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TRANSITION GRAPHS OF ITERATIONS OF INITIAL (2, 2)-AUTOMATA

The iterations of an automaton A naturally produces a sequence of finite graphs $G_A(n)$ which describe the transitions in $A^{(n)} = A \circ A \circ \ldots \circ A$ (n times). We consider combinatorial properties of the graphs $G_A(n)$ for initial invertible automata with two states over the binary alphabet. We compute the chromatic number and girth of the graphs $G_A(n)$ and show that all of them are imbalance graphic.

Ітерації автомата A природньо породжують послідовність скінченних графів $G_A(n)$, що описують переходи в автоматах $A^{(n)} = A \circ A \circ \ldots \circ A$ (n разів). Ми розглядаємо комбінаторні властивості графів $G_A(n)$ для ініціальних оборотних автоматів з двома станами над бінарним алфавітом. У статті пораховано хроматичне число і обхват для графів $G_A(n)$ і доведено, що всі вони є імбалансно графічними.

1. Introduction. Let a be a Mealy automaton with the same input-output alphabet. Then we can consider the sequence of its iterations $(a^n)_{n\geq 1}$, where the n-th iteration a^n is the minimization of $a \circ \ldots \circ a$ (n times composition of a with itself). The study of iterations of invertible automata is at the heart of famous examples of Burnside automaton groups and groups of intermediate growth (see [6, 9]).

Important information about a^n is contained in its transition graph whose vertices are the states of a^n and arrows correspond to transitions. We will be interested in the graph $G_a(n)$, which is a simple graph obtained from the transition graph of a^n by ignoring loops, directions, and multiple edges. The graphs $G_a(n)$ for non-initial automata were intensively studied for the last twenty years as Schreier graphs of automaton groups (see [2–4] and the references therein). In particular, the study of spectrum of graphs $G_a(n)$ for certain automaton lead to the solution of the Atiyah problem about the range of L^2-Betti numbers of closed manifolds (see [5]).

In this paper we study the graphs $G_a(n)$ for initial invertible automata with two states over the binary alphabet or just $(2, 2)$-automata for short. This class contains 18 minimal automata, eleven of which have finite order and the corresponding sequence $(G_a(n))_{n\geq 1}$ consists of at most two graphs. The sequence $(G_a(n))_{n\geq 1}$ contains infinitely many graphs for the seven $(2, 2)$-automata of infinite order, which are the adding machine, two states of the cyclic automaton (they generate C_∞), and two states of the lamplighter automaton and its inverse (they generate the lamplighter group $\mathbb{Z}_2 \wr \mathbb{Z}$). The growth function for every $(2, 2)$-automaton a, which computes the number of vertices of $G_a(n)$, i.e., the number of states of a^n, was calculated in [10]. We compute the chromatic number $\chi(G)$ and girth $g(G)$ of these graphs.

Theorem 1. *The chromatic number and girth of the graphs $G_a(n)$ for a $(2, 2)$-automaton a are the following:*

- *if a is trivial or acts as permutation of every letter, then $G_a(n)$ is acyclic and $\chi(G_a(n)) = 1$ for all $n \geq 1$;*
- *if a has order two and do not act as permutation of every letter, then $G_a(n)$ is acyclic and $\chi(G_a(n)) = 1 + (n \bmod 2)$ for all $n \geq 1$;*

- *if a is the adding machine or a state of the cyclic automaton, then $G_a(n)$ is acyclic and $\chi(G_a(n)) = 2$ for $n = 2^k$, $k \geq 0$ and $g(G_a(n)) = \chi(G_a(n)) = 3$ otherwise;*
- *if a is a state of the lamplighter automaton or its inverse, then $G_a(1)$ is acyclic with $\chi(G_a(1)) = 2$ and $g(G_a(n)) = \chi(G_a(n)) = 3$ for $n \geq 2$.*

Also we consider graph imbalances introduced in [1] as a measure of graph irregularity. A graph is imbalance graphic if its imbalance multiset coincides with a degree multiset for some other graph. This property was studied in [8]. We prove

Theorem 2. *Let a be a $(2,2)$-automaton. Then the graph $G_a(n)$ is imbalance graphic for every $n \in \mathbb{N}$.*

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2. Preliminaries. In this section we recall necessary information on chromatic number, girth and imbalances of graphs (see [7] for more information and references). Throughout the paper we consider only finite simple undirected graphs without loops.

Let G be a graph. The *chromatic number* $\chi(G)$ is the smallest number of colors that can be used to color the vertices of G in such a way that no two adjacent vertices share the same color. We will use the following well-known theorem that gives us an upper bound for the graph chromatic number based on the maximal vertex degree.

Theorem 3 (Brooks, 1941)**.** *Let G be a connected graph with the maximal vertex degree d. Then $\chi(G)$ is at most $d+1$. Moreover, $\chi(G) = d+1$ if and only if G is a complete graph or an odd cycle.*

The *girth* $g(G)$ of a graph G is the length of the smallest cycle in G or infinity if there are no cycles. If the girth is high, then locally around every vertex the graph looks like a tree, and one could expect that its chromatic number is small. However, this is not the case; in 1959 Erdös has proved using probabilistic arguments that for any positive integers χ and γ there exist graphs with chromatic number χ and girth γ. Since then, many explicit constructions of such graphs were proposed.

Edge imbalances of graphs were introduced in [1] as a tool to investigate graph irregularity. Let G be a graph with the vertex set $V(G)$ and the edge set $E(G)$. The degree of a vertex $v \in V(G)$ will be denoted by $d(v)$.

Definition 1. *Let $e \in E(G)$ be an edge which is incident to the vertices u and v. The imbalance of the edge e is defined as $imb(e) = |d(u) - d(v)|$.*

For a graph G we consider the following two multisets: $M(G)$ is the multiset of all edge imbalances and $D(G)$ is the multiset of vertex degrees.

Definition 2. *A multiset M is graphic if there exists some graph Γ such that $D(\Gamma) = M$. A graph G is called imbalance graphic if its imbalance multiset is graphic.*

Every path and every regular graph are imbalance graphic. Examples of graphs that are not imbalance graphic were constructed in [8].

We need the following lemma for further proofs.

Lemma 1. *Let G be a graph such that its imbalance multiset $M(G)$ contains only values* 0, 1 *and* 2 *with the following property. If there are exactly* 1 *or* 2 *imbalances with value* 2 *then there exists an imbalance with value* 1 *in $M(G)$. Then $M(G)$ is graphic.*

Proof. Let us construct the graph Γ such that $D(\Gamma) = M(G)$. Note that every 0 imbalance can be always realized by some isolated vertex. In addition, we know that the sum of all imbalances for any graph is even. This means that in the given situation we have an even number of values 1 in $M(G)$.

If we have at least three values 2 in $M(G)$ then we can realize them by a cycle. Otherwise, we can construct the path of length 1, 2 or 3 depending on how many values 2 there are in $M(G)$. This is possible as the multiset $M(G)$ contains at least two values 1. All other values 1 can be realized by paths of length 1. Therefore, $M(G)$ is graphic.

3. Automata and their transition graphs. In this section we recall necessary information on automata-transducers (see [6] for more details).

We consider *automata* given by triples $A = (X, S, \lambda)$, where X is a finite set (input-output alphabet), S is a finite set of states, and $\lambda : S \times X \to X \times S$ is an output-transition map. An *initial automaton* is an automaton $A = (X, S, \lambda)$ with a fixed initial state $a \in S$. We will denote initial automaton by its initial state a.

Let X^* be the set of all words over X. Then every initial automaton a defines a transformation of X^* as follows. The image of an input word $x_1x_2 \dots x_n$ is defined recursively by the rule:

$$a(x_1x_2 \dots x_n) = y_1b(x_2 \dots x_n), \text{ if } \lambda(a, x_1) = (y_1, b).$$

An automaton a is called *invertible* if the corresponding transformation of X^* is invertible.

Two initial automata over X are called *equivalent* if they define the same transformation of X^*. An initial automaton is called *minimal* if it has the minimal number of states among the equivalent automata. Every automaton can be minimized using the classical Hopcroft's algorithm (1971). Note that every automaton transformation can be defined by a unique minimal automaton. Since we are going to work only with minimal automata, we can identify initial automata and the corresponding transformations of X^*.

Definition 3. *The n-th iteration a^n of an initial automaton a is the minimal automaton which defines the n-th iteration of the transformation defined by a.*

In other words, we define a composition of automata via the composition of corresponding transformations. Note that this agrees with the standard composition of automata, where the output of the first automaton is connected to the input of the second automaton.

Definition 4. *An initial automaton a has finite order if there exists a positive integer n such that a^n defines the trivial transformation of X^*.*

In order to simplify presentation of automata and calculation of automaton composition, people consider wreath recursion notation for automata.

Definition 5. *A state s_2 of an automaton is a projection of a state s_1 if there exists a letter $x \in X$ such that $\lambda(s_1, x) = (y, s_2)$ for some letter $y \in X$.*

Every state s of an automaton over $X = \{1, 2, \ldots, d\}$ can be written using its projections in the *wreath recursion* notation $s = (s_1, s_2, \ldots, s_d)\pi_s$, where $\pi_s : X \to X$ is a map on the alphabet defined by a and $\lambda(s, i) = (\pi_s(i), s_i)$. Note that every automaton can be uniquely given by the system of wreath recursion for all of its states.

Definition 6. *Let a be a minimal initial automaton. For every $n \in \mathbb{N}$ we define the graph $G_a(n)$ with the vertex set $V(G_a(n)) = States(a^n)$, where two vertices s_1 and s_2 are adjacent if one of them is a projection of another.*

In other words, the graph $G_a(n)$ is a simple graph obtained from the transition graph of a^n by ignoring loops, directions, and multiple edges.

Definition 7. *An initial finite automaton a is called imbalance graphic if for every $n \in \mathbb{N}$ the graph $G_a(n)$ is imbalance graphic.*

Let us note that not all automata are imbalance graphic. For example, the following wreath recursion defines an automaton with six states over the binary alphabet $X = \{1, 2\}$ that is not imbalance graphic:

$$\begin{aligned} a &= (a, b)\sigma, & b &= (c, d)\sigma, \\ c &= (d, e), & d &= (e, e)\sigma, \\ e &= (e, f)\sigma, & f &= (f, f), \end{aligned}$$

where σ is the transposition $(1, 2)$. Indeed, the graph $G_a(1)$ has two imbalances of value 2 and all other imbalances are equal to 0. Such a multiset is not graphic.

4. The graphs $G_a(n)$ for initial $(2, 2)$-automata. Up to symmetry and letters interchanging, there are ten minimal non-initial invertible automata with two states $S = \{a, b\}$ over the alphabet $X = \{1, 2\}$:

1) $b = (b, b)$, $a = (a, a)\sigma$;

2) $b = (b, b)$, $a = (b, b)\sigma$;

3) $b = (a, a)$, $a = (a, a)\sigma$;

4) $b = (a, a)$, $a = (b, b)\sigma$;

5) $b = (a, b)$, $a = (a, a)\sigma$;

6) $b = (a, b)$, $a = (b, b)\sigma$;

7) $b = (b, b)$, $a = (b, a)\sigma$ (the adding machine);

8) $b = (a, a)$, $a = (a, b)\sigma$ (the cyclic automaton);

9) $b = (a, b)$, $a = (b, a)\sigma$ (the lamplighter automaton);

10) $b = (a, b)$, $a = (a, b)\sigma$ (inverse to the lamplighter automaton),

where σ is the transposition $(1, 2)$. By fixing an initial state, we get eleven initial $(2, 2)$-automata of finite order and seven automata of infinite order.

For automata of finite order, the order is one or two. This means that it is enough to consider only the graphs $G_a(1)$ and $G_a(2)$. These graphs are acyclic and imbalance graphic. The chromatic number is $\chi(G_a(n)) = 1$ for two of these automata and $\chi(G_a(n)) = 1 + (n \mod 2)$ for the other and for all $n \geq 1$. We have checked Theorems 1 and 2 for automata of finite order.

Further we consider one by one the seven automata of infinite order.

Proposition 1. *Let a be the adding machine. Then the graph $G_a(n)$ is imbalance graphic for every $n \in \mathbb{N}$.*

Proof. The structure of the graphs $G_a(n)$ is described in [10]. For $n = 2^k$ the graph $G_a(n)$ is a path of length $k + 2$. Therefore, it is imbalance graphic.

Now we consider other values of n. Every state a^m, $2 < m < n$, has one or two projections. Since the automaton is minimal, each state except a^n is a projection of some state a^l. Each automaton can be a projection for at most two other states. As a result we get that the vertex which corresponds to the state a^m can have only degrees from the set $\{2, 3, 4\}$. Moreover, the vertex a^n has degree 1 if n is even and 2 otherwise. These facts together with a direct check of states a^0, a and a^2 give us the result that all imbalances of $G_a(n)$ can be equal only to 0, 1 or 2.

Moreover, every automaton a^n contains the trivial state a^0 which is connected only with state a. The imbalance of the corresponding edge is always equal to 2.

On the other hand, the automaton a^n contains a state a^k for an odd $k > 1$. We take the biggest such a value k. If $n = k$ then we get the imbalance 1 for the edge which contains the vertex a^n and its projection with odd power. If $n = 2^s k$ then it is easy to check that for $s > 1$ we will get the imbalance 1 on the edge between n and $2^{s-1}k$. If $s = 1$ then the vertex a^n has degree 1 and connected only with a^k which has degree 3. But a^k has two projections $a^{[\frac{k}{2}]}$ and $a^{[\frac{k}{2}]+1}$. One of these powers is even and the corresponding vertex has degree 2. So we get the imbalance 1 in the graph.

Thus the multiset $M(G)$ satisfies all the conditions of Lemma 1. The statement is proved.

Proposition 2. *Let a be the adding machine. Then $G_a(n)$ is acyclic for $n = 2^k$ and $g(G_a(n)) = 3$ for the other values of n. The chromatic number is*

$$\chi(G_a(n)) = \begin{cases} 2, & \textit{if } n \textit{ is a power of two;} \\ 3, & \textit{otherwise.} \end{cases}$$

Proof. For $n = 2^k$ the graph $G_a(n)$ is a path and $\chi(G_a(n)) = 2$. For the other values of n the graph $G_a(n)$ contains a cycle of length 3 (vertices a^3, a^2 and a^1); therefore, $g(G_a(n)) = 3$ and $\chi(G_a(n)) \leq 3$. We can color the graph $G_a(n)$ by three colors as follows. The state a^n is colored by the first color, its projections should be colored by the second one, while for the next projections we can use the first color again. This approach can be used until we get one of the vertex from the cycle. Therefore, $\chi(G_a(n)) = 3$.

Now we consider the case of the cyclic automaton. The structure of the graphs $G_a(n)$ for the cyclic automaton is described in [10]. Each state of a^n is a projection

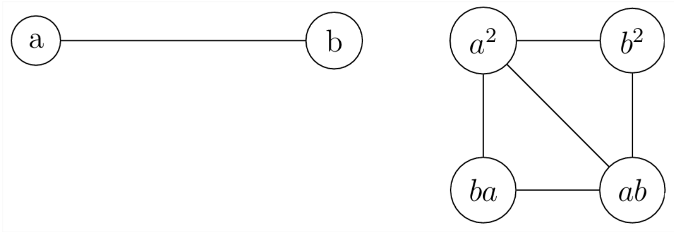

Figure 1. The graphs $G_a(1)$ and $G_a(2)$ for the state a of the lamplighter automaton.

of at most two other states and has only one or two projections. Therefore, the degree of each state (except for maybe a^n) in the graph $G_a(n)$ is 2, 3 or 4. Also we can prove in the same way as for the adding machine that if there exists imbalance with value 2 then there exists an edge with imbalance 1. Therefore, $G_a(n)$ satisfies all the conditions of Lemma 1 and it is imbalance graphic. The chromatic number and girth is calculated in the same way as for the adding machine. For the state b we get the same results, because $G_b(n) = G_{a^2}(n) = G_a(2n)$.

It is left to consider the lamplighter automaton $b = (a, b)$, $a = (b, a)\sigma$ and its inverse $d = (c, d)$, $c = (c, d)\sigma$. Since the transformations defined by c and d are inverse to the transformations defined by a and b respectively, the graphs $G_c(n)$ and $G_a(n)$ are isomorphic, and the graphs $G_b(n)$ and $G_d(n)$ are isomorphic.

Proposition 3. *Let a be a state of the lamplighter automaton. Then the graph $G_a(n)$ is imbalance graphic for every $n \in \mathbb{N}$.*

Proof. It was proved in [10] that for every n the automata a^n and b^n contain exactly 2^n states. Moreover, each state is a word of length n over the alphabet $\{a, b\}$. This means that the graphs $G_a(n)$ and $G_b(n)$ are the same for every $n \in \mathbb{N}$.

First of all we can directly check the cases $n = 1$ and $n = 2$ (see Figure 1). In this case $M(G_a(1)) = \{0\}$ and $M(G_a(2)) = \{0, 1, 1, 1, 1\}$. Both of these multisets are graphic.

Now we consider the case when $n > 2$. Each state of the automaton a^n has two different projections, because one of them is a word over $\{a, b\}$ with the first letter a while the other one has the first letter b. However, for some states one of the projections can coincide with the state.

Let us show that each state s of the automaton a^n can be a projection only for one or two other states. Let the first letter of s as a word over $\{a, b\}$ is a then it can be only the first projection of a state which starts with b or the second projection of the state which starts with a. Then we take the second letter of s. It is easy to see that for each case we will get only one possible second letter for state to contain s as a projection. After repeating of this procedure we get that each state can be projection only for two states. While in some cases s can be a projection of itself as it was described above. The other possible situation is when the two different states s_1 and s_2 are the projection each to other. Let us prove that the last two properties can not hold simultaneously. It is easy to show that the only state that can be a projection of itself are b^n and $b^{n-1}a$. This follows from the fact that the word that corresponds to such a state can not contain subwords of type ab and aa. On the other hand the state a^n is a projection of b^n but not vice versa. Also the state $b^{n-1}a$

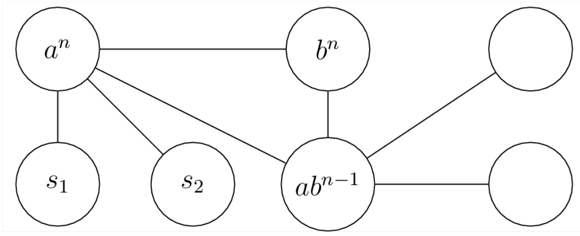

Figure 2. The neighborhood of the vertex b^n in the graph $G(a^n)$.

contains the state $a^{n-1}b$ as a projection but not vice versa.

As a result we have that each vertex in $G_a(n)$, $n > 2$ has degree 2, 3 or 4. Therefore, the imbalance multiset $M(G_a(n))$ can contain only values 0, 1 and 2.

Let us consider the state b^n and corresponding vertex. It is connected with the states a^n and ab^{n-1}. Then the corresponding vertex of the graph $G(a^n)$ has degree 2. Moreover, each of these states is not a projection of itself and a^n has two different projections $(ab)^{[\frac{n}{2}]}a^r$ and $(ba)^{[\frac{n}{2}]}b^r$, where r is the remainder modulo 2. But ab^{n-1} has a^n and b^n as projections. Then it is a projection for some other two different states and the corresponding vertex in the graph $G_a(n)$ has degree 4. This pattern gives us two edges with imbalance 2 (see Figure 2).

As was mentioned above the state $b^{n-1}a$ also is a projection of itself. Thus, the corresponding vertex has the degree 2. Now we can use the following approach. Since $G_a(n)$ is connected and the vertices corresponding to $b^{n-1}a$ and b^n are not adjacent, there is a path of the length not less than two between them. Moreover, such a path contains either the vertex a^n or the vertex ab^{n-1}. Both of these vertices have degree 4. Then on the path between such a vertex and a vertex for $b^{n-1}a$ there is at least one edge with imbalance 2 or 1.

Hence the multiset $M(G_a(n))$ satisfies all the conditions of Lemma 1. The statement is proved, what completes the proof of Theorem 2.

Proposition 4. *Let a be a state of the lamplighter automaton. Then $G_a(1)$ is acyclic with $\chi(G_a(1)) = 2$ and $g(G_a(n)) = \chi(G_a(n)) = 3$ for $n \geq 2$.*

Proof. By directly check we have that $G_a(1)$ is acyclic with $\chi(G_a(1)) = 2$, while $\chi(G_a(2)) = 3$. It was shown in the previous proof that every graph $G_a(n)$ for $n > 2$ contains a triangle and therefore $g(G_a(n)) = 3$. In particular, $\chi(G_a(n)) \geq 3$. On the other hand, the maximal vertex degree is 4 and the graphs are neither complete nor the odd cycle. Hence, $\chi(G_a(n)) \leq 4$ by the Brook's theorem.

Let us prove that $\chi(G_a(n)) = 3$. Let a state ω be a word of length $n \geq 2$ over $\{a, b\}$ and it has two projections ω_1 and ω_2 with the same length and these projections are not coincide with ω (this is true for all states except for b^n and ab^{n-1}). Let us assume by induction that the graph $G_a(n)$ can be colored in three colors. Now we consider the graph $G_a(n+1)$. Note, that we can match a vertex ω of $G_a(n)$ with two vertices $a\omega$ and $b\omega$ of $G_a(n+1)$. Moreover, both of these states have the same projections $a\omega_1$ and $b\omega_2$. If we will color $a\omega$ and $b\omega$ in the same color as ω is colored in the graph $G_a(n)$ for every word ω, then the number of colors will not be changed. But we get two edges with the same colored vertices when $b\omega$ is a projection of $a\omega$. These cases appear when $\omega = \omega_2 \in \{b^n, b^{n-1}a\}$. The first edge connects b^{n+1} and ab^n, while the second connects b^na and $ab^{n-1}a$. Note that the vertices b^{n+1} and $b^n a$ have degree 2 in $G_a(n+1)$, and they always can be colored properly because we use three colors. Then we can left colors for ab^n and $ab^{n-1}a$ by using the colors of b^n and $b^{n-1}a$ respectively. Hence, the chromatic number is 3. The statement is proved. The proof of Theorem 1 is complete.

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