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ON PROPERTIES OF POSETS OF *MM*-TYPE $(1,3,5)$

We introduce the notion of poset of MM-type P, where P is a fixed poset, and calculate the coefficient of transitiveness for all posets of MM-type $(1,3,5)$.

Ми вводимо поняття частково впорядкованої множини MM-типу P, де P — фіксована частково впорядкована множина, і обчислюємо коефіцієнт транзитивності для всіх частково впорядкованої множин MM-типу $(1,3,5)$.

1. Introduction. In [1] P. Gabriel introduced the notion of representation of a finite quiver $Q = (Q_0, Q_1)$ (with Q_0 and Q_1 being the set of vertices Q_0 and the set of arrows, respectively), and also introduced a quadratic form $q_Q : \mathbb{Z}^n \to \mathbb{Z}$, $n = |Q_0|$, called by him the *quadratic Tits form of the quiver* Q:

$$q_Q(z) = q_Q(z_1, \ldots, z_n) := \sum_{i \in Q_0} z_i^2 - \sum_{i \to j} z_i z_j,$$

where $i \to j$ runs through the set Q_1. For the quivers, he received a criterion of finiteness (representation) type in terms of the quivers themselves, and also proved that the quiver Q has finite type over a field k if and only if its Tits form is positive. Criterions of tameness for the quivers were obtained in [2, 3]; in this situation the main role play the non-negative quadratic forms.

The above quadratic form is naturally generalized to a finite poset $S \not\ni 0$:

$$q_S(z) = z_0^2 + \sum_{i \in S} z_i^2 + \sum_{i<j, i,j \in S} z_i z_j - z_0 \sum_{i \in S} z_i.$$

In [4] Yu. A. Drozd showed that a poset S has finite type if and only if its Tits form is weakly positive, i.e., takes positive value on any nonzero vector with nonnegative coordinates (representations of posets were introduced by L. A. Nazarova and A. V. Roiter [5], a criterion of finiteness type in terms of the posets themselves was obtained by M. M. Kleiner [6]). As shown in [7], in the study of tame posets the main role play the weakly non-negative quadratic forms (a criterion of tameness in terms of the posets themselves was first formulated in [8]).

For posets, in contrast to quivers, the sets of those with weakly positive (respectively, weakly non-negative) and with positive (respectively, non-negative) Tits forms do not coincide. Posets with positive and non-negative Tits forms were studied by the authors (from different points of view) in many papers (see e.g. [9–17].

In particular, in [13] it is introduced the notion of P-critical poset: a poset S is called P-*critical* if its Tits quadratic form is not positive, but that of any proper subset of S is positive. If one gives the similar definition for a quiver. then the set of P-critical quivers coincides with the set of extended Dynkin diagrams. So the P-critical posets are analogs of the extended Dynkin diagrams. All such posets are classified in [13].

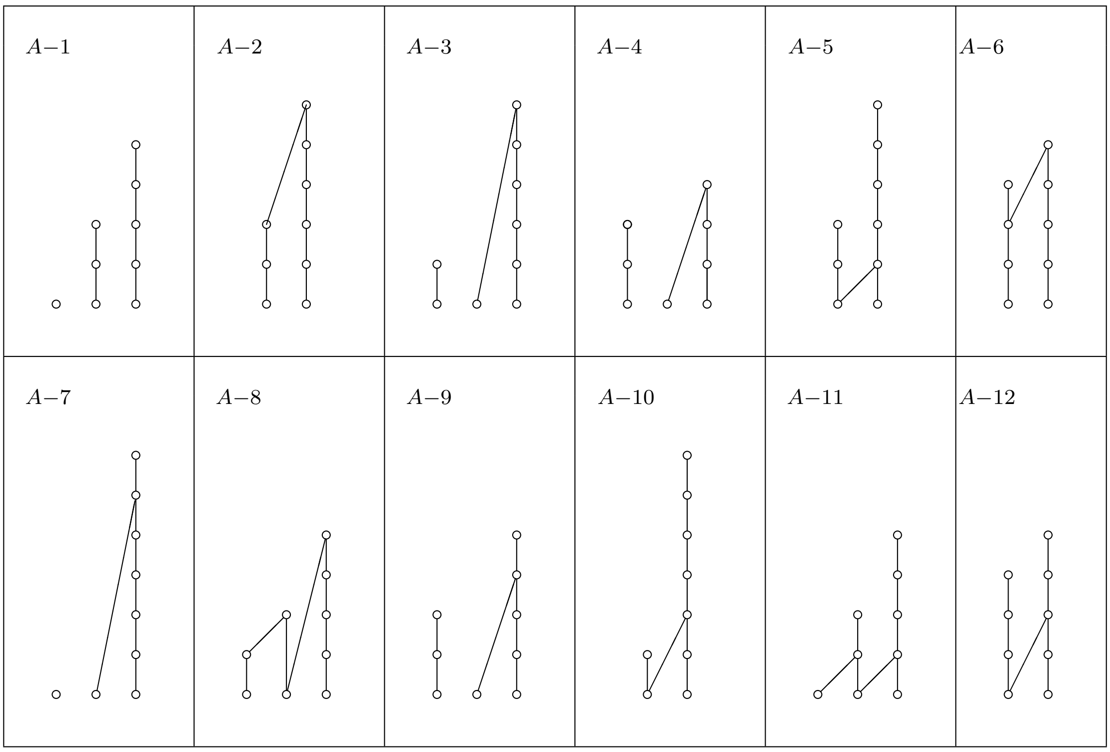
Combinatorial properties of P-critical posets were studied by the author in [18]. In this paper we continue these investigations.

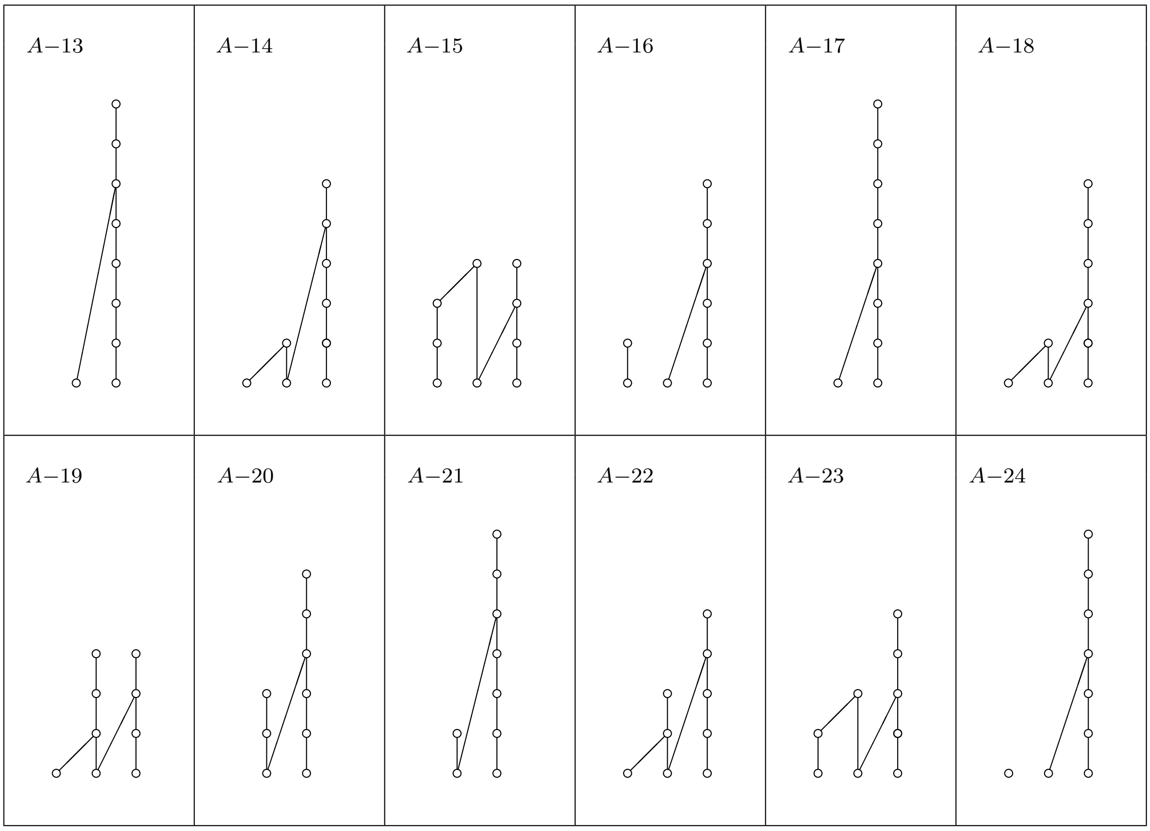
2. Coefficients of transitiveness. In this section we recall the notion of coefficient of transitiveness of a poset [18].

Let S be a finite poset and $S^2_< := \{(x,y)\,|\,x,y \in S, x < y\}$. If $(x,y) \in S^2_<$ and there is no z satisfying $x < z < y$, then one says that x and y are *neighboring*. We put $n_w = n_w(S) := |S^2_<|$ and denote by $n_e = n_e(S)$ the number of pairs of neighboring elements. On the language of the Hasse diagram $H(S)$ (that represents S in the plane), n_e is equal to the number of all its edges and n_w to the number of all its paths, up to parallelity, going bottom-up (two path is called parallel if they start and terminate at the same vertices). The ratio $k_t = k_t(S)$ of the numbers $n_w - n_e$ and n_w we call *the coefficient of transitiveness of* S. If $n_w = 0$ (then $n_e = 0$), we assume $k_t = 0$.

3. The posets of MM-type (1,3,5). We will say that a poset S is of the *MM-type* P, where P is a fixed poset, if S is (min, max)-equivalent to P (the notion of (min, max)-equivalence was introduced in [11]; see also [13]).

The poset $(1,3,5)$ (the disjoint union of chains of length 1, 3 and 5) is the smallest element with trivial group of automorphisms in the set of so-called 1-oversupercritical posets (see [19]). The posets of type $(1,3,5)$ were classified in [19], They are given (up to isomorphism and anti-isomorphism) by the following table.





4. Main result. In [18] the authors calculate the coefficient of transitiveness for all P-critical posets. In this paper we do it for the posets of MM-type $(1, 3, 5)$.

We write all the coefficients of transitiveness k_t up to the second decimal place.

Theorem 1. *The following holds for posets* $(A\text{-}1)$ – $(A\text{-}24)$ *of the* MM*-type* $(1, 3, 5)$:

N	n_e	n_w	k_t	N	n_e	n_w	k_t	N	n_e	n_w	k_t
1	6	13	0,54	9	7	15	0,53	17	8	33	0,76
2	8	21	0,62	10	8	27	0,70	18	8	21	0,62
3	7	17	0,59	11	8	19	0,58	19	8	17	0,53
4	7	13	0,46	12	8	19	0,58	20	8	21	0,62
5	8	23	0,65	13	8	31	0,74	21	8	25	0,68
6	8	19	0,58	14	8	19	0,58	22	8	17	0,53
7	7	23	0,70	15	8	15	0,47	23	8	17	0,53
8	8	15	0,47	16	7	19	0,63	24	7	25	0,72

The proof is carried out by direct calculations.

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