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FINITE LIPSCHITZ MAPPINGS ON FINSLER MANIFOLDS

We consider ring Q-homeomorphisms with respect to p-modulus on Finsler manifolds, $n-1 < p < n$, and establish sufficient conditions for these mappings to be finitely Lipschitzian.

Key words: *Finsler manifolds, ring Q-homeomorphisms, p-modulus, finite Lipschitz mappings.*

1. Introduction.

In this article we continue our study of mappings on Finsler manifolds $(\mathbb{M}^n, \Phi)$ started in [1]. For historical remarks and needed definitions, we refer to [1]. The main tools involve the method of moduli applied to ring Q-homeomorphisms and the method of p-capacities recently developed for Finsler manifolds. For the latter see [2]–[4].

Recall that a mapping $f : D \to D'$ between Finsler manifolds $(\mathbb{M}^n, \Phi)$ and $(\mathbb{M}^n_*, \Phi_*)$, $n \geq 2$, is called *Lipschitz* if there is a finite constant $C > 0$ such that the inequality $d^*_\Phi(f(x), f(y)) \leq C \cdot d_\Phi(x, y)$ holds for all $x, y \in \mathbb{M}^n$, cf. [5]. We say that a continuous mapping $f : D \to D'$ is *finitely Lipschitzian* on the domain D if

$$L(x, f) = \limsup_{y \to x} \frac{d^*_\Phi(f(x), f(y))}{d_\Phi(x, y)} < \infty$$

for all $x \in D$, cf. [6].

The main result of the paper is the following statement.

Theorem 1. *Let D and D' be domains in $(\mathbb{M}^n, \widetilde{\Phi})$ and $(\mathbb{M}^n_*, \widetilde{\Phi}_*)$, $n \geq 2$, respectively. Assume that $Q : D \to [0, \infty]$ is a locally integrable function such that*

$$\limsup_{\varepsilon \to 0} \frac{1}{\sigma_{\widetilde{\Phi}}(B(x_0, \varepsilon))} \int_{B(x_0,\varepsilon)} Q(x)\, d\sigma_{\widetilde{\Phi}}(x) < \infty \tag{1}$$

and $f : D \to D'$ is a ring Q-homeomorphism with respect to a p-modulus at any $x_0 \in D$, $n - 1 < p < n$. Then f is finitely Lipschitzian on D.

The similar results for homeomorphisms and mappings with branching were earlier obtained in $\mathbb{R}^n$, $n \geq 2$, see [7]. The Lipschitzian continuity for mappings in $\mathbb{R}^n$, $n \geq 2$, with a uniformly bounded function Q has been established by Gehring [8]. The same condition for Riemannian manifolds was proved in [9].

2. Definitions and preliminary results.

Recall some needed definitions. By *domain* in a topological space T we mean an open linearly connected set. The domain D is called *locally connected at a point* $x_0 \in \partial D$, if for any neighborhood U of x_0 there is a neighborhood $V \subseteq U$ of x_0 such that $V \cap D$ is connected, cf. [10, c. 232]. Similarly, we say that a domain D is *locally linearly connected* *at a point* $x_0 \in \partial D$, if for any neighborhood U of x_0 there exists a neighborhood $V \subseteq U$ of x_0 such that $V \cap D$ is linearly connected. Recall that the *n-dimensional topological manifold* $\mathbb{M}^n$ is a Hausdorff topological space with a countable base such that every point has a neighborhood homeomorphic to $\mathbb{R}^n$. The manifold of the class C^r with $r \geq 1$ is called *smooth*.

Let further D denote a domain in the Finsler space $(\mathbb{M}^n, \Phi)$, $n \geq 2$, and $T\mathbb{M}^n = \cup\, T_x\mathbb{M}^n$ be a tangent bundle of $(\mathbb{M}^n, \Phi)$ for all $x \in \mathbb{M}^n$. By a *Finsler manifold* $(\mathbb{M}^n, \Phi)$, $n \geq 2$, we mean a smooth manifold of class C^∞ with defined Finsler structure $\Phi(x, \xi)$, where $\Phi(x, \xi) : T\mathbb{M}^n \to \mathbb{R}^+$ is a function satisfying the following conditions:

1) $\Phi \in C^\infty(T\mathbb{M}^n \setminus \{0\})$;

2) $\Phi(x, a\xi) = a\Phi(x, \xi)$ holds for all $a > 0$ and $\Phi(x, \xi) > 0$ holds for $\xi \neq 0$;

3) the $n \times n$ Hessian matrix $g_{ij}(x, \xi) = \frac{1}{2}\frac{\partial^2 \Phi^2(x,\xi)}{\partial \xi_i \partial \xi_j}$ is positive defined at every point of $T\mathbb{M}^n \setminus \{0\}$, cf. [4].

By the *geodesic distance* $d_\Phi(x, y)$ we mean the infimum of lengths of piecewise-smooth curves joining x and y in $(\mathbb{M}^n, \Phi)$, $n \geq 2$. It is well-known that for such metric only two axioms of metric spaces hold, namely identity and triangle inequality axioms. Therefore, the Finsler manifold provides a quasimetric space for which symmetry axiom fails (see, e.g. [11]).

Remark 1. Consider a Finsler structure of the type

$$\widetilde{\Phi}(x, \xi) = \frac{1}{2}(\Phi(x, \xi) + \Phi(x, -\xi)). \tag{2}$$

In this case we obtain a Finsler manifold $(\mathbb{M}^n, \widetilde{\Phi})$ with symmetrized (reversible) function $\widetilde{\Phi}$. Clearly, if $\widetilde{\Phi}$ is reversible, then the induced distance function $d_{\widetilde{\Phi}}$ is reversible, i.e., $d_{\widetilde{\Phi}}(x, y) = d_{\widetilde{\Phi}}(y, x)$, for all pairs of points $x, y \in \mathbb{M}^n$. It is also known that the reversible Finsler metric coincides with the Riemannian one, see, e.g., [11]. Therefore, in our further discussion we can rely on the results of [12].

Let $\gamma : [a, b] \to \mathbb{M}^n$ be a piecewise-smooth curve and $x(t)$ be its parametrization. An *element of length* in $(\mathbb{M}^n, \widetilde{\Phi})$, $n \geq 2$, we define as a differential of path for infinitesimal measured part of a curve $\gamma \in D$ by $ds^2_{\widetilde{\Phi}} = \sum\limits_{i,j=1}^{n} g_{ij}(x, \xi) d\eta_i d\eta_j$; see, e.g. [13]. So, the distance $ds_{\widetilde{\Phi}}$ in the Finsler space, as in the case of a Riemannian space, is determined by a metric tensor. Using the quadratic form $ds_{\widetilde{\Phi}}$, we determine the length of $\gamma \subset D$ by $s_{\widetilde{\Phi}}(\gamma) = \int\limits_\gamma ds_{\widetilde{\Phi}} = \int\limits_{t_1}^{t_2} \widetilde{\Phi}(x, dx)dt$, see, e.g. [11]. The invariance of this integral requires the restrictions 2)-3), given above, on the Lagrangian $\widetilde{\Phi}(x, dx)$.

In the Finsler geometry there are various definitions for the volume: by Holmes-Thompson, Loewner, Busemann and others. In this paper we agree with the volume definition by Busemann (Busemann-Hausdorff). Following [14], an element of *volume* on the Finsler manifold is defined by $d\sigma_\Phi(x) = \frac{|B^n|}{|B^n_x|}\, dx^1...dx^n$, where $|B^n|$ denotes the Euclidean volume of the unit n-ball, whereas $|B^n_x|$ is the Euclidean volume of the set

$B_x^n = \left\{(\xi_1, ..., \xi_n) \in \mathbb{R}^n : \Phi\left(x, \sum\limits_1^n (\xi_i, e_i(x))\right) < 1\right\}$ with an arbitrary basis $\{e_i(x)\}_{i=1}^n$ in $\mathbb{R}^n$ depending on x. It is known that the volume in the Finsler space coincides with its Hausdorff measure induced by metric $d_\Phi(x, y)$, if $\Phi(x, \xi)$ is an invertible function, see, e.g. [14]. In view of Remark 1, we have $d\sigma_{\widetilde{\Phi}}(x) = \sqrt{det\ g_{ij}(x, \xi)}\ dx^1 ... dx^n$, cf. [15].

Let Γ be a family of curves in a domain D. By the family of curves Γ we mean a fixed set of curves γ, and for arbitrary mapping $f : \mathbb{M}^n \to \mathbb{M}_*^n$, $f(\Gamma) := \{f \circ \gamma | \gamma \in \Gamma\}$.

The *p-modulus* of the family Γ, $p \in (1, \infty)$, is defined by

$$M_p(\Gamma) = \inf \int\limits_{\mathbb{M}^n} \rho^p(x)\, d\sigma_{\widetilde{\Phi}}(x)\,, \tag{3}$$

where the infimum is taken over all nonnegative Borel functions ρ such that the condition $\int\limits_\gamma \rho \widetilde{\Phi}(x, dx) = \int\limits_\gamma \rho ds_{\widetilde{\Phi}} \geq 1$ holds for any curve $\gamma \in \Gamma$. The functions ρ, satisfying this condition, are called *admissible* for Γ, cf. [4].

The quantity (3) can be interpreted as an outer measure in the space of curves.

For sets A, B and C from $(\mathbb{M}^n, \widetilde{\Phi})$, $n \geq 2$, by $\Delta(A, B; C)$ we denote a set of all curves $\gamma : [a, b] \to \mathbb{M}^n$, which join A and B in C, i.e. $\gamma(a) \in A$, $\gamma(b) \in B$ and $\gamma(t) \in C$ for all $t \in (a, b)$.

Remark 2. One can apply the following well-known facts: Proposition 1 and Remark 1 in [12] (due to Remark 1), and thus assume that the geodesic spheres $S(x_0, r)$, geodesic balls $B(x_0, r)$ and geodesic rings $A = A(x_0, r_1, r_2)$ lie in a normal neighborhood of a point x_0.

Let D and D' be domains in $(\mathbb{M}^n, \widetilde{\Phi})$ and $(\mathbb{M}_*^n, \widetilde{\Phi}_*)$, $n \geq 2$, respectively, and $Q : \mathbb{M}^n \to (0, \infty)$ be a measurable function, $p \in (1, \infty)$, $x_0 \in D$. We say that a homeomorphism $f : D \to D'$ is a *ring Q-homeomorphism* with respect to a p-modulus at the point x_0 if the inequality

$$M_p\left(\Delta(f(S_\varepsilon), f(S_{\varepsilon_0}); D')\right) \ \leq \int\limits_{\mathbb{A}} Q(x) \cdot \eta^p\left(d_{\widetilde{\Phi}}(x, x_0)\right) d\sigma_{\widetilde{\Phi}}(x) \tag{4}$$

holds for every geodesic ring $\mathbb{A} = \mathbb{A}(x_0, \varepsilon, \varepsilon_0)$, $0 < \varepsilon < \varepsilon_0 < d_0 = dist(x_0, \partial D)$, and for every measurable function $\eta : (\varepsilon, \varepsilon_0) \to [0, \infty]$, such that $\int\limits_\varepsilon^{\varepsilon_0} \eta(r) dr \geq 1$. Here $S_\varepsilon = S(x_0, \varepsilon), S_{\varepsilon_0} = S(x_0, \varepsilon_0)$. We also say that f is a *ring Q-homeomorphism* with respect to a p-modulus in the domain D if f is a ring Q-homeomorphism at every point $x_0 \in D$.

Let us recall that the idea to introduce the ring Q-homeomorphisms goes back to Gehring's ring definition of quasiconformality in $\mathbb{R}^n$, $n = 3$, see [16]. These homeomorphisms first appeared in the plane for study of the Beltrami equations (see, e.g. [17]), and later in $\mathbb{R}^n, n \geq 2$, cf. [18]. Further, the notion of ring homeomorphisms was extended to boundary points of domains in the plane [19] and then in the space [20]. It is well known that the theory of boundary behavior is one of the difficult and interesting parts of the mapping theory; see the monographs [19, 6] and references therein. Note also that the ring Q-homeomorphisms have rich applications in the theory of boundary behavior of Sobolev and Orlic–Sobolev classes of mappings on Riemannian manifolds; see [21]. The notion of ring Q-homeomorphisms at boundary points with respect to p-modulus for $p = 2$ was introduced and applied for study the Beltrami equations with a degenerate condition of strong ellipticity in [22]. Later a criterium for arbitrary homeomorphisms to be ring Q-homeomorphisms with respect to p-modulus, $p \neq n$, at interior points of domains in the n-dimensional Euclidean space $\mathbb{R}^n$ was established in [23].

3. p-capacities and Finsler manifolds.

By a *condenser* we mean a pair $\mathcal{E} = (A, G)$, where $A \subset \mathbb{M}^n$ is open and $G \subset \mathbb{M}^n$ is a non-empty compact set contained in A. We shall say that $\mathcal{E}$ is a *ringlike condenser* if $B = A \backslash G$ is a geodesic ring, i.e. B is a domain whose complement $\overline{D} \backslash B$ has exactly two components. We shall say that $\mathcal{E}$ is a *bounded condenser* if A is bounded. A condenser $\mathcal{E} = (A, G)$ lies in a domain D if $A \subset D$.

Each condenser has *p-capacity* (where $p \geq 1$) defined by the equality

$$\operatorname{cap}_p \mathcal{E} = \operatorname{cap}_p (A, G) = \inf_u \int\limits_{A \backslash G} |\nabla u|^p \, d\sigma_{\widetilde{\Phi}}(x), \tag{5}$$

where the infimum is taken over all Lipschitz functions u with compact support in A. In the local coordinates, the *gradient* at a point $x \in \mathbb{M}^n$ is defined by $(\nabla f)^i = g^{ij} \frac{\partial f}{\partial x^j}$, $1 \leq i \leq n$, where the matrix g^{ij} is the inverse matrix of the matrix g_{ij}; see [24].

Recall that in $\mathbb{R}^n, n \geq 2$, for $1 < p < n$,

$$\operatorname{cap}_p \mathcal{E} \geq n \Omega_n^{\frac{p}{n}} \left(\frac{n-p}{p-1} \right)^{p-1} [m(G)]^{\frac{n-p}{n}}, \tag{6}$$

see, e.g. (8.9) in [25]. Finally, for $n - 1 < p \leq n$ in $\mathbb{R}^n$, the following lower bound

$$\left(\operatorname{cap}_p \mathcal{E} \right)^{n-1} \geq \gamma \frac{d(G)^p}{m(A)^{1-n+p}}, \tag{7}$$

where $d(G)$ is the diameter of the compact set G and γ is a positive constant depending only on n and p (see Proposition 6 in [26]) holds.

4. Proof of Theorem 1.

It suffices to show the following. Let $Q : D \to [0, \infty]$ be a locally integrable function and $f : D \to D'$ be a ring Q-homeomorphism with respect to a p-modulus $(n - 1 < p < n)$ at an arbitrary point $x_0 \in D$ satisfying

$$Q_0 = \limsup_{\varepsilon \to 0} \frac{1}{\sigma_{\widetilde{\Phi}}(B(x_0, \varepsilon))} \int_{B(x_0, \varepsilon)} Q(x) \, d\sigma_{\widetilde{\Phi}}(x) < \infty.$$

We show that

$$L(x_0, f) = \limsup_{x \to x_0} \frac{d^*_{\widetilde{\Phi}}(f(x_0), f(x))}{d_{\widetilde{\Phi}}(x_0, x)} \leq \lambda_{n,p}\, Q_0^{\frac{1}{n-p}},$$

where $\lambda_{n,p}$ is a positive constant depending only on n and p.

Consider a geodesic ring $\mathbb{A} = \mathbb{A}(x_0, \varepsilon_1, \varepsilon_2) \subset D$ with $0 < \varepsilon_1 < \varepsilon_2$ such that $\mathbb{A}(x_0, \varepsilon_1, \varepsilon_2)$ lies in a normal neighborhood at x_0 (see Remark 2). Of course, if $f : D \to D'$ is open and $\mathcal{E} = (A, G)$ is a condenser in D, then $f(\mathcal{E}) = (f(A), f(G))$ is also condenser in D', see Lemma A.1 in [6] and [27]. Then $\left(f(B\,(x_0, \varepsilon_2)), \overline{f(B\,(x_0, \varepsilon_1))}\right)$ is the ringlike condenser in D', in view of Remark 1. Follow the Theorem 2 in [4] we have

$$\mathrm{cap}_p\,(f(B(x_0, \varepsilon_2)), \overline{f(B(x_0, \varepsilon_1))}) = M_p(\triangle(\partial f(B(x_0, \varepsilon_2)), \partial f(B(x_0, \varepsilon_1); f(\mathbb{A}))).$$

This equality is invariant with respect to change of the local coordinates. Since f is a homeomorphism, then

$$\triangle\,(\partial f(B\,(x_0, \varepsilon_2)), \partial f(B\,(x_0, \varepsilon_1)); f(\mathbb{A})) = f((\triangle\,(\partial B(x_0, \varepsilon_2)), \partial B(x_0, \varepsilon_1); \mathbb{A}))\,.$$

Letting

$$\eta(t) \;=\; \begin{cases} \frac{1}{\varepsilon_2 - \varepsilon_1}, & t \in (\varepsilon_1, \varepsilon_2), \\ 0, & t \in \mathbb{R} \setminus (\varepsilon_1, \varepsilon_2), \end{cases}$$

and applying the definition of ring Q-homeomorphisms with respect to p-module, we obtain

$$\mathrm{cap}_p\,(f(B(x_0, \varepsilon_2)), \overline{f(B(x_0, \varepsilon_1))}) \leq \frac{1}{(\varepsilon_2 - \varepsilon_1)^p} \int\limits_{\mathbb{A}(x_0, \varepsilon_1, \varepsilon_2)} Q(x)\; d\sigma_{\widetilde{\Phi}}(x). \tag{8}$$

Choose $\varepsilon_1 = 2\varepsilon$ and $\varepsilon_2 = 4\varepsilon$, then

$$\mathrm{cap}_p\,(f(B(x_0, 4\varepsilon)), f\overline{(B(x_0, 2\varepsilon))}) \leq \frac{1}{(2\varepsilon)^p} \int\limits_{B(x_0, 4\varepsilon)} Q(x)\, d\sigma_{\widetilde{\Phi}}(x). \tag{9}$$

Due to Remark 1 (see also proposition 5.11 (d) [28]), inequality (6) holds in sufficiently small neighborhoods of the point x_0 with respect to the normal coordinates, i.e.

$$\mathrm{cap}_p\,(f(B(x_0, 4\varepsilon)), f\overline{(B(x_0, 2\varepsilon))}) \geq C_{n,p}\left[\sigma_{\widetilde{\Phi}}(fB(x_0, 2\varepsilon))\right]^{\frac{n-p}{n}}, \tag{10}$$

where $C_{n,p}$ is a positive constant depending only on n and p. Combining (9) and (10) and taking into account the local n-regularity of measures (see Lemma 2.1 in [1]), we obtain

$$\frac{\sigma_{\widetilde{\Phi}}(f(B(x_0, 2\varepsilon)))}{\sigma_{\widetilde{\Phi}}(B(x_0, 2\varepsilon))} \leq c_{n,p}\left[\frac{1}{\sigma_{\widetilde{\Phi}}(B(x_0, 4\varepsilon))} \int_{B(x_0, 4\varepsilon)} Q(x)\, d\sigma_{\widetilde{\Phi}}(x)\right]^{\frac{n}{n-p}}, \tag{11}$$

where $c_{n,p}$ is a positive constant depending only on n and p.

Now choosing in (8), $\varepsilon_1 = \varepsilon$ and $\varepsilon_2 = 2\varepsilon$, we have

$$\mathrm{cap}_p\,\left(f(B(x_0,2\varepsilon)),\overline{f(B(x_0,\varepsilon))}\right) \le \frac{1}{\varepsilon^p}\int\limits_{B(x_0,2\varepsilon)} Q(x)\,d\sigma_{\widetilde{\Phi}}(x)\,. \tag{12}$$

Arguing similar to above, one gets from (7) the following lower bound

$$\left(\mathrm{cap}_p\,\left(f(B(x_0,2\varepsilon)),\overline{f(B(x_0,\varepsilon))}\right)\right)^{n-1} \ge \widetilde{C}_{n,p}\frac{d^p_{\widetilde{\Phi}}(f(B(x_0,\varepsilon)))}{\sigma^{1-n+p}_{\widetilde{\Phi}}(f(B(x_0,2\varepsilon)))}, \tag{13}$$

where $\widetilde{C}_{n,p}$ is a positive constant that depends only on n and p. Combining (12) and (13) and taking again into account the Lemma 2.1 in [1], we obtain

$$\frac{d^*_{\widetilde{\Phi}}(f(B(x_0,\varepsilon)))}{\varepsilon} \le \gamma_{n,p}\left(\frac{\sigma_{\widetilde{\Phi}}(f(B(x_0,2\varepsilon)))}{\sigma_{\widetilde{\Phi}}(B(x_0,2\varepsilon))}\right)^{\frac{1-n+p}{p}} \times$$

$$\times\left(\frac{1}{\sigma_{\widetilde{\Phi}}(B(x_0,2\varepsilon))}\int\limits_{B(x_0,2\varepsilon)} Q(x)d\sigma_{\widetilde{\Phi}}(x)\right)^{\frac{n-1}{p}}, \tag{14}$$

where $\gamma_{n,p}$ is a positive constant depending only on n and p. The estimates (11) and (14) imply

$$\frac{d^*_{\widetilde{\Phi}}(f(B(x_0,\varepsilon)))}{\varepsilon} \le \lambda_{n,p}\left(\frac{1}{\sigma_{\widetilde{\Phi}}(B(x_0,4\varepsilon))}\int_{B(x_0,4\varepsilon)} Q(x)\,d\sigma_{\widetilde{\Phi}}(x)\right)^{\frac{n(1-n+p)}{p(n-p)}} \times$$

$$\times\left[\frac{1}{\sigma_{\widetilde{\Phi}}(B(x_0,2\varepsilon))}\int_{B(x_0,2\varepsilon)} Q(x)\,d\sigma_{\widetilde{\Phi}}(x)\right]^{\frac{n-1}{p}}.$$

Letting $\varepsilon \to 0$, we obtain the desired estimate

$$L(x_0,f) = \limsup_{x\to x_0}\frac{d^*_{\widetilde{\Phi}}(f(x_0),f(x))}{d_{\widetilde{\Phi}}(x_0,x)} \le \limsup_{\varepsilon\to 0}\frac{d^*_{\widetilde{\Phi}}(f(B(x_0,\varepsilon)))}{\varepsilon} \le \lambda_{n,p}\,Q_0^{\frac{1}{n-p}}$$

with a positive constant $\lambda_{n,p}$ depending on n and p.

Since x_0 was chosen arbitrary, the proof of Theorem 1 is completed.

Corollary 1. *Let D and D' be domains in $(\mathbb{M}^n, \widetilde{\Phi})$ and $(\mathbb{M}^n_*, \widetilde{\Phi}_*)$, $n \ge 2$, respectively, and $f : D \to D'$ be a ring Q-homeomorphism with respect to a p-modulus, $n-1 < p < n$. Assume that $Q(x)$ is bounded almost everywhere (a.e.) in D by a positive constant K. Then f is locally Lipschitzian and, moreover,*

$$L(x_0,f) \le \lambda_{n,p}K^{\frac{1}{n-p}},$$

where $\lambda_{n,p}$ is a constant depending only on n and p.

Remark 4. Condition (1) in Theorem 1 is sufficient. However, it cannot be omitted. Here we refer to an example of homeomorphism in $\mathbb{R}^n$ [7] which does not satisfy (1) and fails to be finitely Lipschizian.

Remark 5. Finitely Lipschitz mappings possess the property of the absolute continuity on surfaces of any dimension (see, e.g. [6]).

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Е. С. Афанасьева

Конечно липшицевы отображения на финслеровых многообразиях.

Рассматриваются кольцевые Q-гомеоморфизмы относительно p-модуля на финслеровых многообразиях, $n-1<p<n$, устанавливаются достаточные условия конечной липшицевости этих отображений.

Ключевые слова: *Финслеровы многообразия, кольцевые Q-гомеоморфизмы, p-модули, конечно липшицевы отображения.*

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Кінцево ліпшицеві відображення на фінслерових многовидах.

Розглядаються кільцеві Q-гомеоморфізми відносно p-модуля на фінслерових многовидах, $n-1<p<n$, та встановлюються достатні умови кінцевої ліпшицевості таких відображень.

Ключові слова: *Фінслерові многовиди, кільцеві Q-гомеоморфізми, p-модулі, кінцево ліпшицеві відображення.*

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