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NUMERICAL INTEGRATION OF THE EQUATIONS OF THE THEORY OF CREEP, WHICH TAKES INTO ACCOUNT THE MICROSTRAINS

The algorithm of numerical integration of the constitutive relationships of the theory of creep, which takes into account the microstrains, was developed based on the backward Euler method. The effectiveness of this method comparing with the direct Euler method is shown.

Keywords: *theory of creep, microstrains, reverse Euler method.*

Introduction. A number of approaches to the construction of the theory of creep, which takes into account the microstrains, were considered in [3, 4]. One of the versions of the theory was analyzed in details In [1] and a comparison with known experimental data was given [5]. As a result of calculations, which were carried out by Euler method, it was found that the descriptions of the experiments need comparatively large number of material constants, incoming in the constitutive relationships. This fact makes direct use of the theory impractical. In this paper it is proposed to use a modified local law, which in conjunction with the use of reverse Euler method for numerical calculations allows significantly increases the effectiveness of the theory.

Problem definition. Resolving equations of the theory of creep, which takes into account the microstrains. Staying within the main hypotheses of plasticity theory, which takes into account the microstrains, we assume that the represented macrovolume consists of a set of interrelated microparticles, which stress-strain state is determined by microstresses and microstrains.

Following [1], we assume that each microparticle is in homogeneous stress-strain state. The relative amount of grain (the ratio of grain volume to the volume of represented macrovolume) is denoted by Δv_α Initial yield of grain is equal to $\tau_{0\alpha} \in T$. The plastic strain $\boldsymbol{\varepsilon}_\alpha^p = \lambda_\alpha \boldsymbol{\mu}_\alpha$, homogeneous within the scope of this volume, is characterized by the λ_n ($\lambda_\alpha = \sqrt{\boldsymbol{\varepsilon}_\alpha^p : \boldsymbol{\varepsilon}_\alpha^p}$) and direction $\boldsymbol{\mu}_\alpha \in \Omega$. Material is considered to be initially isotropic, and because of this, we assume that all possible directions of microplastic strain distributed evenly in Ω.

For the simplicity we consider the case of small strains. Constructing the local law of microparticles strains with number α the microstrain tensor $\boldsymbol{\varepsilon}_\alpha$ is considered as a sum of elastic $\boldsymbol{\varepsilon}_\alpha^e$ and plastic $\boldsymbol{\varepsilon}_\alpha^p$ components

$$\boldsymbol{\varepsilon}_\alpha = \boldsymbol{\varepsilon}_\alpha^e + \boldsymbol{\varepsilon}_\alpha^p. \tag{1}$$

Macroplastic strain is obtained by averaging the local viscoplastic strain throughout the represented macrovolume

$$\bar{\boldsymbol{\varepsilon}}^p = \sum_{\alpha=1}^{N} \lambda_\alpha \boldsymbol{\mu}_\alpha \Delta\nu_\alpha \quad \left(\sum_{\alpha=1}^{N} \Delta\nu_\alpha = 1 \right).$$

Assuming that the plastic strain doesn't affect the elasticity of the material, we can write

$$\bar{\boldsymbol{\sigma}} = \mathbf{G}_e : (\bar{\boldsymbol{\varepsilon}} - \bar{\boldsymbol{\varepsilon}}^p), \tag{2}$$

where $\bar{\boldsymbol{\sigma}}$ – tensor of macrostresses; $\mathbf{G}_e$ – fourth rank tensor (elastic stiffness matrix), which is written

$$\mathbf{G}_e = 2G\left[\frac{1}{2}\mathbf{I} + \frac{\nu}{1-2\nu}\mathbf{i} \otimes \mathbf{i} \right],$$

where G – the shear modulus; ν – the Poisson ratio; $\mathbf{I}, \mathbf{i}$ – fourth-rank and second-rank unit tensors. It is assumed that the elastic stiffness matrix of microparticles coincides with the elastic stiffness matrix of macrovolume.
Acting in a stress grain, deviator $\mathbf{s}_\alpha$ is represented (as it's usually done in the theory of microstrains) as the sum of

$$\mathbf{s}_\alpha = \boldsymbol{\tau}_\alpha + \boldsymbol{\rho}_\alpha,$$

where $\boldsymbol{\tau}_\alpha$ and $\boldsymbol{\rho}_\alpha$ – dissipative and "elastic" components of resistance to plastic strains. In the projection to the direction $\boldsymbol{\mu}_n$ we obtain

$$s_\alpha = \tau_\alpha + \rho_\alpha, \tag{3}$$

where it's designated that

$$s_\alpha = \mathbf{s}_\alpha : \boldsymbol{\mu}_\alpha, \tau_\alpha = \boldsymbol{\tau}_\alpha : \boldsymbol{\mu}_\alpha, \rho_\alpha = \boldsymbol{\rho}_\alpha : \boldsymbol{\mu}_\alpha. \tag{4}$$

It's should be noted that in the general case the local yield limit (yield limit of an individual grain) depends on the size and shape of a particular grain. In [1, 6] it was shown that taking into account the actual distribution of the grain size of the sample material (which also means the fluidity limit distribution of the different grains) can significantly improve the plastic strains description of the polycrystalline materials.

To describe the viscoplastic flow in [3, 4], it was proposed to set the tensors of dissipative $\boldsymbol{\tau}_\alpha$ and internal forces $\boldsymbol{\rho}_\alpha$ with the use of plastic flow rule, which is sensitive to strain rate, in the form of:

$$\dot{\boldsymbol{\varepsilon}}_\alpha^p = \dot{\lambda}_\alpha \boldsymbol{\mu}_\alpha;$$

$$\boldsymbol{\tau}_\alpha = \tau_\alpha(\lambda_\beta, \dot{\lambda}_\alpha)\boldsymbol{\mu}_\alpha;$$

$$\boldsymbol{\rho}_\alpha = \rho_\alpha(\lambda_\beta, \dot{\lambda}_\beta)\boldsymbol{\mu}_\alpha, \tag{5}$$

where $\dot{\lambda}_\alpha = \sqrt{\dot{\boldsymbol{\varepsilon}}_\alpha^p : \dot{\boldsymbol{\varepsilon}}_\alpha^p}$ – viscoplastic strain rate of the n-th microparticle.

Deviator of the internal microstress of microparticle, considering the results of [1, 3], are presented in the following form:

$$\rho_\alpha = \rho(\lambda_\alpha)\varphi(\dot{\lambda}_\alpha),\ \tau_\alpha = \tau_0(\bar{\lambda}) + \tau_1(\bar{\lambda})\psi(\dot{\lambda}_\alpha). \tag{6}$$

The evolution equation for parameter $\rho(\lambda_\alpha)$ represented as

$$\dot{\rho}(\lambda_\alpha) = a(\boldsymbol{\mu}_\alpha : \boldsymbol{\mu}'_\alpha)R_1\dot{\lambda}_\alpha + R_2\dot{\bar{\boldsymbol{\varepsilon}}}_p : \boldsymbol{\mu}_\alpha + R_3\dot{\bar{\lambda}}, \tag{7}$$

where it's designated that

$$\dot{\bar{\lambda}} = \sum_{\alpha=1}^{N} \dot{\lambda}_\alpha \Delta\nu_\alpha;$$

$$a(\boldsymbol{\mu}_\alpha : \boldsymbol{\mu}'_\alpha) = \frac{1}{2}[1 - \eta + (1+\eta)sign(\boldsymbol{\mu}_\alpha : \boldsymbol{\mu}'_\alpha)], \tag{8}$$

and R_1, R_2, R_3 – material functions, which depend on the distribution of yield limits, as it is in [1]; $\boldsymbol{\mu}'_\alpha$ – direction of the active local plastic strain $(\dot{\lambda}_\alpha > 0)$. Should be noted that in the process of plastic strain the value $\rho(\lambda_\alpha)$ changes not only in the direction of active microplastic strain, but also in all other directions.

To establish the connection of local micro- and macroscopic strain laws we use relationships of the Kroner type

$$\bar{\mathbf{s}} - \mathbf{s}_\alpha = A_0\left(\boldsymbol{\varepsilon}_\alpha^p - \bar{\boldsymbol{\varepsilon}}^p\right),$$

where A_0 – the constant of the material. From the last relationship in the projection to the $\boldsymbol{\mu}_\alpha$ direction, taking into account (2), (3)–(8), we find that

$$\bar{\sigma} : \mu_\alpha = \tau_0(\bar{\lambda}) + \tau_1(\bar{\lambda})\psi(\dot{\lambda}_\alpha) + \rho(\lambda_\alpha)\varphi(\dot{\lambda}_\alpha) + A_0(\lambda_\alpha - \bar{\varepsilon}^p : \mu_\alpha). \tag{9}$$

The last relationship is the local yield condition in the $\boldsymbol{\mu}_\alpha$ direction and it's sensitive to a strain rate. From (9) we see that function $\psi(\dot{\lambda}_\alpha)$ determines the effect of speed on the local yield limit. Function $\varphi(\dot{\lambda}_\alpha)$ determines the same effect on the internal stresses.

In [3] the special case was considered, where we have

$$\varphi\left(\dot{\lambda}_\alpha\right)=1,\psi\left(\dot{\lambda}_\alpha\right)=\left(\dot{\lambda}_\alpha\right)^m,$$

so it was assumed that the strain rate affects only the local yield limit. In this case from (9) we get

$$\bar{\mathbf{s}}:\boldsymbol{\mu}_\alpha-A_0(\lambda_\alpha-\boldsymbol{\mu}_\alpha:\bar{\boldsymbol{\varepsilon}}^p)-\rho(\lambda_\alpha)-\tau_0=\tau_1 f(\dot{\lambda}_\alpha{}^m),$$

so we get the local law of the Pezhiny viscoplastic theory type

$$\dot{\lambda}_\alpha=\left\langle\frac{\bar{\mathbf{s}}:\boldsymbol{\mu}_\alpha-A_0(\lambda_\alpha-\boldsymbol{\mu}_\alpha:\bar{\boldsymbol{\varepsilon}}^p)-\rho_\alpha(\lambda_\alpha)-\tau_0}{\tau_1}\right\rangle^{1/m},$$

where $\langle..\rangle$ – McCauley brackets, which are given by formula $\langle x\rangle=0{,}5(x+|x|)$.

Comparison of the theoretical results with given in [5] experimental data showed that satisfactory results can be achieved only with obligatory accounting of the yield limits distribution.

Considering the modified local law, we assume

$$\varphi\left(\dot{\lambda}_\alpha\right)=\psi\left(\dot{\lambda}_\alpha\right)=f^{-1}\left(\frac{\dot{\lambda}_\alpha}{\dot{\lambda}_0}\right),$$

where f^{-1} – the function that is inverse to the given one $f=f\left(\dot{\lambda}_n/\dot{\lambda}_0\right)$, which depends on the relative velocity of microplastic strain and τ_1, r_1, r_2, r_3 – material constants.

In this case, under the (9) we find

$$\bar{\boldsymbol{\sigma}}:\boldsymbol{\mu}_\alpha-A_0(\lambda_\alpha-\bar{\boldsymbol{\varepsilon}}^p:\boldsymbol{\mu}_\alpha)=[\tau_1+\rho(\lambda_\alpha)]f^{-1}\left(\frac{\dot{\lambda}_\alpha}{\dot{\lambda}_0}\right).$$

The last equation is represented as

$$\dot{\lambda}_\alpha=\dot{\lambda}_0 f\left(\chi_\alpha(t)\right), \tag{10}$$

where it's designated that:

$$\chi_\alpha(t)=\left\langle\frac{p_\alpha(t)}{g_\alpha(t)}\right\rangle, g_\alpha(t)=\tau_1+\rho(\lambda_\alpha(t));$$
$$p_\alpha(t)=[\bar{\boldsymbol{\sigma}}(t)+A_0\bar{\boldsymbol{\varepsilon}}^p(t)]:\boldsymbol{\mu}_\alpha-A_0\lambda_\alpha(t).$$

Considering the formulas (1), (8) and (7), the relationships (10) allow to determine the velocity of microplastic strain at a random time.

A numerical method for integrating the defining relationships. We suppose that at some moment of time t we know the values $\lambda_\alpha(t)$, $\bar{\boldsymbol{\sigma}}(t)$, $\bar{\boldsymbol{\varepsilon}}(t)$, $\bar{\boldsymbol{\varepsilon}}^p(t)$, $\rho_\alpha(t)$. Defining the deformation process means that we define $\bar{\boldsymbol{\varepsilon}} = \bar{\boldsymbol{\varepsilon}}(\tau)$ at some moment of time $\tau = t + \Delta t$ and we pose the problem of determining $\lambda_\alpha(\tau)$, $\bar{\boldsymbol{\sigma}}(\tau)$, $\bar{\boldsymbol{\varepsilon}}^p(\tau)$, $\rho_\alpha(\tau)$ at the same moment of time.

We use the inverse Euler method. Its main idea is that performance of the theory relationships is required not at the beginning step, as in the direct method, but at the end of the step, so that to improve the accuracy. Method is iterative and it's implemented by the predictor-corrector scheme. Thus, recording the increment of total and plastic strains on the step at the moment of time in the form

$$\bar{\boldsymbol{\varepsilon}}^p(\tau) = \bar{\boldsymbol{\varepsilon}}^p(t) + \Delta\bar{\boldsymbol{\varepsilon}}^p(\tau),$$

and using Hooke law (2),which is recorded at the moment of time τ , we get

$$\bar{\boldsymbol{\sigma}}(\tau) = \bar{\boldsymbol{\sigma}}(t) + \mathbf{G}_e : \Delta\bar{\boldsymbol{\varepsilon}}(\tau) - \mathbf{G}_e : \Delta\bar{\boldsymbol{\varepsilon}}^p(\tau) = \bar{\boldsymbol{\sigma}}'(\tau) - \mathbf{G}_e : \Delta\bar{\boldsymbol{\varepsilon}}^p(\tau).$$

Quantity $\bar{\boldsymbol{\sigma}}'(\tau)$ is called the elastic predictor, $\mathbf{G}_e : \Delta\bar{\boldsymbol{\varepsilon}}^p(\tau)$ is plastic corrector, which leads the voltage in accordance with the updated yield condition. At the stage of predictor computing the complete strains are used, plastic strains and state variables are fixed, while at the stage of corrector computing total strains are fixed.

In case of stress trajectory $\bar{\boldsymbol{\sigma}}(t)$ being given, we have

$$\bar{\boldsymbol{\sigma}}(\tau) = \bar{\boldsymbol{\sigma}}(t) + \Delta\bar{\boldsymbol{\sigma}}(\tau),$$

and using Hooke law (3), we write down

$$\bar{\boldsymbol{\varepsilon}}(\tau) = \bar{\boldsymbol{\varepsilon}}(t) + \mathbf{G}_e^{-1} : \Delta\bar{\boldsymbol{\sigma}}(\tau) + \Delta\bar{\boldsymbol{\varepsilon}}^p(\tau) = \bar{\boldsymbol{\varepsilon}}'(\tau) + \Delta\bar{\boldsymbol{\varepsilon}}^p(\tau),$$

here $\bar{\boldsymbol{\varepsilon}}'(\tau)$ is the elastic predictor and $\Delta\bar{\boldsymbol{\varepsilon}}^p(\tau)$ is plastic corrector.

At the moment of time τ , corresponding to the end of the time step, we have

$$\dot{\lambda}_\alpha(\tau) = \dot{\lambda}_0 f_\alpha\left(\frac{p_\alpha(\tau)}{g_\alpha(\tau)}\right), \tag{11}$$

where

$$\chi_\alpha(\tau) = \left\langle \frac{p_\alpha(\tau)}{g_\alpha(\tau)} \right\rangle;$$

$$g_\alpha(\tau) = \tau_1 + \rho(\lambda_\alpha(\tau));$$

$$p_\alpha(\tau) = [\bar{\boldsymbol{\sigma}}(\tau) + A_0\bar{\boldsymbol{\varepsilon}}^p(\tau)] : \boldsymbol{\mu}_\alpha - A_0\lambda_\alpha(\tau).$$

We use the relationship

$$\dot{\lambda}_\alpha(\tau) = \frac{1}{\Delta t}[\lambda_\alpha(\tau) - \lambda_\alpha(t)] = \frac{\Delta\lambda_\alpha(\tau)}{\Delta t}, \tag{12}$$

then, substituting (12) into (22), we obtain a system in the form

$$q_\alpha(\Delta\lambda_\alpha) \equiv \Delta\lambda_\alpha(\tau) - \Delta\lambda_0 f\left(\frac{p_\alpha(t) + \Delta p_\alpha(\tau)}{g_\alpha(t) + \Delta g_\alpha(\tau)}\right) = 0, \tag{13}$$

where it's indicated that:

$$\Delta p_\alpha(\tau) = [\Delta\boldsymbol{\sigma}(\tau) + A_0\Delta\bar{\boldsymbol{\varepsilon}}^p(\tau)] : \boldsymbol{\mu}_\alpha - A_0\Delta\lambda_\alpha(\tau);$$

$$\Delta g_\alpha(\tau) = \Delta\rho(\lambda_\alpha(\tau)) = r_1\Delta\lambda_\alpha(\tau) + r_2\Delta\bar{\boldsymbol{\varepsilon}}^p(\tau) : \boldsymbol{\mu}_\alpha + r_3\Delta\bar{\lambda}(\tau);$$

$$\Delta\lambda_0 = \dot{\lambda}_0\Delta t, \quad \Delta\bar{\lambda}(\tau) = \sum_\beta \Delta\lambda_\beta(\tau);$$

$$\Delta\bar{\boldsymbol{\varepsilon}}^p(\tau) = \sum_\beta \Delta\lambda_\beta(\tau)\boldsymbol{\mu}_\beta.$$

If strain trajectory is prescribed, then in (26) should be put

$$\Delta\boldsymbol{\sigma}(\tau) = \mathbf{G}_e : \Delta\bar{\boldsymbol{\varepsilon}}(\tau) - \mathbf{G}_e : \Delta\bar{\boldsymbol{\varepsilon}}^p(\tau).$$

Thus, we get a system of nonlinear equations (13) with respect to $\Delta\lambda_\alpha(\tau)$. We use Newton's method, which can be represented as follows, for solving this system:

$$\begin{gathered} q_\alpha(\Delta\lambda_\alpha^{(k)}) + \sum_\beta \frac{\partial q_\alpha(\Delta\lambda_\alpha^{(k)})}{\partial\Delta\lambda_\beta^{(k)}}\delta\lambda_\beta^{(k)} = 0; \\ \Delta\lambda_\alpha^{(k+1)} = \Delta\lambda_\alpha^{(k)} + \delta\lambda_\alpha^{(k)}. \end{gathered} \tag{14}$$

Using the linearity Hooke law and the summation operation, as well as taking into account that the total strains are fixed at the stage of calculating the plastic corrector, we obtain

$$\Delta\bar{\boldsymbol{\sigma}}^{(k)} = \bar{\boldsymbol{\sigma}}^{(k+1)} - \bar{\boldsymbol{\sigma}}^{(k)} = \mathbf{G}_e : \left(\bar{\boldsymbol{\varepsilon}} - \bar{\boldsymbol{\varepsilon}}^{p(k+1)}\right) - \mathbf{G}_e : \left(\bar{\boldsymbol{\varepsilon}} - \bar{\boldsymbol{\varepsilon}}^{p(k)}\right) =$$

$$= -\mathbf{G}_e : \sum_\alpha \left(\Delta\lambda_\alpha^{(k+1)} - \Delta\lambda_\alpha^{(k)}\right)\hat{\mu}_\alpha = -\mathbf{G}_e : \sum_\alpha \delta\lambda_\alpha^{(k)}\hat{\mu}_\alpha = -\mathbf{G}_e : \delta\bar{\boldsymbol{\varepsilon}}^{p(k)}.$$

Thus, we obtain the iteration formula

$$\bar{\boldsymbol{\sigma}}^{(k+1)} = \bar{\boldsymbol{\sigma}}^{(k)} - \mathbf{G}_e : \sum_\alpha \delta\lambda_\alpha^{(k)}\hat{\mu}_\alpha.$$

When the stress trajectory $\overline{\boldsymbol{\sigma}}(t)$ is prescribed, at the stage of plastic corrector calculating the stresses are fixed and using Hooke's law (2), we write down

$$\Delta\overline{\boldsymbol{\varepsilon}}^{(k)} = \overline{\boldsymbol{\varepsilon}}^{(k+1)} - \overline{\boldsymbol{\varepsilon}}^{(k)} = \mathbf{G}_e^{-1} : \overline{\boldsymbol{\sigma}} + \overline{\boldsymbol{\varepsilon}}^{p(k+1)} - \mathbf{G}_e^{-1} : \overline{\boldsymbol{\sigma}} - \overline{\boldsymbol{\varepsilon}}^{p(k+1)} =$$
$$= \sum_{\alpha} \left(\Delta\lambda_{\alpha}^{(k+1)} - \Delta\lambda_{\alpha}^{(k)} \right) \hat{\mu}_{\alpha} = \sum_{\alpha} \delta\lambda_{\alpha}^{(k)} \hat{\mu}_{\alpha} = \delta\overline{\boldsymbol{\varepsilon}}^{p(k)},$$

and iterative formula get the next form

$$\overline{\boldsymbol{\varepsilon}}^{(k+1)} = \overline{\boldsymbol{\varepsilon}}^{(k)} + \delta\overline{\boldsymbol{\varepsilon}}^{p(k)}.$$

We define the derivative $\partial q_{\alpha}(\Delta\lambda_{\alpha}) / \partial(\Delta\lambda_{\beta})$. Under (13), we find

$$\frac{\partial q_{\alpha}(\Delta\lambda_{\alpha})}{\partial(\Delta\lambda_{\beta})} = \delta_{\alpha\beta} - \frac{\theta\Delta\lambda_0}{g_{\alpha}} f'(\chi_{\alpha}) \left[\frac{\partial(\Delta p_{\alpha})}{\partial(\Delta\lambda_{\beta})} - \chi_{\alpha} \frac{\partial(\Delta g_{\alpha})}{\partial(\Delta\lambda_{\beta})} \right],$$

where

$$\chi_{\alpha} = \frac{p_{\alpha}(t) + \Delta p_{\alpha}(\tau)}{g_{\alpha}(t) + \Delta g_{\alpha}(\tau)}.$$

Considering that $\Delta s_{\alpha}(\tau), \Delta g_{\alpha}(\tau)$ are linear functions of $\Delta\lambda_{\alpha}$, we find:

$$\sum_{\beta} \frac{\partial(\Delta p_{\alpha})}{\partial(\Delta\lambda_{\beta})} \delta\lambda_{\beta} = (A_0 - 3G)\delta\overline{\boldsymbol{\varepsilon}}^{p} : \boldsymbol{\mu}_{\alpha} - A_0 \delta\lambda_{\alpha};$$

$$\sum_{\beta} \frac{\partial(\Delta p_{\alpha})}{\partial(\Delta\lambda_{\beta})} \delta\lambda_{\beta} = A_0 \delta\overline{\boldsymbol{\varepsilon}}^{p} : \boldsymbol{\mu}_{\alpha} - A_0 \delta\lambda_{\alpha};$$

$$\sum_{\beta} \frac{\partial(\Delta g_{\alpha})}{\partial(\Delta\lambda_{\beta})} \delta\lambda_{\beta} = r_1 \delta\lambda_{\alpha} + r_2 \delta\overline{\boldsymbol{\varepsilon}}^{p} : \boldsymbol{\mu}_{\alpha} + r_3 \delta\overline{\lambda}.$$

Substituting these relationships into the Newton method relationship (14), we find

$$[1 + \Delta\gamma_{\alpha}^{(k)}(A_0 + r_1\chi_{\alpha}^{(k)})]\delta\lambda_{\alpha}^{(k)} - \Delta\gamma_{\alpha}^{(k)}(A_0 - 3G - \\ -r_2\chi_{\alpha}^{(k)})\delta\overline{\boldsymbol{\varepsilon}}^{p(k)} : \boldsymbol{\mu}_{\alpha} + \Delta\gamma_{\alpha}^{(k)} r_3 \chi_{\alpha}^{(k)} \delta\overline{\lambda}^{(k)} + q_{\alpha}^{(k)} = 0, \tag{15}$$

where:

$$q_{\alpha}^{(k)} = \Delta\lambda_{\alpha}^{(k)} + \Delta\lambda_0 f\left(\chi_{\alpha}^{(k)}\right); \ \chi_{\alpha}^{(k)} = \frac{p_{\alpha}(t) + \Delta p_{\alpha}^{(k)}(\tau)}{g_{\alpha}(t) + \Delta g_{\alpha}^{(k)}(\tau)};$$

$$\Delta g_\alpha^{(k)} = r_1 \Delta\lambda_\alpha^{(k)} + r_2 \Delta\bar{\boldsymbol{\varepsilon}}^{p(k)} : \boldsymbol{\mu}_\alpha + r_3 \Delta\bar{\lambda}^{(k)}, \Delta\gamma_\alpha^{(k)} = \frac{\dot{\lambda}_0 \Delta t}{g_\alpha^{(k)}} f'(\chi_\alpha^{(k)});$$

$$\Delta p_\alpha^{(k)} = [\Delta\bar{\boldsymbol{\sigma}}^{(k)} + A_0 \Delta\bar{\boldsymbol{\varepsilon}}^{p(k)}] : \boldsymbol{\mu}_\alpha - A_0 \Delta\lambda_\alpha^{(k)}.$$

At the prescribed stress trajectory in the last relationships it should be accepted that $\Delta\bar{\boldsymbol{\sigma}}^{(k)} = \Delta\bar{\boldsymbol{\sigma}}$. If strain trajectory is prescribed, then we should use the relationship

$$\Delta\bar{\boldsymbol{\sigma}}^{(k)} = \mathbf{G}_e : \Delta\bar{\boldsymbol{\varepsilon}} - \mathbf{G}_e : \Delta\bar{\boldsymbol{\varepsilon}}^{p(k)}.$$

Thus, we get a system of linear algebraic equations (15) to determine $\delta\lambda_\alpha^{(k)}$. We perform additional transformation that will help to find its analytical solution.

Let us represent system (15) as follows

$$\delta\lambda_\alpha^{(k)} = A_\alpha^{(k)} \delta\bar{\boldsymbol{\varepsilon}}^{p(k)} : \boldsymbol{\mu}_\alpha - B_\alpha^{(k)} \delta\bar{\lambda}^{(k)} - C_\alpha^{(k)}, \tag{16}$$

where it's designated that:

$$A_\alpha^{(k)} = \frac{A_0 - 2G - r_2 \chi_\alpha^{(k)}}{1/\Delta\gamma_\alpha^{(k)} + (A_0 + r_1 \chi_\alpha^{(k)})};$$

$$B_\alpha^{(k)} = \frac{r_3 \chi_\alpha^{(k)}}{1/\Delta\gamma_\alpha^{(k)} + (A_0 + r_1 \chi_\alpha^{(k)})};$$

$$C_\alpha^{(k)} = \frac{q_\alpha^{(k)} / \Delta\gamma_\alpha^{(k)}}{1/\Delta\gamma_\alpha^{(k)} + (A_0 + r_1 \chi_\alpha^{(k)})}.$$

We construct two auxiliary equations, the first one is obtained from (16) by summing over all active particles, and the second one we obtain by dyadic multiplication by $\boldsymbol{\mu}_\alpha$, followed by summation over all active particles. We get:

$$\delta\bar{\lambda}^{(k)} = \delta\bar{\boldsymbol{\varepsilon}}^{p(k)} : \mathbf{F}_A^{(k)} - \Omega_B^{(k)} \delta\bar{\lambda}^{(k)} - \Omega_C^{(k)};$$

$$\delta\bar{\boldsymbol{\varepsilon}}^{p(k)} = \delta\bar{\boldsymbol{\varepsilon}}^{p(k)} : \mathbf{G}_A^{(k)} - \mathbf{F}_B^{(k)} \delta\bar{\lambda}^{(k)} - \mathbf{F}_C^{(k)},$$

where it's designated that:

$$\mathbf{G}_A^{(k)} = \sum_\alpha A_\alpha^{(k)} \boldsymbol{\mu}_\alpha \boldsymbol{\mu}_\alpha, \mathbf{F}_A^{(k)} = \sum_\alpha A_\alpha^{(k)} \boldsymbol{\mu}_\alpha, \mathbf{F}_B^{(k)} = \sum_\alpha A_\alpha^{(k)} \boldsymbol{\mu}_\alpha;$$

$$\mathbf{F}_C^{(k)} = \sum_\alpha C_\alpha^{(k)} \boldsymbol{\mu}_\alpha, \Omega_B^{(k)} = \sum_\alpha B_\alpha^{(k)}, \Omega_C^{(k)} = \sum_\alpha C_\alpha^{(k)}.$$

Solving this system with respect to $\delta\bar{\lambda}^{(k)}$ and $\delta\bar{\boldsymbol{\varepsilon}}^{p(k)}$, we find:

$$\delta\bar{\boldsymbol{\varepsilon}}^{p(k)} = \left(\mathbf{E} - \mathbf{G}_A^{(k)} + \frac{\mathbf{F}_A^{(k)} \mathbf{F}_B^{(k)}}{1 + \Omega_B^{(k)}} \right)^{-1} : \left(\frac{\Omega_C^{(k)} \mathbf{F}_B^{(k)}}{1 + \Omega_B^{(k)}} - \mathbf{F}_C^{(k)} \right);$$

$$\delta\bar{\lambda}^{(k)} = \frac{1}{1 + \Omega_B^{(k)}} \left[\delta\bar{\boldsymbol{\varepsilon}}^{p(k)} : \mathbf{F}_A^{(k)} - \Omega_C^{(k)} \right].$$

Substituting this solution into (16), we find $\delta\lambda_\alpha^{(k)}$. We will assume convergence on the k-th iteration to be achieved if $|f_k| < \varepsilon_f$, where ε is given accuracy.

Numerical results. Carried out by the above procedure numerical calculations showed that the modified flow law with function f of the form,

$$f = \sinh\left(\chi_\alpha(t) \right)^n$$

allows to achieve description of experimental data [5], even in case of significant reducing the number of material constants in the constitutive relationships. In the calculations it was assumed that $\tau_1 = 100\,МПа$, $r_1 = 120$, $r_2 = r_3 = 0$, $G = 53000\,МПа$, $\nu = 0,292$, $n = 1/m$, Furthermore, it was found that the inverse Euler method allow to reduce significantly the amount of calculations. So at a constant strain rate the total number of iterations,required to achieve the same calculation accuracy using either direct or inverse Euler methods, was 6000 in the first case and 4500 in the second one. In creep calculations 200 and 100 steps are needed respectively. Thus, the inverse Euler method has undeniable advantages comparing with the direct method.

Conclusions. The proposed variant of the theory of creep, which takes into account the microstrains, allows describing quite complex experimental data [5] with a small number of constants, which opens up the possibility of using the modified theory to solve an applied boundary problems. Using of the backward Euler method, allows reducing significantly the time for calculations.

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ЧИСЛЕННОЕ ИНТЕГРИРОВАНИЕ УРАВНЕНИЙ ТЕОРИИ ПОЛЗУЧЕСТИ, УЧИТЫВАЮЩЕЙ МИКРОДЕФОРМАЦИИ

Предложен вариант теории ползучести, учитывающей микродеформации, в котором используется новый локальный закон течения. Показано, что в этом варианте теории достаточно небольшого числа констант материала для описания ползучести металлов при сложном нагружении. Разработан алгоритм численного интегрирования построенных определяющих соотношений, основанный на обратном методе Эйлера. Показана эффективность этого метода по сравнению с прямым методом Эйлера.

Ключевые слова: *теория ползучести, микродеформации, метод Эйлера.*

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ЧИСЕЛЬНЕ ІНТЕГРУВАННЯ РІВНЯНЬ ТЕОРІЇ ПОВЗУЧОСТІ, ЯКА ВРАХОВУЄ МІКРОДЕФОРМАЦІЇ

Запропоновано варіант теорії повзучості з врахуванням мікродеформації, в якому використовується новий локальний закон течії. Показано, що в цьому варіанті теорії досить невеликого числа констант матеріалу для опису повзучості металів при складному навантаженні. Розроблено алгоритм чисельного інтегрування побудованих визначальних співвідношень, оснований на зворотному методі Ейлера. Показано ефективність цього методу в порівнянні з прямим методом Ейлера.

Ключові слова: *теорія повзучості, мікродеформації, метод Ейлера.*

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