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## FREE VIBRATIONS OF LONGITUDINALLY REINFORCED CONICAL SHELL WITH SPRING ASSOCIATED MASS IN MEDIUM

We consider a circular closed truncated medium-contacting conical shell of constant thickness regularly reinforced with longitudinal ribs supporting the associated mass connected with the shell in diametrically opposite points by means of two springs of the same rigidity.

*Keywords:* conical shell, vibrations, energetic method, Winkler model.

**Introduction.** Conical shells are widely used in aviation, engineering and machine building. One of the first papers on study of stability of conical shells was the paper of Kh.M. Mushtari [1]. In the [2,3] the equations of motion were obtained for conical shells reinforced with stiffening ribs under linear-elastic deformation with regard to transverse shifts. Mathematical model of deformation of general form reinforced orthotropic shells based on the functional of total energy of deformation was represented in the [4]. The [5] was devoted to construction of mathematical model of deformation of conical shell constructions based on the functional of total energy of deformation with regard to orthotropy of the material, geometrical nonlinearity and transversal shifts.

**Problem statement.** The problem of definition of natural frequencies of vibrations of such a shell is solved in linear statement by the energetic method. Discrete arrangement of reinforcing longitudinal ribs is taken into account. Such a system of coordinates is accepted: variable radius  $r$  and dihedral angle  $\varphi$  between diametrical planes (Fig. 1).

The shell's potential energy is calculated by means of the following expression [10]

$$\begin{aligned} \Pi_1 = & \frac{Eh}{12(1-\nu^2)} \int_{r_2}^{\eta} \int_0^{2\pi} \left( \varepsilon_1^2 + \varepsilon_2^2 + 2\nu\varepsilon_1\varepsilon_2 + \frac{1-\nu}{2} \psi^2 \right) \frac{rdrd\varphi}{\sin\gamma} + \\ & + \frac{D}{2} \int_{r_2}^{\eta} \int_0^{2\pi} \left( \chi_1^2 + \chi_2^2 + 2\nu\chi_1\chi_2 + 2(1-\nu)\tau^2 \right) \frac{rdrd\varphi}{\sin\gamma}, \end{aligned}$$

where according to [9] it is accepted

$$\varepsilon_1 = \frac{\partial u}{\partial r} \sin \gamma, \quad \varepsilon_2 = \frac{u}{r} \sin \gamma + \frac{\partial v}{r \partial \varphi} + \frac{w}{r} \cos \gamma, \quad \psi = \frac{\partial u}{r \partial \varphi} + \frac{\partial v}{\partial r} \sin \gamma - \frac{v}{r} \sin \gamma,$$

$$\chi_1 = -\frac{\partial^2 w}{\partial r^2} \sin^2 \gamma; \quad \chi_2 = -\frac{u}{r^2} \sin \gamma \cos \gamma - \frac{w}{r^2} \cos^2 \gamma - \frac{\partial^2 w}{r^2 \partial \varphi^2} - \frac{\partial w}{r \partial r} \sin^2 \gamma;$$

$$\tau = -\frac{\cos \gamma}{r} \left( \frac{\partial u}{r \partial \varphi} - \frac{\partial v}{\partial r} \sin \gamma + \frac{v}{r} \sin \gamma \right) - \frac{2 \sin \gamma}{r} \left( \frac{\partial^2 w}{\partial r \partial \varphi} - \frac{\partial w}{r \partial \varphi} \right);$$

$E$  is modulus of elasticity;  $h$  is the shell's thickness;  $\nu$  is the Poisson's ratio;  $r_1, r_2$  are the radii of the greater and less base of the shell;  $\gamma$  is an angle between the generator and the axis of the shell;  $u, v, w$  are the components of the vector of displacement of the points of the shell's medium surface along the generator in tangential direction and along the normal to

the median surface;  $D = \frac{Eh^3}{12(1-\nu^2)}$ .

Potential energy  $\Pi_2$  of deformation of longitudinal ribs equals [6]:

$$\Pi_2 = \frac{1}{2} \sum_{i=1}^{k_1} \int_{r_2}^{r_1} \left[ EF_1 \left( \frac{\partial u}{\partial r} \sin \gamma \right)^2 + EI_1 \left( \frac{\partial^2 w}{\partial r^2} \sin^2 \gamma \right)^2 + E \tilde{I}_1 \left( \frac{\partial^2 \vartheta}{r^2 \partial \varphi^2} \right)^2 + GI_{1kp} \left( \frac{\partial^2 w}{r \partial r \partial \varphi} \sin \gamma \right)^2 \right]_{\varphi=\varphi_i} \frac{dr}{\sin \gamma}.$$

Here:  $F_1, I_1, \tilde{I}_1, I_{1kp}$  is the area and inertia moments of cross-section of the longitudinal bar with respect to tangential and radial axes, respectively, and also inertia moment in torsion;  $G$  is the shear modulus;  $k_1$  is the amount of longitudinal bars;  $\varphi_i$  are the coordinates of their arrangement.

The kinetic energy  $T_1$  of longitudinal ribs is calculated by means of the expression

$$T_1 = \frac{\gamma_1 h}{2g} \int_{r_2}^{r_1} \int_0^{2\pi} \left[ \left( \frac{\partial w}{\partial t} \right)^2 + \left( \frac{\partial u}{\partial t} \right)^2 + \left( \frac{\partial \vartheta}{\partial t} \right)^2 \right] r dr d\varphi + \frac{\gamma_1 F_1}{2g} \sum_{i=1}^{k_1} \int_0^{2\pi} \left[ \left( \frac{\partial w}{\partial t} \right)^2 + \left( \frac{\partial u}{\partial t} \right)^2 + \left( \frac{\partial \vartheta}{\partial t} \right)^2 \right]_{\varphi=\varphi_i} \frac{dr}{\sin \gamma},$$

wherein  $\gamma_1$  is specific weight of the shell and ribs material;  $g$  is acceleration of gravitational force.

Influence of medium on the shell is determined as of external surface loads applied to the shell and is calculated as a work done by loads when taking the system from deformed state to initial undeformed one and is represented in the form

$$A = \int_{r_2}^{r_1} \int_0^{2\pi} q_r r dr d\varphi \quad (1)$$

We shall take into account the motion of mass only in the plane of cross-section along the axis  $z$  (Fig. 1). The motion of the mass from this plane and deformation of springs caused by displacements of the points of their fastening to the shell in the direction of displacement vector  $u$  are ignored. As the problem is solved in linear statement, i.e. in assumption that the displacement of the points of the deformable system is small, such assumptions are rightful.

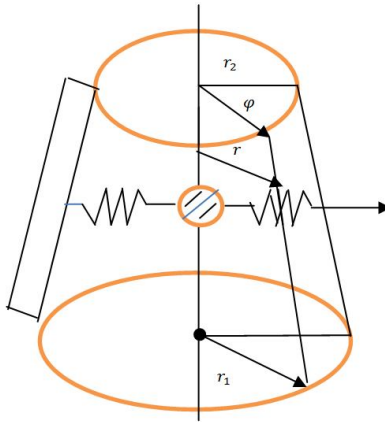


Fig. 1 – Longitudinally reinforced, medium-contacting conical shell with mass

Potential energy  $\Pi_3$  of springs and kinetic energy  $T_2$  of the mass are equal to

$$\Pi_3 = (z - w_0 \cos \gamma)^2 c; \quad T_2 = \frac{1}{2} M \dot{z}^2, \quad \text{respectively,}$$

where  $w_0$  is the displacement of the spring's fastening points, arranged in diametrical coordinate plane  $\varphi = 0$ .

**Problem solution.** We represent the displacements of the shell as follows [7]:

$$\begin{aligned}
 w &= \frac{r^2}{r_1^2} \sin \frac{m\pi(r-r_2)}{r_1-r_2} \sum_{n=1}^{\infty} A_n(t) \cos n\varphi; \\
 \vartheta &= \frac{r^2}{r_1^2} \sin \frac{m\pi(r-r_2)}{r_1-r_2} \sum_{n=1}^{\infty} B_n(t) \sin n\varphi; \\
 u &= \frac{r^2}{r_1^2} \cos \frac{m\pi(r-r_2)}{r_1-r_2} \sum_{n=1}^{\infty} D_n(t) \cos n\varphi,
 \end{aligned} \tag{2}$$

where  $n$  is the amount of waves in peripheral direction;  $m$  is the amount of half waves along the generator.

These expressions satisfy such conditions of elastic built-in of the shell along the edges under which  $w = \vartheta = 0$ . Expressions (1) are convenient in the sense that owing to the multiplier  $r^2$ , the integrals encountered when calculating the energy, are taken in quadratures and this simplifies the problem solution. Besides this, expressions (2) qualitatively reflect the fact that the crests of half-waves of the form of vibrations along the generator of the conical shell are somewhat displaced to the side of its greater base.

It should be noted that for even  $n$  the springs are deformed, the mass does not move; for odd  $n$  the mass performs vibrations along the axis  $z$ . In the sequel we shall consider this case.

Substituting the found expressions for kinetic and potential energy in Lagrange's equation of second kind [8] we get the system of ordinary differential equations infinite with respect to  $n$

$$\begin{aligned}
 M\ddot{z} + 2c(z - \alpha_0 \sum_{n=1,3,\dots}^{\infty} A_n) &= 0; \\
 2\ddot{A}_1 a + 2A_1(l_1 + \alpha_0^2 c + \beta_0) + B_1 f_1 + D_1 q_1 - 2z\alpha_0 c + 2c\alpha_0^2 (A_3 + A_5 + A_7 + \dots) &= 0; \\
 2\ddot{B}_1 a + 2B_1 d_1 + A_1 f_1 + D_1 s_1 &= 0; \quad 2\ddot{D}_1 b + 2D_1 e_1 + A_1 q_1 + B_1 s_1 &= 0; \tag{3} \\
 2\ddot{A}_3 a + 2A_3(l_3 + \alpha_0^2 c + \beta_0) + B_3 f_3 + D_3 q_3 - 2z\alpha_0 c + \\
 + 2c\alpha_0^2 (A_1 + A_5 + A_7 + \dots) &= 0; \\
 2\ddot{B}_3 a + 2B_3 d_3 + A_3 f_3 + D_3 s_3 &= 0; \quad 2\ddot{D}_3 b + 2D_3 e_3 + A_3 q_3 + B_3 s_3 &= 0;
 \end{aligned}$$

where

$$\begin{aligned}
l_n = & \Omega_1 R_1 \cos^2 \gamma + \frac{h^3}{3} \left\{ \sin^4 \gamma \left( \frac{117}{8} R_2 + \frac{31}{32} \alpha^2 R_4 + \frac{1}{192} \alpha^4 R_6 \right) + \right. \\
& + \frac{1}{4} R_2 \left[ \left( 1 - n^2 \right)^2 - \sin^4 \gamma \right] + \frac{1}{8} \left( \frac{1}{4} \alpha^2 R_4 + 3 R_2 \right) \sin^4 \gamma + \\
& + \nu \left[ \frac{1}{2} \alpha^2 R_4 \sin^4 \gamma - R_2 \sin^4 \gamma - R_2 \left( 1 - n^2 + \sin^2 \gamma \right) \sin^2 \gamma - \right. \\
& \left. - 2 \left( 1 - n^2 + \sin^2 \gamma \right) \frac{1}{32} \alpha R_4 - \frac{3}{2\alpha} R_2 \right] - (1 - \nu) \left( 7 R_2 - \frac{1}{4} R_4 \alpha \right) n^2 \sin^2 \gamma \left. \right\} + \\
& + \frac{k_1 \sin \gamma}{4 r_1^4} \left[ E I_1 \left( 10 R_1 + 4 \alpha^2 R_3 + 0,01 \alpha^4 R_5 - \frac{1}{6} \alpha^2 R_3 + \frac{R_1}{\alpha} + \frac{3}{8} \alpha^3 \Omega \right) \sin^2 \gamma + \right. \\
& \left. + G I_{kp} \left( \frac{9}{4} R_2 + \frac{\alpha^2}{64} R_3 + R_1 \right) \right];
\end{aligned}$$

$$\begin{aligned}
e_n = & \frac{1}{2} \Omega_1^2 \sin^2 \gamma \left\{ \frac{1}{\alpha^2} R_2 \left[ 9 - 12 \nu + \frac{3(1 - \nu) n^2}{4} \right] + R_4 \left[ \nu + \frac{7}{8} + \frac{1}{16} (1 - \nu) n^2 \right] + \right. \\
& + \frac{1}{48} \alpha^2 R_6 \left. \right\} + \frac{h^3}{96} \Omega_1 \left[ \frac{1}{4} \sin^2 2\gamma + 2(1 - \nu) n^2 \cos^2 \gamma \right] R_2 + \\
& + \frac{E k_1}{4 r_1^4 \sin \gamma} \left[ F_1 \left( \frac{1}{2} \alpha \Omega + \frac{1}{4} R_3 + \frac{4 R_1}{\alpha^2} + \frac{\alpha^2}{40} R_5 \right) \sin \gamma + \frac{1}{2} I_1 R_1 n^4 \right];
\end{aligned}$$

$$\begin{aligned}
d_m = & \Omega_1 \left\{ \frac{1}{2} n^2 \left( \frac{1}{8} R_4 - \frac{3}{2\alpha^2} R_2 \right) + \frac{1 - \nu}{2} \left( \frac{1}{4} R_4 - \frac{3}{\alpha^2} R_2 + \frac{\alpha^2}{48} R_6 \right) \right. \\
& \left. + \frac{h^2 (1 - \nu) \sin 2\gamma}{12} \frac{1}{128} R_4 - \frac{11}{32} R_2 \right\};
\end{aligned}$$

$$f_n = -\Omega_1 \left[ \left( \frac{1}{8} R_4 - \frac{3}{2\alpha^2} R_2 \right) n \cos \gamma + \frac{1}{12} h^2 n \sin 2\gamma (1 - \nu) \left( \frac{9}{4} R_2 + \frac{1}{16} \alpha^2 R_4 \right) \right];$$

$$\begin{aligned}
q_n = & \Omega_1 \frac{1}{4} \sin 2\gamma \left( \Omega + \frac{\nu \alpha R_5}{10} \right) - \frac{h^2}{24} \left\{ \left( 1 - n^2 + \sin \gamma \right) \frac{1}{\alpha} R_2 + \left[ \frac{\alpha}{4} R_3 \left( \nu + \frac{1}{4} \right) + \right. \right. \\
& \left. \left. + \frac{1}{\alpha} R_4 (1 + 5\nu) + \Omega \nu \alpha^2 \right] \frac{1}{2} \sin^2 \gamma + \frac{\alpha}{8} R_3 (1 - \nu) n^2 \right\} \sin 2\gamma;
\end{aligned}$$

$$a = \beta_1 - \beta_2; \quad b = \beta_1 + \beta_2;$$

$$\beta_1 = \frac{\gamma_1}{2gr_1^4} \left( \frac{\pi h R_6}{12 \sin \gamma} + \frac{F_1 k_1 R_5}{20 \sin \gamma} \right); \quad \beta_2 = \frac{\gamma_1}{2gr_1^4} \left[ \frac{\pi h}{\sin \gamma} \left( \frac{5}{2\alpha^2} R_4 + \frac{30}{\alpha^4} R_2 \right) + \frac{F_1 k_1 \Omega}{\alpha \sin \gamma} \right];$$

$$\Omega = \frac{r_1^3}{\alpha} - \frac{6r_1}{\alpha^3} - \left( \frac{r_2^3}{\alpha} - \frac{6r_2}{\alpha^3} \right); \quad \Omega_1 = \frac{Eh\pi}{(1-\nu^2)r_1^4 \sin \gamma}; \quad \alpha = \frac{2\pi m}{r_1 - r_2};$$

$$R_p = r_1^p - r_2^p \quad (p=1, \dots, 6);$$

$$\beta_0 = \frac{k\pi}{r_1^2} \left\{ \frac{1}{12} (r_2^6 - r_1^6) + \frac{5(r_2 - r_1)(r_1^2 - r_2^2)}{4m\pi} r_1^2 + r_2^2 - 12 \right\}.$$

Accepting the solution of the infinite system in the form  $z = z^* \sin \omega t$ ,  $A_n = A_n^* \sin \omega t$ ,  $B_n = B_n^* \sin \omega t$ ,  $D_n = D_n^* \sin \omega t$ , where  $\omega$  is the natural frequency of vibrations of the studied system, and substituting these solutions in equation (3), we get an infinite homogeneous system of algebraic equations with respect to unknowns  $z^*$ ,  $A_n^*$ ,  $B_n^*$ ,  $D_n^*$ .

Assuming that the influence of the medium on the shell is subjected to the Winkler model, i.e.  $q_r = kw$ , we can represent (2) in the form:

$$A = \int_{r_2}^{r_1} \int_0^{2\pi} k(1 - \beta \omega^2) w^2 r dr d\varphi \quad (4)$$

where  $k$ ,  $\beta$  are constants.

Equate to zero the determinant of this system

$$\begin{vmatrix} \lambda_1 + R - \omega^2 + \frac{\beta_0}{2a} & R & R & \dots \\ R & \lambda_3 + R - \omega^2 + \frac{\beta_0}{2a} & R & \dots \\ R & R & \lambda_5 + R - \omega^2 + \frac{\beta_0}{2a} & \dots \end{vmatrix} = 0 \quad (5)$$

where

$$\lambda_n = \frac{l_n}{a} + \frac{f_n}{2a\Delta} \left[ \frac{q_n s_n}{4ab} - \frac{f_n}{2a} \left( \frac{e_n}{b} - \omega^2 \right) \right] + \frac{q_n}{2a\Delta} \left[ \frac{f_n s_n}{4ab} - \frac{q_n}{2ab} \left( \frac{d_n}{a} - \omega^2 \right) \right];$$

$$R = -\frac{\alpha_0^2 \omega^2}{a} \left( \frac{c}{M} - \omega^2 \right).$$

As a result, we get an equation that allows to determinate natural frequencies of vibrations of the medium-filled shell with associated mass. The influence of the associated mass is included into the expression for  $R$ . Consider the limit cases. For  $c=0$  and  $R=0$  we get a frequency equation for a medium-filled shell without associated mass

$$\lambda_n + R - \omega^2 + \frac{\beta_0}{2a} = 0.$$

For  $M=0$  the results are similar. If  $c=\infty$ , then  $R=-\alpha_0^2 M \omega^2 / a$  mass is associated to the shell by means of absolutely rigid bars; for  $M=\infty$   $R=\alpha_0^2 c / a$  is a shell with inner elastic bonds.

Natural frequencies of the system's vibrations were determined by numerical solution of equation (5). The followings for accepted for the problem parameters:  $r_1=160\text{ mm}$ ,  $r_2=85\text{ mm}$ , longitudinal bar of angle profile with dimensions  $5 \times 5 \times 1$  (in  $\text{mm}$ ),  $k_1=32$ ,  $m=1$ . The longitudinal bars were fastened to the inner surface of the shell. The associated mass whose quantity varies in the research process, was fastened in the middle of the shell in diametrically opposite points.

**Conclusions.** The curve reflecting the dependence of natural frequency  $f^* = \omega / (2\pi)$  of vibrations of medium-filled shell with no associated mass on the number of waves  $n$  in peripheral direction is depicted in Fig. 2. It is seen that with increasing the number of waves  $n$  in peripheral direction, the frequencies of vibrations of medium-filled shell with no associated mass at first decreases, attains minimum and begins to increase.

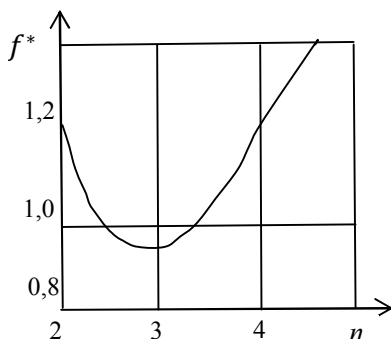


Fig. 2 – Dependence of natural frequency of shell's vibrations on the number of waves in peripheral direction

The curves reflecting the dependence of minimum natural frequency of the system's vibrations on quantity of associated mass, relative rigidity of

springs  $\bar{c} = c/D$  for relative bed coefficient  $\bar{k} = k/D = 0,1$ , were depicted in Fig. 3. The similar dependence for  $\bar{k} = 0,3$  is in Fig. 4.

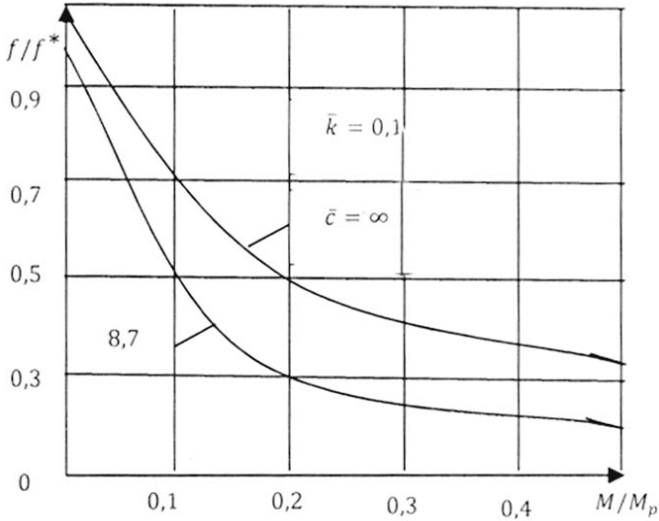


Fig. 3 – Dependence of natural frequency of shell's vibrations on the quantity of associated mass

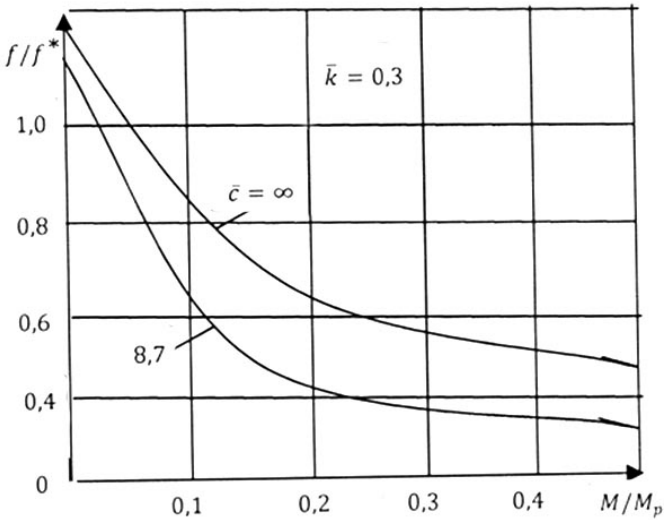


Fig. 4 – Dependence of natural frequency of shell's oscillations on the quantity of associated mass



Analysis of curves shows that influence of associated mass on minimum natural frequency of shell's vibrations is very essential. With decreasing the rigidity of bonds between the mass and shell, this influence increases. Comparison of curves from Fig. 3 and 4 shows that with increasing relative bed coefficient, minimum natural frequency of the systems vibration increases. This is connected with the fact that the Winkler model does not take into account inertial actions of medium on vibrations process.

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### **ВІЛЬНІ КОЛИВАННЯ В СЕРЕДОВИЩІ ПРОДОЛЬНО ПОСИЛЕНОЇ КОНІЧНОЇ ОБОЛОНКИ З МАСОЮ, ЩО ПРИЄДНАНА ПРУЖИНАМИ**

Розглядається контактуюча з середовищем кругова замкнена усічена конічна оболонка постійної товщини, яка регулярно підкріплена поздовжніми ребрами і несе приєднану масу, що пов'язана з оболонкою в діаметрально протилежних точках за допомогою двох пружин однакової жорсткості.

*Ключові слова:* конічна оболонка, коливання, енергетичний метод, модель Вінклера.

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### **СВОБОДНЫЕ КОЛЕБАНИЯ В СРЕДЕ ПРОДОЛЬНО ПОДКРЕПЛЕННОЙ КОНИЧЕСКОЙ ОБОЛОЧКИ С МАССОЙ, ПРИСОЕДИНЕННОЙ НА ПРУЖИНАХ**

Рассматривается контактирующая со средой круговая замкнутая усеченная коническая оболочка постоянной толщины, регулярно подкрепленная продольными ребрами и несущая присоединенную массу, связанную с оболочкой в диаметрально противоположных точках с помощью двух пружин одинаковой жесткости.

*Ключевые слова:* коническая оболочка, колебания, энергетический метод, модель Винклера.

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