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ON THE DIRICHLET PROBLEM FOR QAUSILINEAR POISSON EQUATIONS

We study the Dirichlet problem for the quasilinear partial differential equations of the form $\triangle u(z) = h(z)\cdot f(u(z))$ in the unit disk $\mathbb{D}\subset\mathbb{C}$ with functions $h:\mathbb{D}\to\mathbb{R}$ in the class $L^p(\mathbb{D})$, $p>1$, and continuous functions $f:\mathbb{R}\to\mathbb{R}$ with nondecreasing $|f|$ of $|t|$ such that $f(t)/t\to 0$ as $t\to\infty$. On the basis of the potential theory and applying the Leray–Schauder approach, under arbitrary continuous boundary data $\varphi:\partial\mathbb{D}\to\mathbb{R}$ we prove the existence of continuous solutions u of the problem in the class $W^{2,p}_{\rm loc}$. Moreover, $u\in W^{1,q}_{\rm loc}(\mathbb{D})$ for some $q>2$ and u is locally Hölder continuous. If in addition φ is Hölder continuous, then u is Hölder continuous in $\mathbb{D}$. Furthermore, $u\in C^{1,\alpha}_{\rm loc}(\mathbb{D})$ with $\alpha=(p-2)/p$ if $p>2$.

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1. Introduction.

Let $\mathbb{D}$ be the unit disk in the complex plane $\mathbb{C}$. For z and $w\in\mathbb{D}$ with $z\neq w$, set

$$G(z,w) := \log\left|\frac{1-z\bar{w}}{z-w}\right| \quad \text{and} \quad P(z,e^{it}) := \frac{1-|z|^2}{|1-ze^{-it}|^2} \tag{1}$$

be the **Green function** and **Poisson kernel** in $\mathbb{D}$. If $\varphi\in C(\partial\mathbb{D})$ and $g\in C(\overline{\mathbb{D}})$, then a solution to the **Poisson equation**

$$\triangle f(z) \ = \ g(z) \tag{2}$$

satisfying the boundary condition $f|_{\partial\mathbb{D}}=\varphi$ is given by the formula

$$f(z) \ = \ \mathcal{P}_{\varphi}(z) - \mathcal{G}_g(z) \tag{3}$$

where

$$\mathcal{P}_{\varphi}(z) \ = \ \frac{1}{2\pi}\int\limits_0^{2\pi} P(z,e^{it})\,\varphi(e^{-it})\,dt\ , \quad \mathcal{G}_g(z) \ = \ \int\limits_{\mathbb{D}} G(z,w)\,g(w)\,dm(w)\ , \tag{4}$$

see e.g. [8], p. 118–120. Here $m(w)$ denotes the Lebesgue measure in $\mathbb{C}$.

In the next section, we give the representation of solutions of the Poisson equation in the form of the Newtonian (normalized antilogarithmic) potential that is more convenient for our research and, on this basis, we prove the existence and representation theorem for solutions of the Dirichlet problem to the Poisson equation under the corresponding conditions of integrability of sources g.

2. Potentials and the Poisson equation.

Correspondingly to 3.1.1 in [18], given a finite Borel measure ν on $\mathbb{C}$ with compact support, its **potential** is the function $p_\nu : \mathbb{C} \to [-\infty, \infty)$ defined by

$$p_\nu(z) \; = \; \int\limits_{\mathbb{C}} \log|z-w| \, d\nu(w) \; . \tag{5}$$

Remark 1. Note that the function p_ν is subharmonic by Theorem 3.1.2 and, consequently, it is locally integrable on $\mathbb{C}$ by Theorem 2.5.1 in [18]. Moreover, p_ν is harmonic outside of the support of ν.

This definition can be extended to finite **charges** ν with compact support (named also **signed measures**), i.e., to real valued sigma-additive functions on Borel sets in $\mathbb{C}$, because of $\nu = \nu^+ - \nu^-$ where ν^+ and ν^- are Borel measures by the well–known Jordan decomposition, see e.g. Theorem 0.1 in [14].

The key fact is the following statement, see e.g. Theorem 3.7.4 in [18].

Proposition 1. *Let ν be a finite charge with compact support in $\mathbb{C}$. Then*

$$\triangle p_\nu \; = \; 2\pi \cdot \nu \tag{6}$$

in the distributional sense, i.e.,

$$\int\limits_{\mathbb{C}} p_\nu(z) \, \triangle\psi(z) \, dm(z) \; = \; 2\pi \int\limits_{\mathbb{C}} \psi(z) \, d\nu(z) \qquad \forall \; \psi \in C_0^\infty(\mathbb{C}) \; . \tag{7}$$

Here as usual $C_0^\infty(\mathbb{C})$ denotes the class of all infinitely differentiable functions $\psi : \mathbb{C} \to \mathbb{R}$ with compact support in $\mathbb{C}$, $\triangle = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is the Laplace operator and $dm(z)$ corresponds to the Lebesgue measure in $\mathbb{C}$.

Corollary 1. *In particular, if for every Borel set B in $\mathbb{C}$*

$$\nu(B) \; := \; \int\limits_{B} g(z) \; dm(z) \tag{8}$$

where $g : \mathbb{C} \to \mathbb{R}$ is an integrable function with compact support, then

$$\triangle N_g \; = \; g \; , \tag{9}$$

where

$$N_g(z) \; := \; \frac{1}{2\pi} \int\limits_{\mathbb{C}} \log|z-w| \, g(w) \; dm(w) \; , \tag{10}$$

in the distributional sense, i.e.,

$$\int\limits_{\mathbb{C}} N_g(z) \, \triangle\psi(z) \, dm(z) \; = \; \int\limits_{\mathbb{C}} \psi(z) \, g(z) \, dm(z) \qquad \forall \; \psi \in C_0^\infty(\mathbb{C}) \; . \tag{11}$$

Here the function g is called a **density of charge** ν and the function N_g is said to be the **Newtonian potential** of g.

The next statement on continuity in the mean of functions $\psi : \mathbb{C} \to \mathbb{R}$ in $L^q(\mathbb{C})$, $q \in [1, \infty)$, with respect to shifts is useful for the study of the Newtonian potential, see e.g. Theorem 1.4.3 in [20], cf. also Theorem III(11.2) in [19]. Here we give its direct proof arguing by contradiction.

Lemma 1. *Let $\psi \in L^q(\mathbb{C})$, $q \in [1, \infty)$, have a compact support. Then*

$$\lim_{\Delta z \to 0} \int\limits_{\mathbb{C}} |\psi(z + \Delta z) - \psi(z)|^q \, dm(z) \ = \ 0 \, . \tag{12}$$

The **shift of a set** $E \subset \mathbb{C}$ **by a complex vector** $\Delta z \in \mathbb{C}$ *is the set*

$$E + \Delta z \ := \ \{ \ \xi \in \mathbb{C} : \ \xi \ = \ z \ + \ \Delta z \ , \ z \in E \ \} \, .$$

Proof. Let us assume that there is a sequence $\Delta z_n \in \mathbb{C}$, $n = 1, 2, \ldots$, such that $\Delta z_n \to 0$ as $n \to \infty$ and, for some $\delta > 0$ and $\psi_n(z) := \psi(z + \Delta z_n)$, $n = 1, 2, \ldots$,

$$I_n \ := \ \left[\int\limits_{\mathbb{C}} |\psi_n(z) - \psi(z)|^q \, dm(z) \right]^{\frac{1}{q}} \ \geq \ \delta \qquad \forall \ n = 1, 2, \ldots \ . \tag{13}$$

Denote by K the closed disk in $\mathbb{C}$ centered at 0 with the minimal radius R that contains the support of ψ. By the Luzin theorem, see e.g. Theorem 2.3.5 in [5], for every prescribed $\varepsilon > 0$, there is a compact set $C \subset K$ such that $g|_C$ is continuous and $m(K \setminus C) < \varepsilon$. With no loss of generality, we may assume that $C \subset K_*$ where K_* is a closed disk in $\mathbb{C}$ centered at 0 with a radius $r \in (0, R)$ and, moreover, that $C_n \subset K$, where $C_n := C - \Delta z_n$, for all $n = 1, 2, \ldots$. Note that $m(C_n) = m(C)$ and then $m(K \setminus C_n) < \varepsilon$ and, consequently, $m(K \setminus C_n^*) < 2\varepsilon$, where $C_n^* := C \cap C_n$, because $K \setminus C_n^* = (K \setminus C_n) \cup (K \setminus C)$.

Next, setting $K_n = K - \Delta z_n$, we see that $K \cup K_n = C_n^* \cup (K \setminus C_n^*) \cup (K_n \setminus C_n^*)$ and that $K_n \setminus C_n^* + \Delta z_n = K \setminus C_n^*$. Hence by the triangle inequality for the norm in L^p the following estimate holds

$$I_n \ \leq \ 4 \ \cdot \left[\ \int\limits_{K \setminus C_n^*} |\psi(z)|^q \, dm(z) \right]^{\frac{1}{q}} \ + \ \left[\int\limits_{C_n^*} |\psi_n(z) - \psi(z)|^q \, dm(z) \right]^{\frac{1}{q}} \quad \forall \ n = 1, 2, \ldots$$

By construction the both terms from the right hand side can be made to be arbitrarily small, the first one for small enough ε because of absolute continuity of indefinite integrals and the second one for all large enough n after the choice of the set C. Thus, the assumption (13) is disproved. □

Theorem 1. *Let $g : \mathbb{C} \to \mathbb{R}$ be in $L^p(\mathbb{C})$, $p > 1$, with compact support. Then N_g is continuous. A collection $\{N_g\}$ is equicontinuous on compacta if the collection $\{g\}$ is bounded by the norm in $L^p(\mathbb{C})$ with supports in a fixed disk K. Moreover, under these conditions, on each compact set in $\mathbb{C}$*

$$\|N_g\|_C \ \leq \ M \cdot \|g\|_p \ . \tag{14}$$

The corresponding statement on the continuity of integrals of potential type in $\mathbb{R}^n$, $n \geq 3$, can be found in [20], Theorem 1.6.1.

Proof. By the Hölder inequality with $\frac{1}{q} + \frac{1}{p} = 1$ we have that

$$|N_g(z) - N_g(\zeta)| \ \leq \ \frac{\|g\|_p}{2\pi} \cdot \left[\int\limits_K | \log|z-w| - \log|\zeta - w| \,|^q \ dm(w) \right]^{\frac{1}{q}} \ =$$

$$= \ \frac{\|g\|_p}{2\pi} \cdot \left[\int\limits_{\mathbb{C}} | \, \psi_\zeta(\xi + \Delta z) - \psi_\zeta(\xi) \, |^q \ dm(\xi) \right]^{\frac{1}{q}}$$

where $\xi = \zeta - w$, $\Delta z = z - \zeta$, $\psi_\zeta(\xi) := \chi_{K+\zeta}(\xi) \log|\xi|$. Thus, the first conclusion follows by Lemma 1 because $\log|\xi| \in L^q_{\rm loc}(\mathbb{C})$ for all $q \in [1, \infty)$.

The second conclusion follows by the continuity of the integral from the right hand side in the above estimate with respect to the parameter $\zeta \in \mathbb{C}$. Indeed,

$$\|\psi_\zeta - \psi_{\zeta_*}\|_q \ = \ \left\{ \int\limits_{\Delta} |\log|\xi||^q \ dm(\xi) \right\}^{\frac{1}{q}}$$

where Δ denotes the symmetric difference of the disks $K + \zeta$ and $K + \zeta_*$. Thus, the statement follows from the absolute continuity of the indefinite integral.

The third conclusion similarly follows through the direct estimate

$$|N_g(\zeta)| \ \leq \ \frac{\|g\|_p}{2\pi} \cdot \left[\int\limits_K | \log|\zeta - w| \,|^q \ dm(w) \right]^{\frac{1}{q}} \ = \ \frac{\|g\|_p}{2\pi} \cdot \left[\int\limits_{\mathbb{C}} | \, \psi_\zeta(\xi) \, |^q \ dm(\xi) \right]^{\frac{1}{q}} .$$

□

Proposition 2. *There exist functions $g \in L^1(\mathbb{C})$ with compact support whose potentials N_g are not continuous, furthermore, $N_g \notin L^\infty_{\rm loc}$.*

Proof. Indeed, let us consider the function

$$g(z) = \omega(|z|),\ z \in \overline{\mathbb{D}}, \qquad g(z) \equiv 0,\ z \in \mathbb{C} \setminus \overline{\mathbb{D}}\ ,$$

where

$$\omega(t) = 1/t^2(1-\ln t)^{\alpha},\ t \in (0,1],\ \alpha \in (1,2), \qquad \omega(0) = 0\ .$$

Setting $\Omega(t) = t \cdot \omega(t)$, we see that, firstly,

$$\int\limits_{\overline{\mathbb{D}}} |g(w)|\ dm(w) \ = \ 2\pi \int\limits_0^1 \Omega(t)\ dt \ = \ 2\pi \int\limits_0^1 \frac{d\ln t}{(1-\ln t)^{\alpha}} \ = \ \frac{2\pi}{\alpha - 1}$$

and, secondly,

$$\mathrm{I} \ := \ N_g(0) \ = \ \int\limits_0^1 \Omega(t) \ln t\ dt \ = \ \left[\ln t \int\limits_0^t \Omega(\tau)\ d\tau \right]_0^1 \ - \ \int\limits_0^1 \left(\frac{1}{t} \int\limits_0^t \Omega(\tau)\ d\tau \right)\ dt \ =$$

$$= \ \frac{1}{\alpha-1} \cdot \left(\left[\frac{\ln t}{(1-\ln t)^{\alpha-1}} \right]_0^1 \ + \ \int\limits_0^1 \frac{dt}{t(1-\ln t)^{\alpha-1}} \right) \ =$$

$$= \ \frac{1}{\alpha-1} \cdot \left[(1-\ln t)^{1-\alpha} \ - \frac{3-\alpha}{2-\alpha} \cdot (1-\ln t)^{2-\alpha} \right]_0^1 \ = \ -\infty\ .$$

□

The following theorem on the Newtonian potentials is important to obtain solutions of the Dirichlet problem to the Poisson equation of higher regularities.

Theorem 2. *Let $g : \mathbb{C} \to \mathbb{R}$ have compact support. If $g \in L^1(\mathbb{C})$, then $N_g \in L^r_{\mathrm{loc}}$ for all $r \in [1, \infty)$, $N_g \in W^{1,q}_{\mathrm{loc}}$ for all $q \in [1,2)$, moreover, $N_g \in W^{2,1}_{\mathrm{loc}}$ and*

$$4 \cdot \frac{\partial^2 N_g}{\partial z \partial \overline{z}} \ = \ \triangle N_g \ = \ 4 \cdot \frac{\partial^2 N_g}{\partial \overline{z} \partial z} \ = \ g\ . \tag{15}$$

If $g \in L^p(\mathbb{C})$, $p > 1$, then $N_g \in W^{2,p}_{\mathrm{loc}}$, $\triangle N_g = g$ a.e. and, moreover, $N_g \in W^{1,q}_{\mathrm{loc}}$ for $q > 2$, consequently, N_g is locally Hölder continuous. If $g \in L^p(\mathbb{C})$, $p > 2$, then $N_g \in C^{1,\alpha}_{\mathrm{loc}}$ where $\alpha = (p-2)/p$.

In this connection, recall the definition of the formal complex derivatives:

$$\frac{\partial}{\partial z} \ := \ \frac{1}{2} \left\{ \frac{\partial}{\partial x} \ - \ i \cdot \frac{\partial}{\partial y} \right\}\ , \quad \frac{\partial}{\partial \overline{z}} \ := \ \frac{1}{2} \left\{ \frac{\partial}{\partial x} \ + \ i \cdot \frac{\partial}{\partial y} \right\}\ , \quad z = x + iy\ .$$

The elementary algebraic calculations show that the Laplacian

$$\triangle \ := \ \frac{\partial^2}{\partial x^2} \ + \ \frac{\partial^2}{\partial y^2} \ = \ 4 \cdot \frac{\partial^2}{\partial z \partial \overline{z}} \ = \ 4 \cdot \frac{\partial^2}{\partial \overline{z} \partial z}.$$

Proof. Note that N_g is the convolution $\varphi * g$, where $\varphi(\zeta) = \log|\zeta|$, and that

$$\frac{\partial}{\partial z}\log|z-w| \;=\; \frac{1}{2}\cdot\frac{1}{z-w}\ , \qquad \frac{\partial}{\partial \overline{z}}\log|z-w| \;=\; \frac{1}{2}\cdot\frac{1}{\overline{z}-\overline{w}}\ .$$

Note also that $N_g \in L^r_{\rm loc}$ for all $r \in [1,\infty)$, see e.g. Corollary 4.5.2 in [9]. Moreover, $\frac{\partial\varphi * g}{\partial z} = \frac{\partial\varphi}{\partial z} * g$ and $\frac{\partial\varphi * g}{\partial\overline{z}} = \frac{\partial\varphi}{\partial\overline{z}} * g$, see e.g. (4.2.5) in [9]. Hence

$$\frac{\partial N_g(z)}{\partial z} \;=\; \frac{1}{4}\cdot Tg(z)\ , \qquad \frac{\partial N_g(z)}{\partial\overline{z}} \;=\; \frac{1}{4}\cdot \overline{T}g(z)\ ,$$

where Tg and $\overline{T}g$ are the well-known integral operators

$$Tg(z) \;:=\; \frac{1}{\pi}\int\limits_{\mathbb{C}} g(w)\,\frac{dm(w)}{z-w}\ , \qquad \overline{T}g(z) \;:=\; \frac{1}{\pi}\int\limits_{\mathbb{C}} g(w)\,\frac{dm(w)}{\overline{z}-\overline{w}}\ .$$

Thus, all conclusions for $g \in L^1(\mathbb{C})$ follow by Theorems 1.13–1.14 in [21]. If $g \in L^p(\mathbb{C})$, $p>1$, then $N_g \in W^{1,q}_{\rm loc}$, $q>2$, by Theorem 1.27, (6.27) in [21], consequently, N_g is locally Hölder continuous, see e.g. Theorem 8.22 in [6], and $N_g \in W^{2,p}_{\rm loc}$ by Theorems 1.36–1.37 in [21]. If $g \in L^p(\mathbb{C})$, $p>2$, then $N_g \in C^{1,\alpha}_{\rm loc}$ with $\alpha = \frac{p-2}{p}$ by Theorem 1.19 in [21]. $\Box$

By Theorem 2 and the known Poisson formula, see e.g. I.D.2 in [12], we come to the following consequence on the existence, regularity and representation of solutions for the Dirichlet problem to the Poisson equation in the unit disk $\mathbb{D}$ where we assume the charge density g to be extended by zero outside $\mathbb{D}$.

Corollary 2. *Let $\varphi : \partial\mathbb{D} \to \mathbb{R}$ be a continuous function and $g : \mathbb{D} \to \mathbb{R}$ belong to the class $L^p(\mathbb{D})$, $p>1$. Then the function $U := N_g - \mathcal{P}_{N^*_g} + \mathcal{P}_\varphi$, $N^*_g := N_g|_{\partial\mathbb{D}}$, is continuous in $\overline{\mathbb{D}}$ with $U|_{\partial\mathbb{D}} = \varphi$, belongs to the class $W^{2,p}_{\rm loc}(\mathbb{D})$ and $\triangle U = g$ a.e. in $\mathbb{D}$. Moreover, $U \in W^{1,q}_{\rm loc}(\mathbb{D})$ for some $q>2$ and U is locally Hölder continuous. If in addition φ is Hölder continuous, then U is Hölder continuous in $\overline{\mathbb{D}}$. If $g \in L^p(\mathbb{D})$, $p>2$, then $U \in C^{1,\alpha}_{\rm loc}(\mathbb{D})$, where $\alpha = (p-2)/p$.*

Remark 2. The Hölder continuity of U for Hölder continuous φ follows from the corresponding result for the integral of the Cauchy type over the unit circle, see e.g. Theorem 1.10 in [21], because of the Poisson kernel $P(z,e^{it}) = {\rm Re}\,\frac{e^{it}+z}{e^{it}-z}$. Note also by the way that a generalized solution of the Dirichlet problem to the Poisson equation in the class $C(\overline{\mathbb{D}}) \cap W^{1,2}_{\rm loc}(\mathbb{D})$ is unique at all, see e.g. Theorem 8.30 in [6]. One can show that the integral operators in Theorem 2 and Corollary 2 are completely continuous (it is clear from the corresponding theorems in [21] mentioned under the proof of Theorem 2), cf. e.g. [10] and [11]. However, for our goals it is sufficient that the operator $N_g : L^p(\mathbb{D}) \to C(\overline{\mathbb{D}})$ is completely continuous by Theorem 1 for $p>1$, see the proof of Theorem 3 further.

3. The case of the quasilinear Poisson equations.

The case is reduced to the Poisson equation by the Leray–Schauder approach.

Theorem 3. *Let $\varphi : \partial\mathbb{D} \to \mathbb{R}$ be a continuous function, $h : \mathbb{D} \to \mathbb{R}$ be a function in the class $L^p(\mathbb{D})$, $p > 1$, and let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function with the nondecreasing function $|f|$ of $|t|$ such that*

$$\lim_{t\to+\infty} \frac{f(t)}{t} = 0 . \tag{16}$$

Then there is a continuous function $U : \overline{\mathbb{D}} \to \mathbb{R}$ with $U|_{\partial\mathbb{D}} = \varphi$, $U|_{\mathbb{D}} \in W^{2,p}_{\rm loc}$ and

$$\triangle U(z) = h(z) \cdot f(U(z)) \qquad \text{for a.e. } z \in \mathbb{D} . \tag{17}$$

Moreover, $U \in W^{1,q}_{\rm loc}(\mathbb{D})$ for some $q > 2$ and U is locally Hölder continuous. If in addition φ is Hölder continuous, then U is Hölder continuous in $\overline{\mathbb{D}}$. Furthermore, if $p > 2$, then $U \in C^{1,\alpha}_{\rm loc}(\mathbb{D})$, where $\alpha = (p-2)/p$.

In particular, the latter statement in Theorem 3 implies that $U \in C^{1,\alpha}_{\rm loc}(\mathbb{D})$ for all $\alpha = (0,1)$ if h is bounded.

Proof. If $\|h\|_p = 0$ or $\|f\|_C = 0$, then the Poisson integral $\mathcal{P}_\varphi$ gives the desired solution of the Dirichlet problem for equation (17), see e.g. I.D.2 in [12]. Hence we may assume further that $\|h\|_p \neq 0$ and $\|f\|_C \neq 0$.

By Theorem 1 and the maximum principle for harmonic functions, we obtain the family of operators $F(g;\tau) : L^p(\mathbb{D}) \to L^p(\mathbb{D})$, $\tau \in [0,1]$:

$$F(g;\tau) := \tau h \cdot f(N_g - \mathcal{P}_{N_g^*} + \mathcal{P}_\varphi) , \; N_g^* := N_g|_{\partial\mathbb{D}} , \qquad \forall\, \tau \in [0,1] \tag{18}$$

which satisfies all groups of hypothesis H1–H3 of Theorem 1 in [15].

H1). First of all, $F(g;\tau) \in L^p(\mathbb{D})$ for all $\tau \in [0,1]$ and $g \in L^p(\mathbb{D})$ because by Theorem 1 $f(N_g - \mathcal{P}_{N_g^*} + \mathcal{P}_\varphi)$ is a continuous function and, moreover, by (14)

$$\|F(g;\tau)\|_p \le \|h\|_p \, |f\,(2M\,\|g\|_p + \|\varphi\|_C)\,| < \infty \qquad \forall\, \tau \in [0,1] .$$

Thus, by Theorem 1 in combination with the Arzela–Ascoli theorem, see e.g. Theorem IV.6.7 in [4], the operators $F(g;\tau)$ are completely continuous for each $\tau \in [0,1]$ and even uniformly continuous with respect to the parameter $\tau \in [0,1]$.

H2). The index of the operator $F(g;0)$ is obviously equal to 1.

H3). By inequality (14) and the maximum principle for harmonic functions, we have the estimate for solutions $g \in L^p$ of the equations $g = F(g;\tau)$:

$$\|g\|_p \le \|h\|_p \, |f\,(2M\,\|g\|_p + \|\varphi\|_C)\,| \le \|h\|_p \, |f(3M\,\|g\|_p)|$$

whenever $\|g\|_p \ge \|\varphi\|_C/M$, i.e. then it should be

$$\frac{|f(3M\,\|g\|_p)|}{3M\,\|g\|_p} \ge \frac{1}{3M\,\|h\|_p} \tag{19}$$

and hence $\|g\|_p$ should be bounded in view of condition (16).

Thus, by Theorem 1 in [15] there is a function $g \in L^p(\mathbb{D})$ such that $g = F(g;1)$ and, consequently, by Corollaries 2 the function $U := N_g - \mathcal{P}_{N_g^*} + \mathcal{P}_\varphi$ gives the desired solution of the Dirichlet problem for the quasilinear Poisson equation (17). □

Remark 3. As it is clear from the proof, condition (16) can be replaced by the weaker one

$$\limsup_{t\to+\infty} \frac{|f(t)|}{t} \; < \; \frac{1}{3M\|h\|_p} \tag{20}$$

where M is the constant from estimate (14). Moreover, Theorem 3 is valid if f is an arbitrary continuous bounded function.

Theorem 3 together with Remark 3 can be applied to many physical problems. The first circle of such applications is relevant to reaction-diffusion problems. Problems of this type are discussed detaily in [1]. A nonlinear system is obtained for the density u and the temperature T of the reactant. Upon eliminating T the system can be reduced to the equation (17) with $h(z) \equiv \lambda > 0$ and, for isothermal reactions, $f(u) = u^q$ where $q > 0$ is called the order of the reaction. It turns out that the density of the reactant u may be zero in a subdomain called a **dead core.** A particularization of results in Chapter 1 of [3] shows that a dead core may exist just if and only if $0 < q < 1$ and λ is large enough, see also the corresponding examples in [7]. Certain mathematical models of a thermal evolution of a heated plasma also lead to the equation of the type (17), for instance with $f(u) = |u|^{q-1}u$, $0 < q < 1$. Finally, in the theory of the stationary combustion, see e.g. [2, 17] and the references therein, the equation (17) arose with $h \equiv \delta > 0$ and the bounded functions $f(u) = e^{-\beta\cdot u}$, $\beta > 0$, as in Remark 3.

Thus, in Theorem 3 we have established the existence of solutions of more high regularities of the Dirichlet problem for quasilinear Poisson equations than in the monographs [3, 6] and [16], having significant applications. Our approach makes possible to extend the main part of the above results to arbitrary Jordan's domains whose boundaries are smooth, Lipschitz and the so–called quasiconformal boundaries that can be even even locally not rectifiable, and also with the quasihyperbolic boundary condition that, generally speaking, not implying the standard (A)–condition and the known outer cone condition, see e.g. [13]. Furthermore, thanking to a factorization theorem established by us earlier in [7], we are able to extend them to the semi–linear partial differential equations, the linear part of which is written in a divergence (anisotropic !) form. Such extensions of these results will be published elsewhere.

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О задаче Дирихле для квазилинейных уравнений Пуассона.

Изучается задача Дирихле для квазилинейных дифференциальных уравнений в частых производных вида $\triangle u(z) = h(z) \cdot f(u(z))$ в единичном круге $\mathbb{D} \subset \mathbb{C}$ с функциями $h : \mathbb{D} \to \mathbb{R}$ из класса

$L^p(\mathbb{D})$, $p > 1$, и непрерывными функциями $f : \mathbb{R} \to \mathbb{R}$ с неубывающими $|f|$ от $|t|$, такими, что $f(t)/t \to 0$ при $t \to \infty$. На основе теории потенциала, и применяя подход Лере–Шаудера, при произвольных непрерывных граничных данных $\varphi : \partial\mathbb{D} \to \mathbb{R}$, доказано существование непрерывных решений u поставленной задачи в классе $W^{2,p}_{\rm loc}$. Кроме того, показано, что $u \in W^{1,q}_{\rm loc}(\mathbb{D})$ для некоторых $q > 2$ и что u локально непрерывно по Гёльдеру. Если дополнительно φ непрерывно по Гёльдеру, то u непрерывно по Гёльдеру в $\mathbb{D}$. Более того, $u \in C^{1,\alpha}_{\rm loc}(\mathbb{D})$ с $\alpha = (p-2)/p$, если $p > 2$.

Ключевые слова: *полулинейные эллиптические уравнения, квазилинейные уравнения Пуассона, логарифмический потенциал, подход Лере–Шаудера.*

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Про задачу Діріхле для квазілінійних рівняняь Пуасона.

Вивчається задача Діріхле для квазілінійних диференціальних рівнянь в частинних похідних виду $\triangle u(z) = h(z) \cdot f(u(z))$ в одиничному колі $\mathbb{D} \subset \mathbb{C}$ з функціями $h : \mathbb{D} \to \mathbb{R}$ із класу $L^p(\mathbb{D})$, $p > 1$, і непрервними функціями $f : \mathbb{R} \to \mathbb{R}$ з неспадаючими $|f|$ від $|t|$, такими, що $f(t)/t \to 0$ при $t \to \infty$. На основі теорії потенциалу, і застосовуючи підхід Лере–Шаудера, при довільних неперервних граничних даних $\varphi : \partial\mathbb{D} \to \mathbb{R}$, доведено існування неперервних розв'язків u поставленої задачі в класі $W^{2,p}_{\rm loc}$. Крім того, показано, що $u \in W^{1,q}_{\rm loc}(\mathbb{D})$ для деяких $q > 2$ і що u локально неперервне по Гельдеру. Якщо додатково φ неперервне по Гельдеру, то u неперервне по Гельдеру в $\mathbb{D}$. Більш того, $u \in C^{1,\alpha}_{\rm loc}(\mathbb{D})$ з $\alpha = (p-2)/p$, якщо $p > 2$.

Ключові слова: *напівлвнвйні еліптичні рівняння, квазілінійні рівняння Пуасона, логарифмічний потенціал, підхід Лере–Шаудера.*

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