_0\right)\varphi^{-2}\left(\frac{1}{|x-y|} + v_0\right).$$

Therefore, have

$$\sigma(h) = C\left(\varphi\left(\frac{1}{h} + v_0\right)\right)^{-1},$$

and

$$\sigma^{(-1)}(h) = \frac{1}{\varphi^{(-1)}\left(\frac{c}{h}\right) - v_0}$$

because satisfies condition (11).

Example 1. *Consider a concrete example for lemma 4. Let* $U(x) = |x|^p, p \geq 2$. *Then in lemma 4 put* $\varphi(x) = |x|^\alpha, 0 < \alpha \leq 1, v_0 = 0$. *Have*

$$\varphi^{(-1)}\left(\frac{c}{v}\right) = \left(\frac{c}{v}\right)^{\frac{1}{\alpha}}.$$

And integral from (9) has the form

$$\int_0^{\epsilon}\left(\frac{l}{2}\left(\frac{c}{v}\right)^{\frac{1}{\alpha}}\right)^{\frac{1}{p}}dv = \left(\frac{l}{2}\right)^{\frac{1}{p}}c^{\frac{1}{p\alpha}}\int_0^{\epsilon}\frac{1}{v^{\frac{1}{p\alpha}}}dv.$$

Last integral converge when $\frac{1}{p\alpha} < 1$.

We obtain

$$\alpha > \frac{1}{p}.$$

Theorem 4. *Considered problem (1)-(3). Let* $\xi = \{\xi(x), x \in [0;l]\}$ *from (2) be a strictly Orlicz stochastic process belonging to the Orlicz space* $L_U(\Omega)$ *of random variables, where* $U(x)$ *is a* C*-function, such that the function* $V(x) = x^2$ *subordinate to* $U(x)$*, and for* $U(x)$ *holds* g*-condition. Assume that stochastic process* ξ *is separable and mean square continuous,* $E\xi(x) = 0$.

Let

$$Z(t,x) = \sum_{k=1}^{\infty} A_k e^{-\left(\frac{a\mu_k}{l}\right)^2 t} X_k(x),$$

where $X_k(x)$ — *eigenfunctions,* μ_k — *appropriate eigenvalues of the Sturm-Liouville problem*

$$X''(x) - \lambda X(x) = 0,$$

$$X'(0) = 0,$$

$$X'(l) + X(l) = 0.$$

$$A_k = \frac{2}{l}\int_0^l \xi(x) X_k(x)\,dx.$$

Let for $\forall\epsilon > 0$ *and* $c > 0$ *satisfies condition*

$$\sup_{|x-y|\le h} \left(E(\xi(x) - \xi(y))^2\right)^{1/2} \le c|h|^{\alpha}, \tag{20}$$

where $\alpha > \frac{1}{p}$, $p \ge 2$ *and converge series*

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} |EA_iA_j| \mu_i^{2\alpha}\mu_j^{2\alpha}, \tag{21}$$

that for $\forall\vartheta > 0$ *uniformly in* $0 \le x \le l, t \ge \vartheta\,(\vartheta > 0)$ *with probability one series converge (5). In domain* $0 \le x \le l, t \ge \vartheta\,(\vartheta > 0)$ *function* $Z(t,x)$ *with probability one satisfies condition (1) and condition (2). Besides, uniformly in probability* $Z(t,x) \to \xi(x)$ *in domain* $x \in [0;l]$ *when* $t \to 0$. *If for* $\forall\epsilon > 0$ *and* $c > 0$ *series converge (21), that* $Z(t,x) \to \xi(x)$ *uniformly in* $x \in [0;l]$ *when* $t \to 0$ *with probability one.*

Proof. Uniform convergence with probability one series (5) when $0 \le x \le l$, $0 < t < \epsilon$ proved in lemma 2. In lemma 4 proved that $Z(t,x) \to \xi(x)$ when $t \to 0$. From lemma 4 and condition (21) follows that $Z(t,x) \to \xi(x)$ when $t \to 0$ uniformly in $0 \le x \le l$ with probability one.

Lemma 5. *Let for eigenfunctions* $X_k(x), x \in [0,l]$ *from problem (1)-(3) satisfies condition*

$$|X_k(x) - X_k(s)| \le D\frac{\mu_k}{l}|x - s| \tag{22}$$

(where D from (18)) and for $N = 1$ next series converge

$$\sum_{i=N}^{\infty}\sum_{j=N}^{\infty}|EA_iA_j|\,\mu_i^{\beta}\mu_j^{\beta}=T_N, \tag{23}$$

where $\frac{2}{p}<\beta\le 1$, that for $t_1,t_2\in[0,T]\,,\,x_1,x_2\in[0,l]\,,\,h\in[0,l]$ there is an inequality

$$\sup_{\max(|t_1-t_2|,|x_1-x_2|)\le h}\left(E\left(Z_N(t_1,x_1)-Z_N(t_2,x_2)\right)^2\right)^{\frac{1}{2}}\le 2\max(2,D)h^{\beta}\sqrt{T_N}, \tag{24}$$

where $Z_N\left(t,x\right)=\sum\limits_{k=N}^{\infty}A_ke^{-\left(\frac{a\mu_k}{l}\right)^2t}X_k\left(x\right)$.

Proof. The validity of the theorem follows from (19) if $\varphi(u)=u^{\beta}$.

Theorem 5. *Let satisfies conditions of lemma 5, $(\xi(x),x\in[0,l])$ initial condition from (2) — separable and strictly Orlicz stochastic process belonging to the $L_p(\Omega)$, denote $B=\{x\in[0,l]\,,t\in[0,T]\}$, $T>0$, that for $\forall\epsilon>0$:*

$$P\left\{\sup_{x,t\in B}|Z_N(t,x)|>\epsilon\right\}\le\frac{\left|\hat{B}_N(\theta)\right|^p}{\epsilon^p}, \tag{25}$$

where

$$\hat{B}_N(\theta)=\frac{1}{\theta\left(1-\theta\right)}\int\limits_0^{\omega_{0N}\theta}\left(\frac{l}{2}\left(\frac{u}{c}\right)^{\frac{1}{\alpha}}+1\right)\left(\left(\frac{T}{2}\left(\frac{u}{c}\right)^{\frac{1}{\alpha}}+1\right)\right)^{\frac{1}{p}}du,$$

$\theta\in(0,1)$, $\omega_{0N}=R_Nh^{\beta}$, $R_N=2C_{\Delta}\sqrt{T_N}\max(2,L)$, C_{Δ} — constant from definition 5.

Proof. Proof of this theorem follows from theorem 3. Therefore in this theorem $\sigma(h)=ch^{\alpha},\alpha>\frac{1}{p},h\in[0,l]$ and for all $\epsilon>0$ converge integral

$$\int\limits_0^{\epsilon}\left(\sigma^{(-1)}(u)\right)^{-\frac{1}{p}}du=\int\limits_0^{\epsilon}\frac{c^{\frac{1}{\alpha p}}}{u^{\frac{1}{\alpha p}}}du,$$

and that is condition (25) satisfies.

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Одержано 12.09.2013