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A. I. Plakosh (National Academy of Sciences of Ukraine, Institute of Mathematics)

ON WEAK EQUIVALENCE OF REPRESENTATIONS OF KLEIN FOUR-GROUP

We give a classification of the integral p-adic representations of the Klein four-group up to weak equivalence.

Ми даємо класифікацію цілочислових p-адичних зображень четверної групи Клейна з точністю до слабкої еквівалентності.

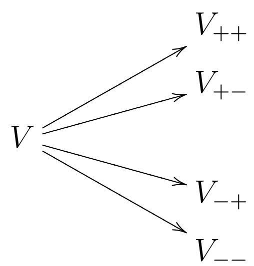
Let $G = \langle\, a, b \mid a^2 = b^2 = 1,\ ab = ba \,\rangle$ be the Klein four-group. Its integral 2-adic representations were descibed by Nazarova [1]. If one wants to apply these results to the description of Chernikov p-groups, as in [3–5], one has to consider the classes of *weak equivalence* of such representations. We recall that two $\mathbb{Z}_2G$-modules M, N are said to be *weakly equivalent* if there is an automorphism σ of the group G and an isomorphism of $\mathbb{Z}_2$-modules $f : M \to N$ such that $f(gv) = \sigma(g)f(v)$ for all $g \in G$, $v \in M$. The aim of this paper is to give a classification of indecomposable integral 2-adic representations of the Klein group up to weak equivalence. So we consider $\mathbb{Z}_2G$-*lattices*, i.e. finitely generated $\mathbb{Z}_2G$-modules torsion free (hence free) as $\mathbb{Z}_2$-modules.

To study weak equivalence, it is convenient to consider another description of the representations, which uses the technique of [5–7].

The group ring $\mathbb{Z}_2G$ is a local Gorenstein ring. Therefore, any indecomposable G-lattice M except $\mathbb{Z}_2G$ itself is indeed an $\boldsymbol{A}$-lattice, where $\boldsymbol{A}$ is the unique minimal over-ring of $\boldsymbol{A}$ [8] that is, in our case, $\boldsymbol{A} = \mathbb{Z}_2G + \mathbb{Z}_2z$, where $z = \frac{1+a+b+ab}{2}$. One easily sees that $\operatorname{rad}\mathbb{Z}_2G = \langle\, x, y \,\rangle$, where $x = a-1$, $y = b-1$, while $\operatorname{rad}\boldsymbol{A} = \langle\, x, y, z \,\rangle$. Actually, $J = \operatorname{rad}\boldsymbol{A}$ is a module over the maximal over-ring $\boldsymbol{R}$ of $\mathbb{Z}_2G$, which is generated, as $\mathbb{Z}_2$-module, by 4 primitive orthogonal idempotents: e_i $(1 \le i \le 4)$, where

$$e_{++} = \frac{1+a+b+ab}{4}, \qquad e_{+-} = \frac{1+a-b-ab}{4},$$
$$e_{-+} = \frac{1-a+b-ab}{4}, \qquad e_{--} = \frac{1-a-b+ab}{4}.$$

We set $\boldsymbol{R}_{ij} = e_{ij}\boldsymbol{R}$, where $i, j \in \{+, -\}$. The $\mathbb{F}_2$-algebra $\boldsymbol{R}/J$ is isomorphic to $\mathbb{F}_2^4$ with the basis consisting of the classes of these idempotents. Hence $\boldsymbol{A}$ is a Bäckström order in the sense of [7]. Therefore, there is a one-to-one correspondence between the $\boldsymbol{A}$-lattices and diagrams of vector spaces (representations of the quiver of type $\tilde{D}_4$)



(1)

such that all arrows are surjective and the induced map $\iota : V_0 \to \oplus_{ij} V_{ij}$ is injective (ibid.). Namely, M corresponds to the diagram with

$$V_0 = M/JM, \quad V_{ij} = e_{ij}(\boldsymbol{R}M/JM) \quad (i, j \in \{+, -\})$$

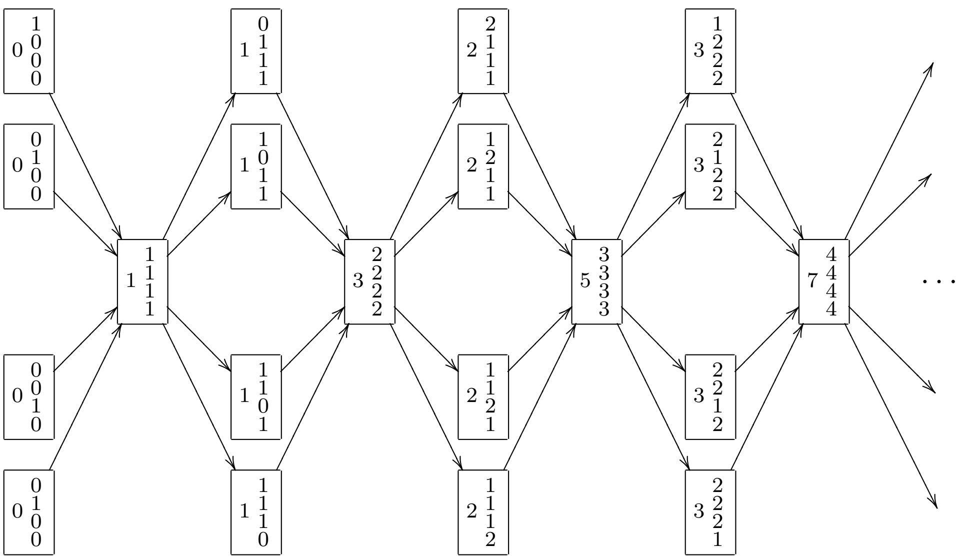
and arrows denote the natural projections of $V_0 \subseteq \boldsymbol{R}M/JM$ onto the components. On the contrary, given such diagram with $\dim V_{ij} = d_{ij}$, we identify the direct sum $\oplus_{ij} V_{ij}$ with N/JN, where $N = \oplus_{ij} \boldsymbol{R}_{ij}^{d_{ij}}$, and take the preimage M of V_0 considered as the subspace of $\oplus_{ij} V_{ij}$ via the map ι. Note that the only indecomposable diagrams of the shape (1) which do not correspond to $\boldsymbol{A}$-lattices are the "trivial" ones, where one of the spaces is 1-dimensional and the others are zero. We call the *dimension* of such a diagram the vector $(d_0, d_{++}, d_{+-}, d_{-+}, d_{--})$, where $d_0 = \dim V_0$, $d_{ij} = \dim V_{ij}$, and usually arrange this quintuple as

$$\boxed{\begin{matrix} & d_{++} \\ & d_{+-} \\ d_0 & \\ & d_{-+} \\ & d_{--} \end{matrix}}\,.$$

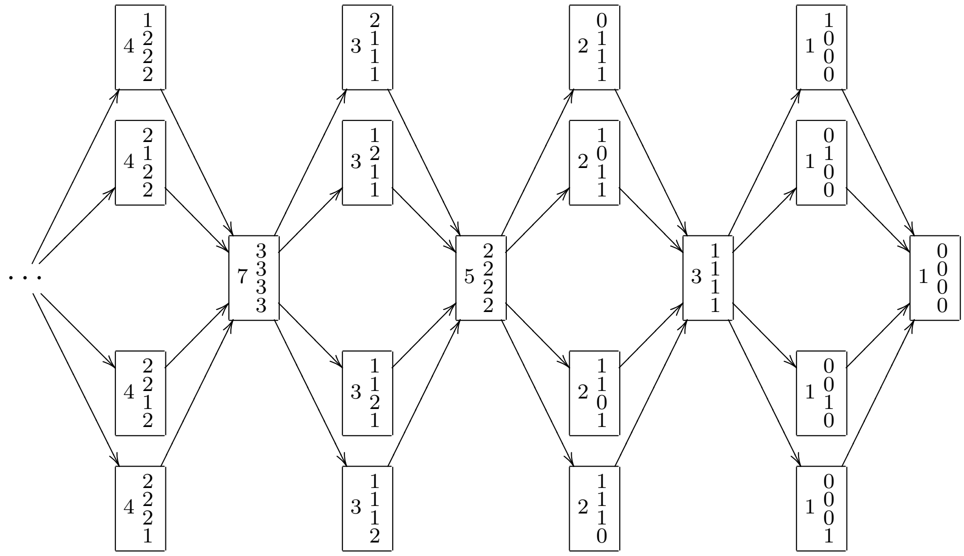
The indecomposable diagrams are arranged into the *Auslander–Reiten quiver* [6]. Its vertices are the just the diagrams and the arrows are the *irreducible maps*, i.e. non-invertible morphisms of the diagrams which cannot be presented as sums of compositions of such morphisms. The structure of this diagram is described in [5] and [6, Sec. 3.6]. It consists of three components:

preprojective **regular** **preinjective**

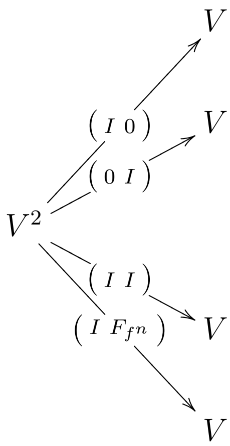
with morphisms going "from left to right". Moreover, there are no arrows (irreducible morphisms) between different components. In the preprojective and preinjective parts the representations are uniquely defined by their dimensions, which are the positive real roots of the Tits form. Namely, the preprojective component is



And the preinjective component is

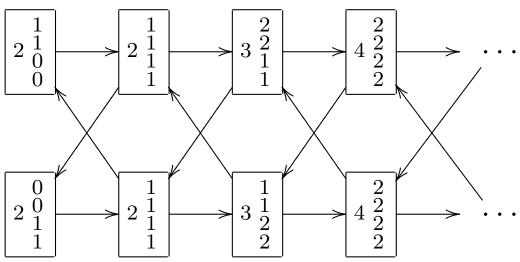


The regular component consists of *tubes*. Most of them are *regular*, i.e. behave as indecomposble finite dimensional modules over the ring $\daleth[[t]]$, where $\daleth$ is an extension of the field $\mathbb{F}_2$. They are in one-to-one correspondence with irreducible polynomials $f(t) \in \mathbb{F}_2[t]$, $f(t) \notin \{t, t-1\}$. The dimensions of the representations in regular tubes are $\boxed{2nd \begin{smallmatrix} nd \\ nd \\ nd \\ nd \end{smallmatrix}}$, where $d = \deg f(t)$, and the corresponding diagrams are

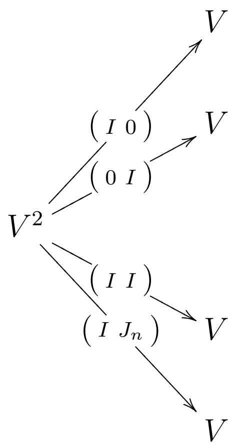


where $V = \daleth^{dn}$ and F_{f^n} is the Frobenius matrix with the minimal polynomial $f^n(t)$.

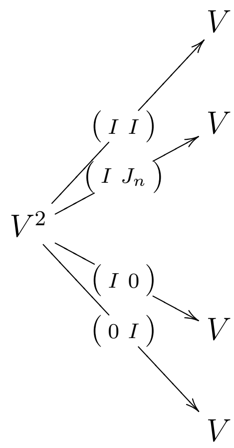
There are also 3 special tubes T_k $(2 \le k \le 4)$ of period 2. The tube T_4 is



where the representations of the dimension $\boxed{2n \begin{smallmatrix} n \\ n \\ n \\ n \end{smallmatrix}}$ in the first (in the second) row are, respectively,



and



where $V = ⅂^n$, and J_n is the nilpotent Jordan $n \times n$ matrix. The other representations are also defined by their dimensions. The tubes T_2 and T_3 are obtained from T_4 just by transposing V_{--} with, respectively, V_{+-} or V_{-+} and also transposing the corresponding maps.

The group of automorphisms of G is identified with S_3. If we consider its action on the representations, it permutes the components M_{ij}, where $i, j \in \{+, -\}$, $(ij) \neq (++)$. Thus it also permutes the components V_{ij} of the corresponding diagrams. Therefore, for the preprojective and preinjective components it leaves untouched the central and the first rows and permutes the other three. It also permutes the special tubes T_k mapping the first row to the first and the second row to the second. One can also check that the action of S_3 on the regular tubes coincides with its classical action on polynomials. Namely, the transposition of $(-+)$ and $(--)$ maps $f(t)$ to $c^{-1}t^d f(1/t)$, where $d = \deg f,\ c = f(0)$; the transposition of $(+-)$ and $(-+)$ maps $f(t)$ to $c^{-1}(t-1)^d f(1/(t-1))$.

It accomposhes the description of indecomposable representations of G up to weak equivalence.

References

1. *Nazarova L. A.* Inegral representations of the four-group // Dokl. Akad. Nauk SSSR. – 1961. – **140**, no 5. – P. 1011–1014.
2. *Gudivok P. M., Vashchuk F. G., Drobotenko V. S.* Chernikov p-groups and integral p-adic representations of finite groups // Ukrain. Mat. Zh. – 1992. – **44**, no 6. – P. 742–753.
3. *Gudivok P. M., Shapochka I. V.* On the Chernikov p-groups // Ukrain. Mat. Zh. – 1999. – **51**, no. 3. – P. 291–304.
4. *Shapochka I. V.* On the classification of nonisomorphic extensions of a divisible Abelian 2-group with the minimality condition by the Abelian group of type $(2, 2)$ // Nauk. Visn. Uzhhorod. Univ. Ser. Mat. Inform. – 2010. – Vol. 5. – P. 77–84.
5. *Dlab V., Ringel C. M.* Indecomposable representations of graphs and algebras // Mem. Amer. Math. Soc. – 1976. – 6, P. 1–57.
6. *Ringel C. M.*: Tame algebras and integral quadratic forms // Lecture Notes in Math. – Berlin: Springer-Verlag, 1984. – 1099 p.
7. *Roggenkamp K.W.* Auslander-Reiten Species of Backstrom Orders // J. Algebra. – 1983. – 85. – P. 440–476.
8. *Bass H.* On the ubiquity of Gorenstein rings // Math. Z. – 1963. – **82**. – P. 8–28.

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