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FIXED AND RESIDUAL MODULES

The article deals with the properties of fixed and residual endomorphism submodules of modules over arbitrary associative rings with 1. It is shown how they can be used to represent formal matrices images of group homomorphisms generated by elementary transvections when 2 or 3 elements are circulating in the ring. The homomorphisms with condition (*) are described with the help of this approach.

У роботі розглядаються властивості нерухомих та лишкових підмодулів ендоморфізмів модулів над довільними асоціативними кільцями з одиницею. Покозано як з їх допомогою можна зображати формальними матрицями гомоморфні образи груп породжених елементарними трансвекціями у випадках коли елементи 2 або 3 є оборотними в кільці. За допомогою цього підходу описані гомоморфізми з умовою (*).

Introduction. Let R and K be associative ring with 1. $E\,(n, R)$ is the subgroup of $GL\,(n, R)$, generated by all elementary transvections $t_{ij}(r) = 1 + re_{ij}$, $r \in R$, $1 \leq i \neq j \leq n$, $t_{ij} = t_{ij}\,(-1)\,t_{ji}\,(1)\,t_{ij}\,(-1)$.

The group homomorphisms of $\Lambda : G \to GL\,(W)$, $E\,(n, R) \subseteq G \subseteq GL\,(n, R)$, $n \geq 4$ described in [1], if W is its left K-module of finite dimension, Λ is an isomorphism, $G = GL\,(n, R)$ and [2, 3], if W is an arbitrary (not necessarily free) left K-module and Λ is an homomorphism with condition (*).

Recall that a homomorphism Λ satisfies the condition (*) if for any nonzero nilpotent element $m \in EndW$, $m^2 = 0$ there are natural numbers s_1 and, which are working in K and $h \in G$ such that $\Lambda h = 1 + s_1 m$ and of equality $\Lambda h \cdot \Lambda g = \Lambda g \cdot \Lambda h$, $g \in G$ it follows that $h^{s_2} g = gh^{s_2}$.

It turns out that while describing homomorphism with the condition (*) among which are isomorphisms, key role is played by fixed and residual submodules and modules that they generate. Since the possibility of such an approach is seen endless, it is justified to have a more thorough study of the properties of fixed and residual submodules. The article reflects the efforts of the authors on the above-mentioned direction.

1. General properties of fixed and residual submodules. Let V be arbitrary R-module over the associative ring R with 1, σ an arbitrary endomorphism of module V.

Residual and fixed submodules of V module endomorphism σ will be called submodules $R\,(\sigma) = (\sigma - 1)\,V$ and $P\,(\sigma) = \ker\,(\sigma - 1)$ respectively. Then $R\,(\sigma) = \{(\sigma - 1)\,v \mid v \in V\}$ and $P\,(\sigma) = \{v \in V \mid \sigma v = v\}$, also $R\,(1 - \sigma) = \sigma V$ and $P\,(1-\sigma) = \ker \sigma$.

It is easy to see that if σ is an automorphism of module V, then with the equality $\sigma^{-1} - 1 = (\sigma - 1)\,(-\sigma^{-1})$ it follows that

$$R\,(\sigma^{-1}) = R\,(\sigma) \text{ and } P\,(\sigma^{-1}) = P\,(\sigma)\ .$$

Then $\sigma V_0 = (\sigma - 1 + 1)V_0 \subseteq R(\sigma) + V_0$, $\sigma^{-1} V_0 = (\sigma^{-1} - 1 + 1)V_0 \subseteq R(\sigma^{-1}) + V_0 = R(\sigma) + V_0$, if V_0 is a submodule of V. In particular if $R\,(\sigma) \subseteq V_0$ then $\sigma^{\pm 1} V_0 \subseteq V_0$ and $\sigma V_0 = V_0$.

If g is an arbitrary endomorphism of module V such that $g\sigma = \sigma^{\pm 1} g$, then $g\,(\sigma - 1) = (\sigma^{\pm 1} - 1)\,g$ and $(\sigma - 1)\,g = g\,(\sigma^{\pm 1} - 1)$. That is why

$$gR(\sigma) \subseteq R(\sigma^{\pm 1}) = R(\sigma) \text{ and } gP(\sigma) \subseteq P(\sigma^{\pm 1}) = P(\sigma).$$

It is followed that if g is an automorphism of module V such that $g\sigma g^{-1} = \sigma^{\pm 1}$, then

$$gR(\sigma) = R(\sigma) \text{ and } gP(\sigma) = P(\sigma).$$

This statement also follows from the general formulas

$$gR(\sigma) = R(g\sigma g^{-1}) \text{ and } gP(\sigma) = P(g\sigma g^{-1}),$$

which due to the equality $g\sigma g^{-1} - 1 = g(\sigma - 1)g^{-1}$ is true for any endomorphism σ of module V and any isomorphism g of module V.

There are the obvious inclusions

$$R(\sigma_1\sigma_2) \subseteq R(\sigma_1) + R(\sigma_2),\ P(\sigma_1\sigma_2) \supseteq P(\sigma_1) \bigcap P(\sigma_2),$$

arising from the equalities $\sigma_1\sigma_2 - 1 = (\sigma_1 - 1)\sigma_2 + \sigma_2 - 1 = \sigma_1(\sigma_2 - 1) + \sigma_1 - 1$.

In particular if $[\sigma_1,\sigma_2] = \sigma_1\sigma_2\sigma_1^{-1}\sigma_2^{-1}$, then

$$R([\sigma_1,\sigma_2]) = R(\sigma_1) + R(\sigma_2\sigma_1\sigma_2^{-1}) \subseteq R(\sigma_1) + \sigma_2 R(\sigma_1) \subseteq R(\sigma_1) + R(\sigma_2),$$

$$P([\sigma_1,\sigma_2]) \supseteq P(\sigma_1) \bigcap P(\sigma_2).$$

It is obviously that $(\sigma_1 - 1)(\sigma_2 - 1) = (\sigma_2 - 1)(\sigma_1 - 1)$ if and only if $\sigma_1\sigma_2 = \sigma_2\sigma_1$ and $(\sigma_1 - 1)(\sigma_2 - 1) = (\sigma_2 - 1)(\sigma_1 - 1) = 0$ if and only if $\begin{cases} R(\sigma_1) \subseteq P(\sigma_2); \\ R(\sigma_2) \subseteq P(\sigma_1). \end{cases}$

Then, if $\begin{cases} R(\sigma_1) \subseteq P(\sigma_2); \\ R(\sigma_2) \subseteq P(\sigma_1), \end{cases}$ then $\sigma_1\sigma_2 = \sigma_2\sigma_1$.

If $\sigma_1\sigma_2 = \sigma_2\sigma_1$, then $(\sigma_1 - 1)(\sigma_2 - 1)V = (\sigma_2 - 1)(\sigma_1 - 1)V \subseteq R(\sigma_1) \bigcap R(\sigma_2)$, $(\sigma_1 - 1)(\sigma_2 - 1)(P(\sigma_1) + P(\sigma_2)) = (\sigma_2 - 1)(\sigma_1 - 1)(P(\sigma_1) + P(\sigma_2)) = 0$. Then, if $\begin{cases} \sigma_1\sigma_2 = \sigma_2\sigma_1; \\ R(\sigma_1) \bigcap R(\sigma_2) = 0 \text{ or } P(\sigma_1) + P(\sigma_2) = V, \end{cases}$ then $(\sigma_1 - 1)(\sigma_2 - 1) = (\sigma_2 - 1)(\sigma_1 - 1) = 0$. Then $\begin{cases} R(\sigma_1) \subseteq P(\sigma_2) \\ R(\sigma_2) \subseteq P(\sigma_1) \end{cases}$. It is easy to see that $\sigma^2 = 1$ if and only if $\sigma(\sigma - 1) = -(\sigma - 1)$ if and only if $\sigma\big|_{R(\sigma)} = -1$. It turns out that fixed and residual submodules of finite order, which is reversible in the ring are matched with the Peirce decomposition of idempotents which they defined.

Lemma 1. *Let R be an associative ring with 1, V is left R- module (not necessarily free), $\sigma \in GL(V)$, $\sigma^m = 1$, $m \in R^*$, $e = (1 + \sigma + \dots + \sigma^{m-1})m^{-1}$. Then $e^2 = e$ is an idempotent, $V = R(\sigma) \oplus P(\sigma)$, where $P(\sigma) = \{v \in V \,|(\sigma - 1)v = 0\} = eV$ and*

$$R(\sigma) = \{v \in V\,|(1 + \sigma + \dots + \sigma^{m-1})v = 0\} = (1 - e)V.$$

Proof. Because $e\sigma^i = \sigma^i e = e$ to all $i \geq 0$, then

$$e^2 = e(1 + \sigma + \dots + \sigma^{m-1})m^{-1} = e$$

is an idempotent and the Peirce decomposition is used $V = eV \oplus (1 - e)V$, where $v = ev + (1 - e)v$, $v \in V$. It is clear that

$$eV = \{v \in V\,|(1 - e)v = 0\} = \ker(1 - e)$$

and $(1 - e)V = \{v \in V\,|ev = 0\} = \ker e$. Because $e(1 - \sigma) = (1 - \sigma)e = 0$ and $1 - e = (1 - \sigma)t$, where $t \in EndV$ and $\sigma t = t\sigma$, then $eV \subseteq P(\sigma) \subseteq \ker(1 - e) = eV$ and $(1 - e)V \subseteq R(\sigma) \subseteq \ker e = (1 - e)V$. Thus it is proved that $P(\sigma) = eV = \ker(1 - e) = \{v \in V\,|(\sigma - 1)v = 0\}$, $R(\sigma) = (1 - e)V = \ker e = \left\{v \in V\,\middle|\left(1 + \sigma + \dots + \sigma^{m-1}\right)v = 0\right\}$.

Note that $\sigma-1$ is a reversible element to $R(\sigma)$. Indeed, $e-1=(\sigma^{m-1}-1+\cdots+\sigma-1)\,m^{-1}$, $eR(\sigma)=0$, $\sigma^{m-1}-1+\cdots+\sigma-1=-mE$ to $R(\sigma)$. Similarly, $\sigma^{-1}-1$ is a reversible element to $R(\sigma^{-1})=R(\sigma)$.

In particular, if $m=2\in K^*$, then $P(\sigma)=\{v\in V\,|\sigma v=v\}$, $R(\sigma)=\{v\in V\,|\sigma v=-v\}$.

If $m=3\in K^*$, then $\sigma^3=1$, $P(\sigma)=\{v\in V\,|\sigma v=v\}$, $R(\sigma)=\{v\in V\,|(1+\sigma+\sigma^2)\,v=0\}$.

Lemma 2. *Let K be an associative ring with 1, $m\in K^*$, $a,b\in EndV$, $a^m=b^m=1$, $ab=ba$. Then*

$$P(a)\bigcap P(ab)=P(a)\bigcap P(b)=P(b)\bigcap P(ab),$$
$$P(a)\bigcap R(ab)=P(a)\bigcap R(b)\,,\ P(b)\bigcap R(ab)=P(b)\bigcap R(a)\,,$$
$$R(a)\bigcap P(ab)\subseteq R(a)\bigcap R(b)\,,\ R(b)\bigcap P(ab)\subseteq R(b)\bigcap R(a)\,.$$

Proof. From the properties of fixed and residual submodules of the elements of finite order, which are described in Lemma 1 the equalities arise,

$$\begin{cases} P(a)\bigcap P(ab)=P(a)\bigcap P(b); \\ P(b)\bigcap P(ab)=P(a)\bigcap P(b), \end{cases} \quad \begin{cases} P(a)\bigcap R(ab)=P(a)\bigcap R(b); \\ P(b)\bigcap R(ab)=P(b)\bigcap R(a). \end{cases}$$

Let $v\in P(ab)$ be. Then $abv=v$ and $av=b^{-1}v$, $bv=a^{-1}v$. By induction $a^lv=b^{-l}v$, $b^lv=a^{-l}v$ to all $l\geq 0$. That is why $R(a)\bigcap P(ab)\subseteq R(a)\bigcap R(b)$, $R(b)\bigcap P(ab)\subseteq R(b)\bigcap R(a)$.

Corollary 1. *Let K be an associative ring with 1, $m\in K^*$, $a,b\in EndV$, $a^m=b^m=1$, $ab=ba$. If $m=2\in K^*$, then $R(a)\bigcap R(b)=R(a)\bigcap P(ab)=R(b)\bigcap P(ab)$. If $m=3\in K^*$, then $b=a$ on $R(a)\bigcap R(b)\bigcap R(ab)$ and $b=a^2$ on $R(a)\bigcap R(b)\bigcap P(ab)$.*

Proof. In the case of $m=2\in K^*$ the inclusions of Lemma 2 are converted to equality. Indeed, in this case, $R(a)\bigcap R(b)=\{v\in V\,|av=-v,bv=-v\}\subseteq\{v\in V\,|abv=v\}\subseteq P(ab)$. That is why $R(a)\bigcap R(b)=R(a)\bigcap P(ab)=R(b)\bigcap P(ab)$. In particular,

$$R(a)\bigcap R(b)\bigcap R(ab)=0\ ,\ R(a)\bigcap R(b)\bigcap P(ab)=R(a)\bigcap R(b)\ .$$

In the case of $m=3$ it is revealed $(b-a)\,\nu=0$, if $\nu\in R(a)\bigcap R(b)\bigcap R(ab)$ and $(b-a^2)\,\nu=0$, if $\nu\in R(a)\bigcap R(b)\bigcap P(ab)$. Indeed, if $\nu\in R(a)\bigcap R(b)\bigcap R(ab)$, then

$$\left(a^2+a+1\right)\nu=\left(b^2+b+1\right)\nu=\left((ab)^2+ab+1\right)\nu=0$$

So, $(ab-1)\,(a-b)\,\nu=(a^2+ab+a)(a-b)\nu=a\,(a+b+1)\,(a-b)\,\nu=a\left(a^2-b^2+a-b\right)\nu=0$. Consequently there is the equality

$$0=\left((ab)^2+ab+1\right)(a-b)\,\nu=3\,(a-b)\,\nu.$$

Thus it is proved that $(a-b)\,\nu=0$ for $\nu\in R(a)\bigcap R(b)\bigcap R(ab)$.

Obviously, if $\nu\in R(a)\bigcap R(b)\bigcap P(ab)$, then $ab\nu=\nu$ and $(b-a^2)\,\nu=0$.

Lemma 3. *Let a, b be some elements of associative ring K with 1, $3\in K^*$ such that $b^2=1$, $a^2+a+1=0$, $bab^{-1}=a^2$, $e=(1-a)\,(1-b)\,3^{-1}$. Then $e^2=e$, $eae=(1-e)\,a^2\,(1-e)=0$.*

Proof. It is hard not to see that $b\,(1-b)=-(1-b)$ and $(1-b)\,(1-a)\,(1-b)=(1-a-b+ba)\,(1-b)=(1-a+1-a^2)\,(1-b)=3\,(1-b)$. That is why $(3e)^2=(1-a)\,3\,(1-b)=9e$ and $e^2=e$. Similarly, it can be proved that $(1-b)\,a\,(1-a)\ (1-b)=(a-a^2-ba+ba^2)\,(1-b)=(a-a^2+a^2-a)\,(1-b)=$

0 . That is why $9eae = (1-a)(1-b)a(1-a)(1-b) = 0$ and $eae = 0$. So, $ea^2e = e(-1-a)e = -e$. It is easy to see that $(1-b)a^2 + a^2(1-b) = 2a^2 - (a+a^2)b = 2a^2 + b$. That is why $3(ea^2 + a^2e) = (1-a)((1-b)a^2 + a^2(1-b)) = (1-a)(2a^2+b)$. Thus it is proved that

$$3(1-e)a^2(1-e) = 3a^2 - 3(ea^2 + a^2e) + 3ea^2e = 3a^2 - (1-a)(2a^2+b) - 3e = \\ = 3a^2 - (1-a)(2a^2+b+1-b) = 3a^2 - (1-a)(2a^2+1) = 0$$

and $(1-e)a^2(1-e) = 0$.

Lemma 3 implies that $ae = (1-e)ae$ and $a^2(1-e) = ea^2(1-e)$. It is possible to convince that $e_1 = e - ab$ is also an idempotent which satisfies the equality $e_1ae_1 = (1-e_1)a^2(1-e_1) = 0$. Besides that $(a^2b-1)e_1 = 0$. It can be proved that e_1 is unambiguously certain idempotent which is a linear combination of elements of group $\langle a, b\rangle$ with the whole coefficients and satisfies above-mentioned equalities.

2. Image of endomorphism by formal matrices.

Lemma 4. *Let K be an associative ring with $1, 3 \in K^\star$, W be left K-module, a, b be elements $GL(W)$ such that $a^3 = b^2 = 1$, $bab^{-1} = a^{-1}$. Then there are isomorphism modules $g : W \longrightarrow W_g$, which induces isomorphism $\Lambda_g : GL(W) \longrightarrow GL(W_g)$ so that the elements $\Lambda_g a$, $\Lambda_g b$ can be represented by formal matrices*

$$\Lambda_g a = diag\left(\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}, 1\right), \ \Lambda_g b = diag\left(\begin{pmatrix} \alpha & \beta \\ \alpha+\beta & -\alpha \end{pmatrix}, \gamma\right).$$

where $\alpha, \beta \in EndL$, $\gamma \in EndP$, $\alpha\beta = \beta\alpha$, $\alpha^2+\alpha\beta+\beta^2 = 1$, $\gamma^2 = 1$, $W_g = L\oplus L\oplus P$. In particular, if $W = R(a)$, then

$$\Lambda_g a = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}, \ \Lambda_g b = \begin{pmatrix} \alpha & \beta \\ \alpha+\beta & -\alpha \end{pmatrix}.$$

Proof. Let $R(a) = (a-1)W$ and $P(a) = \ker(a-1)$. Submodule $R(a)$ and $P(a)$ is relatively invariant a and b, $a^2+a+1 = 0$ for submodules $R(a)$ and $a = 1$ for submodules $P(a)$. Let $e = (1-a)(1-b)3^{-1}$, where 1 means a unit of $GL(W)$. Obviously, submodules $R(a)$ and $P(a)$ are relatively invariant e. Narrowing items a, b, e for submodules $R(a)$ satisfying lemma 4. Because $eP(a) \subseteq (1-a)P(a) = 0$, then $e^2 = e = 0$ on $P(a)$. Therefore $e^2 = e$ – idempotent on $R(a)$. This means that $e^2 = e$ – idempotent rings $EndW$, which defines the schedule module W,

$$W = R(a) \oplus P(a) = eR(a) \oplus (1-e)R(a) \oplus P(a), \text{ where}$$
$$R(a) = eR(a) \oplus (1-e)R(a).$$

Let $L = eR(a)$, $P = P(a)$. Since $a \neq 1$, then $R(a) \neq 0$. Under the lemma 4 $aeR(a) \subseteq (1-e)R(a)$ and $a^2(1-e)R(a) \subseteq eR(a)$, $(1-e)R(a) \subseteq aeR(a)$. So, $(1-e)R(a) = aeR(a) = aL$. Thus it is proved that $R(a) = L \oplus aL$, $L \neq 0$, $W = L \oplus aL \oplus P$. Let $W_g = L \oplus L \oplus P$ and $g : W \longrightarrow W_g$ be an isomorphism of modules, which is defined by the rules $g(l_1 + al_2 + p) = l_1 + l_2 + p$, where $l_1 \in L$, $1 \leq i \leq 2$, $p \in P$, and $\Lambda_g : GL(W) \longrightarrow GL(W_g)$ – induced g group isomorphism. This means that the elements of the ring $End(W_g)$ can be represented by formal 3×3 matrices

$$\Lambda_g a = diag\left(\begin{pmatrix} 0 & a_1 \\ 1 & a_2 \end{pmatrix}, 1\right), \ \Lambda_g b = diag\left(\begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}, \gamma\right).$$

Given equality $(1+a+a^2)R(a) = 0$ get that $a_1 = a_2 = -1$. With equality $ba = a^{-1}b$ it follows that $b_3 = b_2$, $b_4 = -b_1$ and with equality $b^2 = 1$ get that $b_1b_2 = b_2b_1$, $b_1^2 + b_1b_2 + b_2^2 = 1$. Let $\alpha = b_1$ and $\beta = b_2$. Then

$$\Lambda_g a = diag\left(\left(\begin{array}{cc} 0 & -1 \\ 1 & -1 \end{array}\right), 1\right), \Lambda_g b = diag\left(\left(\begin{array}{cc} \alpha & \beta \\ \alpha+\beta & -\alpha \end{array}\right), \gamma\right)$$

where $\alpha, \beta \in EndL$, $\gamma \in EndP$, $\alpha\beta = \beta\alpha$, $\alpha^2 + \alpha\beta + \beta^2 = 1$. $\gamma^2 = 1$.

If instead of the idempotent e we choose the idempotent $e_1 = e - ab$ it is possible to prove that $\alpha = 0$ as $\beta = 1$.

Lemma 5. *Let K be an associative ring with 1, $3 \in K^*$, V be a left K-module, $a, b \in GL(V)$, $a^3 = b^3 = 1$, $ab = ba$. Then a and b can be represented by the formal matrices $a = diag(z, E, x, y, E)$, $b = diag(E, z_1, x, y^2, E)$, where $x^2 + x + 1 = 0$, $y^2 + y + 1 = 0$, $z^2 + z + 1 = 0$, $z_1^2 + z + 1 = 0$.*

Proof. Submodules $R(a)$, $R(b)$, $P(a)$, $P(b)$ are invariant relatively to the elements a and b and there are decompositions $V = R(a) \oplus P(a)$, $R(a) = R(a) \bigcap R(b) \oplus R(a) \bigcap P(b)$, $P(a) = P(a) \bigcap R(b) \oplus P(a) \bigcap P(b)$ Because of $(ab)^3 = 1$, the decompositions is also occured

$$R(a) \bigcap R(b) = R(a) \bigcap R(b) \bigcap R(ab) \oplus R(a) \bigcap R(b) \bigcap P(ab)$$

This means that there is a decomposition of the module V in to the direct sum of modules (some of which may be zero)

$$V = R(a) \bigcap P(b) \oplus P(a) \bigcap R(b) \oplus R(a) \bigcap R(b) \bigcap R(ab) \oplus$$
$$\oplus R(a) \bigcap R(b) \bigcap P(ab) \oplus P(a) \bigcap P(b)$$

Thus it is proved that the elements a and b have a image which is shown in Lemma 5. Because of $(x-1)(x+2) = -3 = (x^2-1)(x+1)$, then $x - 1$, $x^2 - 1$ and similarly $y - 1$, $y^2 - 1$, $z - 1$, $z^2 - 1$, $z_1 - 1$, $z_1^2 - 1$ are circulating on the respective non-zero submodules. It is followed from Lemma 5 that if such an element $t \in EndV$ commutes with the product $ab = diag(z, z_1, x^2, E, E)$ then t has a form of $t = diag(t_1, t_2)$, where t_1 commutes with $diag(z, z_1, x^2)$. In particular if $t \in GL(V)$, $R(a) \bigcap P(b) = R(b) \bigcap P(a) = 0$, then $V = R(a) \bigcap R(b) \oplus P(a) \bigcap P(b)$, $ab = diag(x^2, E, E)$, $ab^2 = diag(E, y^2, E)$, t_1 commute with x^2 and as it followed that with $x = -x^2 + 1$, $[a, t] = diag(E, *)$, $[b, t] = diag(E, *)$. In this case the elements in the form of $diag(T, 0, 0)$ commute with the elements ab^2 , $[a, t]$, $[b, t]$ for any $T \in End(R(a) \bigcap R(b) \bigcap R(ab))$.

Lemma 6. *Let K be an associative ring with 1, $3 \in K^*$, V be a left K-module, $a, b, c, d, t \in GL(V)$, $a^3 = b^3 = 1$, $ab = ba$, $b \neq a^2$, $cac^{-1} = a^2$, $cbc^{-1} = b^2$, $c^2 = 1$, $dad^{-1} = b$, $d^2 = 1$, $tab = abt$. Let to any $m \in EndV$, $m^2 = 0$ in condition of m commutes with ab^2 , $[a, t]$, $[b, t]$ it is followed that m commutes with a. Then $R(a) \bigcap P(b) \neq 0$.*

Proof. Let $R(a) \bigcap P(b) = 0$. Then $R(b) \bigcap P(a) = d(R(a) \bigcap P(b)) = 0$, $V = R(a) \bigcap R(b) \bigcap R(ab) \oplus R(a) \bigcap R(b) \bigcap P(ab) \oplus P(a) \bigcap P(b)$, $a = diag(x, y, E)$, $b = diag(x, y^2, E)$, where $x^2 + x + 1 = 0$, $y^2 + y + 1 = 0$. Because of $b \neq a^2$, then $R(a) \bigcap R(b) \bigcap R(ab) \neq 0$. According to Lemma 4 we can assume that $x = \left(\begin{array}{cc} 0 & -1 \\ 1 & -1 \end{array}\right)$. Let $m = diag\left(\left(\begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array}\right), 0, 0\right)$. As noted above, m commutes with ab^2 , $[a, t]$, $[b, t]$. According to the condition m commutes with a. However, according to the form m does not commutes with a. From this contradiction it is followed that $R(a) \bigcap P(b) \neq 0$.

Theorem 1. *Let K be an associative ring with 1, $2 \in K^*$, W be left K-module, a,b,c,d be the elements of group $GL(W)$ such that $a^2 = b^2 = 1$, $ab = ba$, $ca = ac$, $cbc^{-1} = ab$, $db = bd$, $dad^{-1} = ab$, $a \neq 1$. Then there is the isomorphism of modules $g : W \to W_g$, which induces isomorphism $\Lambda_g : GL(W) \to GL(W_g)$ so that the elements $\Lambda_g a$, $\Lambda_g b$, $\Lambda_g c$, $\Lambda_g d$ can be represented by formal matrices $\Lambda_g a = diag(-1,-1,1,1)$, $\Lambda_g b = diag(1,-1,-1,1)$, $\Lambda_g c = diag\left(\begin{pmatrix} 0 & \alpha \\ 1 & 0 \end{pmatrix}, \beta, \gamma\right)$, $\Lambda_g d = diag\left(\beta_1, \begin{pmatrix} 0 & \alpha_1 \\ 1 & 0 \end{pmatrix}, \gamma_1\right)$, where $\alpha, \beta, \alpha_1, \beta_1 \in EndL$, $\gamma, \gamma_1 \in EndP$, $W_g = L \oplus L \oplus L \oplus P$.*

Proof. By condition $R(a) \neq 0, bR(a) = R(a), bP(a) = P(a)$. Therefore, there is a decomposition $W = R(a) \oplus P(a) = R(a) \bigcap R(b) \oplus R(a) \bigcap P(b) \oplus P(a) \bigcap R(b) \oplus P(a) \bigcap P(b)$. Let $L = R(a) \bigcap P(b)$, $P = P(a) \bigcap P(b)$. Then $cL = R(a) \bigcap P(ab) = R(a) \bigcap R(b)$ and $dcL = R(ab) \bigcap R(b) = P(a) \bigcap R(b)$. Because of $R(a) = L \oplus cL \neq 0$, than $L \neq 0$ and $W = L \oplus cL \oplus dcL \oplus P$, where $W_g = L \oplus L \oplus L \oplus P$. Let us consider the isomorphism of the modules $g : W \to W_g$, which is defined by the rule $g(l_1 + cl_2 + dcl_3 + p) = l_1 + l_2 + l_3 + p$, where $l_i \in L$, $1 \leq i \leq 3$, $p \in P$ and group homomorphism $\Lambda_g : GL(W) \to GL(W_g)$ which is induced by the isomorphism of the modules $g : W \to W_g$, where $\Lambda_g \sigma = g\sigma g^{-1}$ for all. We represent the elements of the ring by formal matrices. In particular, we find that $\Lambda_g a = diag(-1,-1,1,1)$, $\Lambda_g b = diag(1,-1,-1,1)$, where 1 is a unit of $EndL$ or $EndP$ a ring respectively. Beside this,

$$c^2L = cR(a) \cap P(b) = R(a) \cap P(ab) = L,$$
$$cdcL = cdR(a) \cap R(b) = P(a) \cap R(ab) = cL,$$
$$cP = P(a) \cap P(ab) = P.$$

Therefore

$$\Lambda_g c = diag\left(\begin{pmatrix} 0 & \alpha \\ 1 & 0 \end{pmatrix}, \beta, \gamma\right),$$

where $\alpha, \beta \in EndL$, $\gamma \in EndP$. Similarly proved that $dcL = cL$, $d^2L = L$, $dP = P$ and

$$\Lambda_g d = diag\left(\beta_1, \begin{pmatrix} 0 & \alpha_1 \\ 1 & 0 \end{pmatrix}, \gamma_1\right),$$

where $\alpha_1, \beta_1 \in EndL$, $\gamma_1 \in EndP$.

In particular, if $c^2 = a$, then $\alpha = -1$, $\beta^2 = 1$, $\gamma^2 = 1$. If $c^2 = 1$, then $\alpha = 1$, $\beta^2 = 1$, $\gamma^2 = 1$. Thus, Theorem 1 is proved.

Remark 1. *If G be a group such that $E(n,R) \subseteq G \subseteq GL(n,R)$, where R is an associative ring with $1, n \geq 3$ and $\Lambda : G \longrightarrow GL(W)$ is non-trivial arbitrary homomorphism, who in the group $GL(W)$ as elements a,b,c,d, appearing in the Theorem 1, provided $\Lambda t_{ij}(2) \neq 1$ you can choose*

$$a = \Lambda diag(-1,-1,1,\ldots,1),\ b = \Lambda diag(1,-1,-1,1,\ldots,1),$$
$$c = \Lambda diag\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, 1,\ldots,1\right),\ d = \Lambda diag\left(1, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, 1,\ldots,1\right).$$

This $c^2 = a$, $d^2 = b$. If $\Lambda t_{ij}(2) \neq 1$ for some, and hence for all $1 \leq i \neq j \leq n$, then as elements a,b,c,d elements can be selected $a = \Lambda t_{12}(1)$, $b = \Lambda t_{13}(1)$, $c = \Lambda t_{32}(-1)$, $d = \Lambda t_{23}(-1)$.

In fact, according to the formula $[t_{ij}, t_{jk}(1), t_{ij}(r)] = t_{ik}(-r)$, where $1 \leq$

$i, j, k \leq n$ are pairly different numbers, there is an inequality $a \neq 1$.

Theorem 2. *Let K be an associative ring with 1, $3 \in K^*$, W be a left K-module, a,b,c,d are the elements of group $GL(W)$ such that $a^3 = b^3 = 1$, $ab = ba$, $cac^{-1} = a^{-1}$, $cbc^{-1} = b^{-1}$, $c^2 = 1$, $dad^{-1} = b$, $d^2 = 1$, $dc = cd$, $R(a) \bigcap P(b) \neq 0$. Then there is the isomorphism of modules $g : W \to W_g$, which induces the isomorphism of group $\Lambda_g : GL(W) \to GL(W_g)$ so that the elements $\Lambda_g a$, $\Lambda_g b$, $\Lambda_g c$, $\Lambda_g d$ can be represented by formal matrices $\Lambda_g a = diag\left(\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}, 1, 1, \alpha\right)$, $\Lambda_g b = diag\left(1, 1, \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}, \beta\right)$, $\Lambda_g c = diag\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \gamma\right)$, $\Lambda_g d = diag\left(\begin{pmatrix} 0 & E \\ E & 0 \end{pmatrix}, \delta\right)$ where $\alpha, \beta, \gamma, \delta \in EndP$, $\alpha^2 = \beta^2 = 1$, $\gamma^2 = \delta^2 = 1$, $\alpha\beta = \beta\alpha$, $\gamma\alpha = \alpha^2\gamma$, $\delta\alpha = \beta\delta$, $E = diag(1, 1)$, 1 is a unit $EndL$ or $EndP$ respectively.*

Proof. Let $e = (1 - a)(1 - c) 3^{-1}$, $f = (1 - b)(1 - c) 3^{-1}$ as in the Lemma 4. Then $e^2 = e$, $eae = 0$, $f^2 = f$, $fbf = 0$, $ded^{-1} = f$, $dfd^{-1} = e$. $W = R(a) \bigcap P(b) \oplus R(b) \bigcap P(a) \oplus R(a) \bigcap R(b) \oplus P(a) \bigcap P(b)$, $R(a) = eR(a) \oplus (1 - e) R(a)$, $R(b) = fR(b) \oplus (1 - e) R(b)$ As in the Lemma 4 we have $ceR(a) = (1 - e) R(a)$, $cfR(b) = (1 - f) R(b)$. Under the condition $dR(a) \bigcap P(b) = R(b) \bigcap P(a)$. Let $L = eR(a) \bigcap P(b)$, $P = R(a) \bigcap R(b) \oplus P(a) \bigcap P(b)$. Then $R(a) \bigcap P(b) = L \oplus cL$, $L \neq 0$, $dL = fR(b) \bigcap P(a)$, $R(b) \bigcap P(a) = dL \oplus dcL$. Thus it is proved that $W = L \oplus cL \oplus dL \oplus dcL \oplus P$. Let $W_g = L \oplus L \oplus L \oplus L \oplus P$, $g : W \to W_g$ be an isomorphism of modules, which is defined by the rule $g(l_1 + cl_2 + dl_3 + dcl_4 + p) = l_1 + l_2 + l_3 + l_4 + p$, where $l_i \in L$, $1 \leq i \leq 4$, $p \in P$ and $\Lambda_g : GL(W) \to GL(W_g)$ is an induced group homomorphism. Represent the elements of the ring $End(W_g)$ by formal 5×5 matrices $\Lambda_g a = diag\left(\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}, 1, 1, \alpha\right)$, $\Lambda_g b = diag\left(1, 1, \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}, \beta\right)$, $\Lambda_g c = diag\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \gamma\right)$, $\Lambda_g d = diag\left(\begin{pmatrix} 0 & A^{-1} \\ A & 0 \end{pmatrix}, \delta\right)$ where $\alpha, \beta, \gamma, \delta \in EndP$ and $A \in (EndL)_2$ are formal 2×2 matrix that commute with formal 2×2 matrices $\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, where $1 \in EndL$. Therefore, up to conjugation, in the formal matrix $diag(A, 1, 1)$ we can assume that $A = 1$. Thus, Theorem 2 is proved.

Remark 2. *If G is a group such that $E(n, R) \subseteq G \subseteq GL(n, R)$, where R is an associative ring with 1, $n \geq 4$, and $\Lambda : G \to GL(W)$ is an arbitrary non-trivial homomorphism with condition (*) on $E(n, R)$, then the elements a,b,c,d, which appear in the theorem 2 in group you can choose $a = \Lambda diag\left(\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}, 1, ..., 1\right)$, $b = \Lambda diag\left(1, 1, \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}, 1, ..., 1\right)$, $c = \Lambda diag\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, 1, ..., 1\right)$, $d = \Lambda diag\left(\begin{pmatrix} 0 & E \\ E & 0 \end{pmatrix}, 1, ..., 1\right)$, where $E = diag(1, 1)$ is a single 2×2 matrix.*

In fact, according to the formula $[t_{ij} t_{ij}(-1), t_{jk}(1), t_{ji}(r)] = t_{ik}(-r)$, where $1 \leq i, j, k \leq n$ are pairly different numbers, there is an inequality $a \neq b^2$. As Λ is a homomorphism with the condition (*), so all the other conditions of Lemma 6 are fulfilled. Therefore, if you put $t = \Lambda diag\left(\begin{pmatrix} E & E \\ 0 & E \end{pmatrix}, E\right)$ in Lemma 6, then t commutes with ab, where $a = \Lambda diag\,(A, E, E)$, $b = \Lambda diag\,(E, A, E)$, $A = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} \in GL\,(2, R)$. Let m be an arbitrary element of the ring $EndW$, $m^2 = 0$, which commutes with ab^2 , $[a, t]$, $[b, t]$. It can be considered that $m \neq 0$. Under condition (*) there is an element $h \in GL\,(n, R)$ such that $\Lambda h = 1 + s_1 m$ and h^{s_2} commutes with $diag\,(A, A^2, E)$, $diag\left(\begin{pmatrix} E & A-E \\ 0 & E \end{pmatrix}, E\right)$, $diag\left(\begin{pmatrix} E & 0 \\ A-E & E \end{pmatrix}, E\right)$. In this case, as the test shows, h^{s_2} commutes with $diag\,(A, E, E)$. That is why the element $\Lambda h^{s_2} = 1 + s_1 s_2 m$ and, consequently, the element $m$commute with a. According to the Lemma 6 $R\,(a) \bigcap P\,(b) \neq 0$. Therefore the above mentioned elements a, b, c, d satisfy the conditions of the theorem 2.

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