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Toy model of superconductivity

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The model of hypothetical superconductivity, where the energy gap asymptotically approaches zero as temperature or magnetic field increases, has been proposed. Formally the critical temperature and the second critical field for such a superconductor is equal to infinity. Thus the material is in superconducting state always.

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Critical temperature T_c and critical magnetic fields H_c, H_{c2} are most important characteristics of a superconductor. The critical parameters depends on an effective coupling constant with some collective excitations $g = \nu_F \lambda \lesssim 1$ (here ν_F is a density of states at Fermi level, λ is an interaction constant), on the frequency of the collective excitations ω and on the correlation length ξ_0. The larger coupling constant the larger the critical parameters. For example, for large values of g we have $T_c \propto \omega\sqrt{g}$ [1,2] (or $T_c \propto \omega g$ in BCS theory). Formally the critical temperature can be made arbitrarily large by increasing the electron–phonon coupling constant. However in order to reach room temperature such values of the coupling constant are necessary which are not possible in real materials. Moreover we can increase the frequency ω due nonphonon pairing mechanisms as proposed in [2]. However with increasing of the frequency the coupling constant decreases as $g \propto 1/\omega$, therefore $T_c(\omega \to \infty) = 1.14\omega \exp(-1/g) \to 0$. The second critical magnetic field can be enlarge due to the decrease of the correlation length in "dirty limit" $\xi = \sqrt{\xi_0 l}$ [4], where l is a free length. However the critical field is low near the critical temperature: $H_{c2}(T \to T_c) \to 0$. In a present work we generalize BCS model so that the problem of the critical parameters is removed due to the fact that a ratio between the gap and the critical temperature ($2\Delta/T_c = 3$–7 for presently known materials) is changed to $2\Delta/T_c \to 0$. We consider a system of fermions with Hamiltonian

$$\widehat{H} = \sum_{\mathbf{k},\sigma} \xi(k) a^{+}_{\mathbf{k},\sigma} a_{\mathbf{k},\sigma} - \frac{\lambda}{V} \sum_{\mathbf{k},\mathbf{p}} a^{+}_{\mathbf{p}\uparrow} a^{+}_{-\mathbf{p}\downarrow} a_{-\mathbf{k}\downarrow} a_{\mathbf{k}\uparrow} +$$

$$+ \upsilon \sum_{\mathbf{k}} \left[\frac{\Delta}{|\Delta|} a^{+}_{\mathbf{k}\uparrow} a^{+}_{-\mathbf{k}\downarrow} + \frac{\Delta^{+}}{|\Delta|} a_{-\mathbf{k}\downarrow} a_{\mathbf{k}\uparrow} \right] \equiv \widehat{H}_{BCS} + \widehat{H}_{\text{ext}}, \quad (1)$$

where $\widehat{H}_{BCS}$ is BCS Hamiltonian — kinetic energy + pairing interaction ($\lambda > 0$), energy $\xi(k) \approx v_F(|\mathbf{k}| - k_F)$ is counted from Fermy surface. The term $\widehat{H}_{\text{ext}}$ is the external pair potential, or "source term" [3]. Operators $a^{+}_{\mathbf{k}\uparrow} a^{+}_{-\mathbf{k}\downarrow}$ and $a_{-\mathbf{k}\downarrow} a_{\mathbf{k}\uparrow}$ are creation and annihilation of Cooper pair operators [5], Δ and Δ^{+} are anomalous averages:

$$\Delta^{+} = \frac{\lambda}{V} \sum_{\mathbf{p}} \left\langle a^{+}_{\mathbf{p}\uparrow} a^{+}_{-\mathbf{p}\downarrow} \right\rangle, \quad \Delta = \frac{\lambda}{V} \sum_{\mathbf{p}} \left\langle a_{-\mathbf{p}\downarrow} a_{\mathbf{p}\uparrow} \right\rangle, \quad (2)$$

which are the complex order parameter $\Delta = |\Delta| e^{i\theta}$. The multipliers $\Delta/|\Delta|$ and $\Delta^{+}/|\Delta|$ are introduced into $\widehat{H}_{\text{ext}}$ in order that the energy does not depend on the phase θ ($a \to a e^{i\theta/2}$, $a^{+} \to a^{+} e^{-i\theta/2} \Rightarrow \Delta \to \Delta e^{i\theta}$, $\Delta^{+} \to \Delta^{+} e^{-i\theta}$). Thus both $\widehat{H}_{BCS}$ and $\widehat{H}_{\text{ext}}$ is invariant under the $U(1)$ transformation unlike the source term in [3] where it has a noninvariant form

$$\upsilon \sum \left[a^{+}_{\mathbf{k}\uparrow} a^{+}_{-\mathbf{k}\downarrow} + a_{-\mathbf{k}\downarrow} a_{\mathbf{k}\uparrow} \right].$$

Hence υ is the energy of a Cooper pair relative to uncoupled state of the electrons in the external pair potential H_{ext}. It should be noted that the energy gap $|\Delta|$ is the energy of a Cooper pair relative to uncoupled state of the electrons too. However the field Δ is a self-consistent field as a consequence of attraction between electrons. The field υ is the applied field to the system from the outside.

Using the Fermi commutation relations and the anomalous averages (2), Hamiltonian (1) can be rewritten in a form

$$\widehat{H} = \sum_{\mathbf{k},\sigma} \xi(k) a^{+}_{\mathbf{k},\sigma} a_{\mathbf{k},\sigma} +$$

$$+ \left(1 - \frac{\upsilon}{|\Delta|}\right) \sum_{\mathbf{k}} \left[\Delta^{+} a_{\mathbf{k}\uparrow} a_{-\mathbf{k}\downarrow} + \Delta a^{+}_{-\mathbf{k}\downarrow} a^{+}_{\mathbf{k}\uparrow} \right] + \frac{1}{\lambda} V |\Delta|^2 . \quad (3)$$

Then normal G and anomalous F propagators have forms

$$G = i\frac{i\varepsilon_n + \xi}{(i\varepsilon_n)^2 - \xi^2 - |\Delta|^2 (1-\upsilon/|\Delta|)^2}, \tag{4}$$

$$F = i\frac{\Delta(1-\upsilon/|\Delta|)}{(i\varepsilon_n)^2 - \xi^2 - |\Delta|^2 (1-\upsilon/|\Delta|)^2}, \tag{5}$$

where $\varepsilon_n = \pi T(2n+1)$ [6]. From Eq. (2) we have self-consistency condition for the order parameter

$$\Delta = \lambda \nu_F T \sum_{n=-\infty}^{\infty} \int_{-\omega}^{\omega} d\xi\, iF(\varepsilon_n, \xi) \Rightarrow 1 =$$

$$= g\int_{-\omega}^{\omega} d\xi \frac{1-\upsilon/|\Delta|}{2\sqrt{\xi^2 + |\Delta|^2 (1-\upsilon/|\Delta|)^2}} \times$$

$$\times \tanh\frac{\sqrt{\xi^2 + |\Delta|^2 (1-\upsilon/|\Delta|)^2}}{2T}. \tag{6}$$

Solutions of Eq. (6) are shown in Fig. 1. If the external pair potential is absent, $\upsilon = 0$, we have usual self-consistency equation for the gap Δ: the gap is a function of temperature such that $\Delta(T \geq T_c) = 0$. The larger coupling constant $g = \lambda\nu_F$ the larger T_c. If $\upsilon > 0$ then the pairing of quasiparticles results in increase of the system's energy that suppresses superconductivity and first order phase transition takes place. If $\upsilon < 0$ then the pairing results in decrease of the system's energy. In this case a solution of Eq. (6) is such that the gap Δ does not vanish at any temperature. At large temperature $T >> T_c$ the gap is

$$|\Delta(T \to \infty)| = \frac{g\omega|\upsilon|}{2T}. \tag{7}$$

Then the critical temperature is $T_c = \infty$ (in reality it limited by the melting of the substance). It should be noted that if $\lambda = 0$ then for any υ a superconducting state does not exist ($\Delta = 0$ always). This means electron–electron coupling is the cause of the transition to superconducting state only but not the external pair potential υ.

For investigation of thermodynamic and electrodynamic properties of the system we should to find a free energy. Greatest interest is the case $\upsilon < 0$ in a limit $T \to \infty$. We can see in Fig. 1 that Δ is small in this region. This means that as a starting point we can take the Landau expansion (in a momentum space) [4]:

$$F_s = F_n + A|\Delta|^2 + \frac{B}{2}|\Delta|^4 + q^2 C|\Delta|^2, \tag{8}$$

where

$$A = \nu_F \frac{T - T_c}{T_c}, \quad B = \nu_F \frac{7\zeta(3)}{8\pi^2 T_c^2}, \quad C = \nu_F \xi_0^2, \tag{9}$$

q is a momentum of a Cooper pair, ξ_0 is a coherence length at $T = 0$, F_n is a free energy of a normal state. In a limit $T >> T_c$ we can write a coefficient A as $A = \nu_F T/\widetilde{T}_c > 0$,

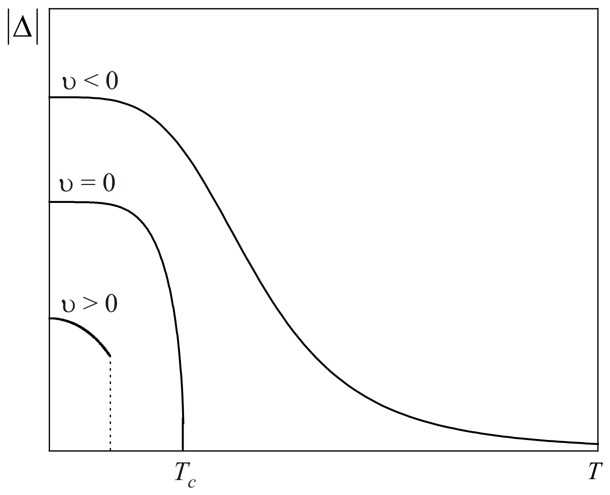

Fig. 1. Energy gaps $\Delta(T)$ as solution of Eq. (6) for three values of the external pair potential υ.

where $\widetilde{T}_c$ is an adjustable parameter now, and we should to add to the free energy a term $\langle \widehat{H}_{\text{ext}} \rangle$. Using the anomalous averages (2) we can obtain

$$\langle \widehat{H}_{\text{ext}} \rangle = \frac{2\upsilon}{\lambda}|\Delta| < 0.$$

Then the free energy has a form

$$F_s = F_n + A|\Delta|^2 + q^2 C|\Delta|^2 + \frac{2\upsilon}{\lambda}|\Delta|. \tag{10}$$

A term $B|\Delta|^4/2$ can be omitted due to the smallness of the gap. Minimization of the free energy with respect to $|\Delta|$ (if $q = 0$) gives

$$|\Delta| = \frac{|\upsilon|}{A\lambda} = \frac{g\omega|\upsilon|}{2T} \Rightarrow A = \frac{2T}{\nu_F \lambda^2 \omega}, \tag{11}$$

where we must suppose $\widetilde{T}_c = g^2\omega/2$ in order to get Eq. (7). Difference of the free energy (8) from the free energy (11) is shown in Fig. 2. We can see that at $\upsilon < 0$ a superconducting phase exists at any temperature.

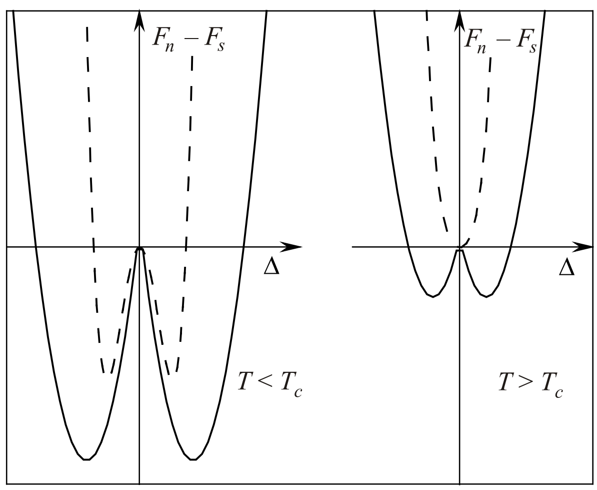

Fig. 2. Free energies Eq. (8) (dash line) and Eq. (10) (solid line) at $T < T_c$ and $T > T_c$ ($q = 0$).

In coordinate space we can write Gibbs free energy as

$$G_s = G_n + a|\Psi|^2 + 2u|\Psi| + \\ + \frac{1}{4m}\left|\left(-i\hbar\nabla - \frac{2e}{c}\mathbf{A}\right)\Psi\right|^2 + \frac{H^2}{8\pi} - \frac{\mathbf{HH}_0}{4\pi}, \qquad (12)$$

where $\mathbf{H}$ is a microscopic magnetic field in each point of a superconductor, $\mathbf{H}_0$ is a strength of an external homogeneous magnetic field, $\mathbf{A} = \operatorname{rot}\mathbf{H}$ is a vector-potential. Coefficient a is proportional to temperature $a = \alpha T > 0$, coefficient u is proportional to the external pair potential $u = \eta\upsilon < 0$. The Eq. (12) is valid at $T >> T_c$ only. For regions $T \sim T_c$ and $T \to 0$ we must replace $\alpha T \to \alpha(T - T_c)$ and take into account a term $b/2|\Psi|^4$.

The free energy (12) can be made dimensionless:

$$G_s = G_n + \frac{H_c^2}{8\pi}\left[|\varphi|^2 - 2|\varphi| + \xi^2\left|\left(i\nabla + \frac{2\pi}{\Phi_0}\mathbf{A}\right)\varphi\right|^2\right] + \\ + \frac{H^2}{8\pi} - \frac{\mathbf{HH}_0}{4\pi}. \qquad (13)$$

Then Lagrange equations are

$$\xi^2|\varphi|\left(i\nabla + \frac{2\pi}{\Phi_0}\mathbf{A}\right)^2\varphi + \varphi|\varphi| - \varphi = 0, \qquad (14)$$

$$\operatorname{rot}\operatorname{rot}\mathbf{A} = -i\frac{\Phi_0}{4\pi\lambda^2}\left(\varphi^+\nabla\varphi - \varphi\nabla\varphi^+\right) - \frac{|\varphi|^2}{\lambda^2}\mathbf{A}, \quad (15)$$

and a boundary condition are

$$\left(i\hbar\nabla + \frac{2\pi}{\Phi_0}\mathbf{A}\right)\mathbf{n}\varphi = 0. \qquad (16)$$

Here φ is a dimensionless order parameter and H_c is a critical magnetic field:

$$|\Psi(\mathbf{A} = 0, \nabla\Psi = 0)| \equiv \Psi_0 = \frac{|u|}{a} \sim \frac{\upsilon}{T} \Rightarrow \varphi = \frac{\Psi}{\Psi_0}, \qquad (17)$$

$$\frac{H_c^2}{8\pi} = \frac{u^2}{a} \Rightarrow H_c \sim \frac{|\upsilon|}{\sqrt{T}}, \qquad (18)$$

$\mathbf{n}$ is a normal to superconductor's surface, $\Phi_0 = \pi\hbar c/e$ is a magnetic flux quantum. Correlation length ξ, magnetic field penetration depth λ and Ginzburg–Landau parameter χ are

$$\xi^2 = \frac{\hbar^2}{4ma} \Rightarrow \xi \sim \frac{1}{\sqrt{T}}, \qquad (19)$$

$$\frac{1}{\lambda^2} = \frac{8\pi e^2}{mc^2}|\Psi_0|^2 \Rightarrow \lambda \sim \frac{T}{|\upsilon|}, \qquad (20)$$

$$\chi = \frac{\lambda}{\xi} \sim \frac{T^{3/2}}{|\upsilon|}. \qquad (21)$$

The proportionality of the penetration depth to the temperature and the inverse proportionality to the external pair potential — Eq. (20) is the expected result. Greater attention should be given to a reduction of the correlation length with temperature — Eq. (19). We can see ξ is determined by properties of a superconductor only (at $T >> T_c$). We know that the correlation length depends on temperature as $\xi = \xi_0/\sqrt{|1 - T/T_c|}$. That is at $T < T_c$ it increases with increasing temperature, at $T = T_c$ it diverges, at $T > T_c$ it decreases with increasing temperature as $1/T$ (at large T). However above the critical temperature the correlation length has physical sense of the size of a superconducting phase nucleus in a normal conductor. Superconducting phase at $T > T_c$ is energetically unfavorable and because it arises fluctuationally by bubble size ξ. Switching of the field υ changes the situation. The field holds fluctuationally arisen superconducting phase nucleuses. This continues until the superconducting phase does not fill the entire volume of the metal. From Eq. (21) we can see the Ginzburg–Landau parameter increases with temperature as $T^{3/2}$ unlike usual superconductors where the parameter is constant. This means that at large temperature all superconductors in the external pair field become type II superconductors.

Besides the critical temperature important characteristics of a superconductor are the first H_{c1} and the second H_{c2} critical fields. The first critical field is half as much than a field of single vortex which can be determined from Eq. (15). Thus we have

$$H_{c1} = \frac{\Phi_0}{4\pi\lambda^2}\ln\chi \sim \frac{\upsilon^2}{T^2}. \qquad (22)$$

Hence critical current of emergence of resistance is

$$I_{c1} = \frac{1}{2}H_{c1}cR \sim \frac{\upsilon^2}{T^2}, \qquad (23)$$

where R is the radius of a wire [7]. For calculation of H_{c2} we can use the method presented in Appendix A. Then Eq. (15) has a form

$$\xi^2|\varphi|\left[-\frac{d^2}{dx^2} + \frac{2\pi i}{\Phi_0}Hx\frac{d}{dy} + \left(\frac{2\pi H}{\Phi_0}\right)^2 x^2\right]\varphi + \varphi|\varphi| - \varphi = 0. \qquad (24)$$

We can consider the order parameter is real $\varphi = \varphi^+$ and average it over the system so that $\langle\varphi(x, y)\rangle = \text{const} = \varphi > 0$. Then we have

$$\xi^4\left(\frac{2\pi H}{\Phi_0}\right)^2\varphi + \varphi - 1 = 0, \qquad (25)$$

and the order parameter is

$$\varphi = \frac{1}{1 + \xi^4\left(2\pi H/\Phi_0\right)^2}. \qquad (26)$$

We can see φ decreases with increasing magnetic field, however the second critical field is infinity like the critical temperature. Superconductor phase exists at any magnetic field. The absence of a phase transition to the normal state with increasing magnetic field can be explained as follows. In Ginzburg–Landau theory transition to normal state takes place when average distance between vortexes becomes the order of the correlation length ξ. A line in center of a vortex is normal. If distance between the centers of vortexes is ξ hence the system is divided into the superconducting regions size of ξ. However the correlation length is size of a normal phase nucleus arising fluctuationally in the superconductor. Thus the fluctuations destroy superconducting phase if distance between centers of vortexes is less than ξ. As mentioned above switching of the field υ changes the situation. The external pair potential holds superconducting phase regions of the size of ξ given by Eq. (19). Thus the superconducting phase is stable at any concentration of the vortexes hence at any magnetic field intensity.

Thus the proposed model of hypothetical superconductivity demonstrates the principal differences from results of BCS and Ginzburg–Landau theory due presence of the external pair potential. In a case of decreasing of Cooper pair's energy by the external field the energy gap tends to zero asymptotically with increasing temperature. The ratio between the gap and the critical temperature is $2\Delta/T_c = 0$ instead of a finite value in BCS theory. Moreover the energy gap tends to zero asymptotically with increasing magnetic field. Thus critical temperature and the second critical magnetic field are equal to infinity formally. Unlike BCS model the Ginzburg–Landau parameter is not constant and it increases with temperature. This means that at large temperature all superconductors in the external pair field become type II superconductors. However the first critical magnetic field and maximal current of a thin wire are finite values and decrease with temperature. This model does not solve the problem of room-temperature superconductivity, however it allows to reformulate the problem. Possible practical realization of the model is proposed in [8], where a source of the external pair potential has been constructed.

Appendix A: Simple method of calculation of the second critical field H_{c2} in Ginzburg–Landau theory

Let a superconductor is in magnetic field $\mathbf{H}\uparrow\uparrow Oz$. It is convenient to choose a calibration $A_y = Hx$. Then Ginzburg–Landau equation has a form [7]

$$\xi^2\left[-\frac{d^2}{dx^2}+\frac{2\pi i}{\Phi_0}Hx\frac{d}{dy}+\left(\frac{2\pi H}{\Phi_0}\right)^2 x^2\right]\varphi-\varphi+|\varphi|^2\varphi=0. \qquad \text{(A.1)}$$

Here unlike the standard method we retained a term φ^3. When the field strength is of about H_{c2} superconductor are pierced by many vortices, so that the order parameter is strongly nonhomogeneous $\varphi=\varphi(x,y)$, it varies over distances of the order of the coherence length ξ. We can consider the order parameter is real $\varphi=\varphi^+$ and average it over the system so that $\langle\varphi(x,y)\rangle=\varphi=\text{const}>0$. In addition, we can suppose $x^2=\xi^2$. Then Eq. (A.1) takes the form

$$\xi^4\left(\frac{2\pi H}{\Phi_0}\right)^2\varphi-\varphi+\varphi^3=0, \qquad \text{(A.2)}$$

and the order parameter is

$$\varphi=\sqrt{1-\xi^4\left(\frac{2\pi H}{\Phi_0}\right)^2}. \qquad \text{(A.3)}$$

We can see φ decreases with increasing magnetic field. At the field

$$H_{c2}=\frac{\Phi_0}{2\pi\xi^2}=\sqrt{2}\chi H_c \qquad \text{(A.4)}$$

second order phase transition takes place $\varphi(H_{c2})=0$. At $H>H_{c2}$ superconducting phase is absent.

Acknowledgment

We are pleased to dedicate this paper about superconductivity to 70th anniversary of Academician Vadim Mikhailovich Loktev. He is always interested and prolifically works in this burning topic, without which the technologies of the future are inconceivable.

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