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LIGHT TAILED ASYMPTOTICS IN AN UNRELIABLE $M/G/1$ RETRIAL QUEUE

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Abstract. We consider the standard unreliable $M/G/1$ retrial queuing system with active and passive breakdowns. The explicit expressions of the probability generating functions of distribution of server state and orbit size are well known from early works. However, some problems particularly related to Cybernetic and Artificial Intellect need to save computational effort. So, we give here another look to solve this problem more simply, but under some light tailed assumptions.

Анотація. Ми розглядаємо стандартні ненадійні системи обслуговування $M/G/1$ з повторними викликами та активними та пасивними відмовами. Точні вирази для генератрис розподілів станів системи обслуговування та розмірів орбіт відомі із попередніх робіт. Однак, в деяких задачах, зокрема пов'язаних із кібернетичним та штучним інтелектом, виникає потреба у зменшенні кількості обчислень. Отже, у цій роботі ми пропонуємо іншу точку зору стосовно розв'язання цієї проблеми простішим чином за певних припущень щодо легкості хвостів розподілів.

Аннотация. Мы рассматриваем стандартные ненадежные системы обслуживания $M/G/1$ с повторными вызовами и активными и пассивными отказами. Точные выражения для генератрис распределений состояний системы обслуживания и размеров орбит известны из ранних работ. Однако, в некоторых задачах, в частности связанных с кибернетическим и искусственным интеллектом, возникает необходимость сокращать количество вычислений. В этой работе мы предлагаем иной и более простой подход к решению этой проблемы.

1. Introduction

Queueing Systems with breakdowns have been the subject of many investigations since they are interesting modeling tools in many modern systems: production, and telecommunication, computer science, finance and so on. Such studies are interesting since they make a gap between Queueing Theory and Reliability theory, in the context of probabilistic framework which is one of logics for uncertainty problems [7], [6].

In last years, a particular interest have been devoted to Retrial Queueing Systems for their interest in many modern system (Call centers, Mobile Systems, CSMA/CD protocol with star topology, random access protocols. . .), see for example [11] and the references in didactical book [9], or bibliographical surveys of [3, 4].

Early works were concerned with analytical or algorithmic resolution of the underlying problems, see for example [6, 5]. Asymptotic methods are interesting in the sense that we can save computational effort for the resolution of the problem under study, see [2]. We refer also to some alternative approaches to study more qualitative properties such

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as stability,see [10] or monotonicity and comparability methods, see [12]. The last paper gives also a short survey about recent advances in Unreliable Retrial Queues.

In this note, we consider the approximation of the queue size of the unreliable $M/G/1$ retrial queue under light tailed assumptions. In the next section we describe the mathematical model and in section 3 we provide some preliminary results. The light tailed approximation is given in section 4.

2. The Mathematical Model

We consider the standard $M/G/1$ retrial queuing system with active and passive breakdowns, with the following parameters and assumptions:

- λ is the arrival rate of primary customers;
- η (resp. μ) is the Poisson rate of passive (resp. active) breakdowns;
- $F(x)$ is probability distribution (p.d.) of the service time;
- $G(x)$ (resp. $H(x)$) is p.d. of duration of passive D_i (resp. active D_b) interruption;
- η is retrial rate of each secondary call i.e. the retrial protocol is such that the probability of a retrial during $(t, t+dt)$ is proportional to the number of units in orbit.
- If a customer whose service is interrupted by an active breakdown need to leave the service zone. The displaced customer must decide either to join the retrial group (with probability c), or leave the system after interruption (with probability $1-c$).

Now, if $A(x)$ is one of the above p.d.'s, we denote by $\tilde{A}(s)$, $\mathrm{Re}(s) \geq 0$ it's Laplace–Stieltjes transform. Let

- $X(t) = 0$, if the server is operative and idle at time t,
- $X(t) = 1$, if the server is operative and busy at time t,
- $X(t) = 2$, if the server is down due to an independent breakdown at time t,
- $X(t) = 3$, if the server is down due to an active breakdown at time t.

Let $Q(t)$ be the number of customers in orbit at time t. We are interested with the limiting probability

$$P_{in} = \lim_{t\to\infty} P\{X(t) = i, Q(t) = n\}, \qquad i = 0, \ldots, 3,\ n \geq 0.$$

The explicit expressions of the probability generating functions of distribution of server state and orbit size are well known from [1] and we recall them in the next section.

3. Preliminary Results

Consider the Standard $M/G/1$ Retrial Queue as described in section 1. We assume that the ergodicity condition

$$\rho = \lambda \frac{1-\tilde{F}(\mu)}{\mu}\Big(1 + \mu\big(E(D_b) + \frac{c}{\mu}\big)\Big) < 1 \tag{1}$$

holds.

Denote by

$$P_{in} = \lim_{t\to\infty} P\{X(t) = i, Q(t) = n\}, \qquad i = 0, \ldots, 3,\ n \geq 0.$$

$$P_i(z) = \sum_{n=0}^{\infty} z^n P_{in}, \qquad i = 0, \ldots, 3,\ |z| < 1.$$

From theorem 4.2. of [1] we know the explicit expression of these generating functions

$$P_0(z) = \frac{K}{A} \exp\left(\frac{\lambda}{\theta}\int_1^z \frac{1 - N(\lambda - \lambda u)}{N(\lambda - \lambda u) - u} du + \frac{\eta}{\theta}\int_1^z \frac{1 - \widetilde{G}(\lambda - \lambda u)}{N(\lambda - \lambda u) - u} du\right) \tag{2}$$

$$P_1(z) = \frac{1-\widetilde{F}(\lambda-\lambda z+\mu)}{\lambda-\lambda z+\mu} \times \frac{\lambda-\lambda z+\eta-\eta\widetilde{G}(\lambda-\lambda z)}{N(\lambda-\lambda z)-z} P_0(z) \tag{3}$$

$$P_2(z) = \eta\frac{1-\widetilde{G}(\lambda-\lambda z)}{\lambda-\lambda z} P_0(z) \tag{4}$$

$$P_3(z) = (1-c+cz)\mu\frac{1-\widetilde{H}(\lambda-\lambda z)}{\lambda-\lambda z} P_1(z) \tag{5}$$

where

$$K = \frac{1-\rho}{\lambda(1+\eta E(D_i))+\eta(1-\rho)}$$

and

$$A = K(1+\eta E(D_i)) + (1-\widetilde{F}(\mu)(1-\eta K)(E(D_b)+\frac{1}{\mu}) \tag{6}$$

These relations are obtained by different ways (Method of imbedded Markov chain, Method of supplementary variable, algorithmic methods. . .), see [1] and its references.

Recall that the function $N(\lambda-\lambda z)$ has the following interpretation. Consider the random variable Φ representing the fundamental period i.e. the duration determined by the competition between the service and the arrival of an active interruption. Thus the Laplace-Stieltjes of the distribution of Φ is given by

$$f(s) = \widetilde{F}(s+\mu) + \frac{\mu\widetilde{H}(s)(1-\widetilde{F}(s+\mu))}{s+\mu}$$

Let N be the number of customers that joins the retrial group during the period $[0, \Phi]$. Then the distribution of N is given by [1]

$$N(\lambda-\lambda z) = E(z^N) = \widetilde{F}(\lambda-\lambda z+\mu) + (1-c+cz)\frac{\mu\widetilde{H}(\lambda-\lambda z)(1-\widetilde{F}(\lambda-\lambda z+\mu)}{\lambda-\lambda z+\mu)}$$

4. Light Tailed Approximation

The behavior of the orbit size distribution is based on the analytical properties of probability generating functions 2–5.

The light tailed asymptotic of the queue size distribution in the standard Unreliable $M/G/1$ Retrial Queue is based on the assumption that the generalized service time distribution Φ has a finite exponential moment i.e.

$$\gamma = \sup\{t \in \mathbb{C} : E(e^{t\Phi}) < \infty, E(e^{tD_i}) < \infty\} > 0. \tag{7}$$

Using this assumption, we can locate the zeros of $N(\lambda-\lambda z)-z$, $|z| \leq \sigma$. More precisely, the analytic function $N(\lambda-\lambda z)-z$, $|z| \leq 1+\frac{\gamma}{\lambda}$ has simple zero at 1 and σ. Furthermore, it has no other zeros on $\{z \in \mathbb{C}\colon |z| \leq \sigma\}$, where $\sigma = \sigma(\lambda, \mu, c, F(\cdot), H(\cdot))$ is the unique root of the equation

$$N(\lambda-\lambda z)-z = \sigma, \qquad 1 < \sigma < 1+\frac{\gamma}{\lambda} \tag{8}$$

The proof is similar to that of lemma 1 of [8]. The function $f(z) = N(\lambda-\lambda z)$, $|z| < 1+\gamma/\lambda$ has a simple zero at $z = 1$ since $f'(1) < 0$. Since $f(z)$ is strictly convex in $z \in (0, 1+\gamma/\lambda)$ and $f(1) = f(\sigma) = 0$, then $f(z) < 0$ for $1 < z < \sigma$. The assertion is a consequence of the Rouché's theorem.

Now, the function

$$g(z) = \frac{1-N(\lambda-\lambda z)}{N(\lambda-\lambda z)-z}$$

is analytic at $z = 1$, since it has a removable singularity at this point. On the other hand this function has a simple pole at $z = \sigma$ and the residue at this point $\mathrm{Re}\,s\ g(\sigma) = \frac{-(\sigma-1)}{-\lambda N'(\lambda-\lambda\sigma)-1}$. So, the function

$$h(z) = \frac{1 - N(\lambda - \lambda z)}{N(\lambda - \lambda z) - z} + \frac{-(\sigma - 1)}{-\lambda N'(\lambda - \lambda\sigma) - 1} \times \frac{1}{z - \sigma}$$

has a removable singularity at $z = \sigma$. Therefore, the function $g(z)$ can be decomposed as a sum of the principal part at $z = \sigma$ and the analytical part. More precisely, we can find $\delta' \in (0, 1 + \gamma/\lambda - \sigma)$ and an analytical function $P(z)$ on $\Theta_{\delta'} = \{z \in \mathbb{C} : |z| < \sigma + \delta'\}$ such that

$$\frac{1 - N(\lambda - \lambda z)}{N(\lambda - \lambda z) - z} = -\frac{\sigma - 1}{-\lambda N'(\lambda - \lambda\sigma) - 1} \times \frac{1}{z - \sigma} + P(z), \qquad |z| < \sigma + \delta',\ z \neq \sigma, \quad (9)$$

where

$$P(z) = \begin{cases} \frac{1-N(\lambda-\lambda z)}{N(\lambda-\lambda z)-z} + \frac{\sigma-1}{-\lambda N'(\lambda-\lambda\sigma)-1} \times \frac{1}{z-\sigma}, |z| < \sigma + \delta', z \neq \sigma, \\ \lim_{u\to\sigma} \frac{1-N(\lambda-\lambda u)}{N(\lambda-\lambda u)-u}, z = \sigma \end{cases}$$

In a similar way, we can find δ'' and an analytical function $Q(z)$ on $\Theta_{\delta''}$ such that

$$\frac{1 - \widetilde{G}(\lambda - \lambda z)}{N(\lambda - \lambda z) - z} = -\frac{\lambda \widetilde{G}'(\lambda - \lambda\sigma)}{-\lambda N'(\lambda - \lambda\sigma) - 1} \times \frac{1}{z - \sigma} + Q(z), \qquad |z| < \sigma + \delta'',\ z \neq \sigma, \quad (10)$$

where

$$Q(z) = \begin{cases} \frac{1-\widetilde{G}(\lambda-\lambda z)}{N(\lambda-\lambda z)-z} + \frac{\lambda\widetilde{G}'(\lambda-\lambda\sigma)}{-\lambda N'(\lambda-\lambda\sigma)-1} \times \frac{1}{z-\sigma}, |z| < \sigma + \delta'', z \neq \sigma, \\ \lim_{u\to\sigma} \frac{1-\widetilde{G}(\lambda-\lambda u)}{N(\lambda-\lambda u)-u}, z = \sigma \end{cases}$$

Next, we take the domain $|z| < \sigma + \delta$, where $\delta = \min(\delta', \delta'')$ gives the range of validity of the two above decompositions. Substituting 9 and 10 into 2, we have

$$\begin{aligned} P_0(z) &= E(z^Q; X = 0) \\ &= \frac{K}{A} \exp\left(\frac{\lambda}{\theta} \int_1^z \frac{1 - N(\lambda - \lambda u}{N(\lambda - \lambda u) - u} du + \frac{\eta}{\theta} \int_1^z \frac{1 - \widetilde{G}(\lambda - \lambda u)}{N(\lambda - \lambda u) - u} du \right) \end{aligned} \quad (11)$$

Thus,

$$\begin{aligned} P_0(z) \quad = \quad & \frac{K}{A} \exp\left(\frac{\lambda}{\theta} \int_1^z \frac{a}{u - \sigma} du \right) \times \exp\left(\frac{\lambda}{\theta} \int_1^z P(u) du \right) \\ & \times \exp\left(\frac{\eta}{\theta} \int_1^z \frac{b}{u - \sigma} du \right) \times \exp\left(\frac{\eta}{\theta} \int_1^z Q(u) du \right) \end{aligned}$$

where

$$a = \frac{\lambda}{\theta} \frac{\sigma - 1}{-\lambda N'(\lambda - \lambda\sigma)}$$

and

$$b = \frac{\eta}{\theta} \frac{\widetilde{G}(\lambda - \lambda\sigma) - 1}{-\lambda N'(\lambda - \lambda\sigma)}.$$

Finally, we get

$$P_0(z) = \frac{K}{A} \left(\frac{\sigma - z}{\sigma - 1} \right)^{-a-b} \Lambda(z), \qquad |z| < \sigma \quad (12)$$

where

$$\Lambda(z) = \exp\left(\frac{\lambda}{\theta} \int_1^z P(u) du + \frac{\eta}{\theta} \int_1^z Q(u) du \right) \quad (13)$$

Since $P(z)$ and $Q(z)$ are analytic on $\{z \in \mathbb{C} : |z| < \sigma + \delta\}$, so is $\Lambda(z)$.

Consider the power series expansion of the function $\Lambda(z)$ at $z=0$, i.e.

$$\Lambda(z)=\sum_{n=0}^{\infty}\varphi_n z^n, \qquad |z|<\sigma+\delta. \tag{14}$$

On the other hand, we note that we have the following power series expansion for the first factor

$$\frac{K}{A}\left(\frac{\sigma-z}{\sigma-1}\right)^{-a-b}=\frac{K}{A}\left(\frac{\sigma-1}{\sigma}\right)^{a+b}\sum_{n=0}^{\infty}\frac{\Gamma(a+b+n)}{\Gamma(a+b)\Gamma(n+1)}\sigma^{-n}z^n. \tag{15}$$

By using verbatim the computation [8] we show

$$\begin{aligned}\lim_{n\to\infty}\frac{P(X=0,Q=n)}{(\Gamma(a+b+n)/\Gamma(a+b)\Gamma(n+1))\sigma^{-n}}&=\frac{K}{A}\left(\frac{\sigma-1}{\sigma}\right)^{a+b}\sum_{n=0}^{\infty}\varphi_n\sigma^n\\&=\frac{K}{A}\left(\frac{\sigma-1}{\sigma}\right)^{a+b}\Lambda(z)=\Gamma(a+b)c_0,\end{aligned} \tag{16}$$

where

$$\begin{aligned}c_0=&\frac{K}{A\Gamma(a+b)}\left(\frac{\sigma-1}{\sigma}\right)^{a+b}\\&\times\exp\left(\int_1^{\sigma}\left(\frac{\frac{\lambda}{\theta}\times\{1-N(\lambda-\lambda u)\}+\frac{\eta}{\theta}\times\{1-\widetilde{G}(\lambda-\lambda u)\}}{\{1-N(\lambda-\lambda u)\}}+\frac{a+b}{u-\sigma}\right)du\right).\end{aligned} \tag{17}$$

Now, taking into account that $\Gamma(a+b+n)/\Gamma(n+1)\propto n^{a+b-1}$ as $n\to\infty$, the relation 16 is equivalent to

$$P(X=0,Q=n)\propto c_0 n^{a+b-1}\sigma^{-n} \tag{18}$$

as $n\to\infty$. Here $g_n\propto h_n$ means that

$$\lim_{n\to\infty}\frac{g_n}{h_n}=1.$$

Similar approximations can be obtained for the probability generating functions 3-5. Consider for example the case of the distribution of the number of customers in orbit when the server is operative and busy $P_1(z)$. Denote by

$$A(z)=\frac{1-\widetilde{F}(\lambda-\lambda z+\mu)}{\lambda-\lambda z+\mu}\times\frac{\lambda-\lambda z+\eta-\eta\widetilde{G}(\lambda-\lambda z)}{N(\lambda-\lambda z)-z}P_{0(z)} \tag{19}$$

Similarly to 10, we can find some $\beta\in\left(0,1+\frac{\gamma}{\lambda}-\sigma\right)$ and an analytical function $S(z)$ on Θ_δ such that

$$\frac{A(z)}{N(\lambda-\lambda z)-z}=-\frac{A'(\sigma)}{-\lambda N'(\lambda-\lambda\sigma)-1}\times\frac{1}{z-\sigma}+S(z), \qquad |z|<\sigma+\beta,\ z\neq\sigma, \tag{20}$$

where

$$S(z)=\begin{cases}\frac{A(z)}{N(\lambda-\lambda z)-z}+\frac{A'(\sigma)}{-\lambda N'(\lambda-\lambda\sigma)-1}\times\frac{1}{z-\sigma}, |z|<\sigma+\beta, z\neq\sigma,\\ \lim_{u\to\sigma}\frac{A(u)}{N(\lambda-\lambda u)-u}, z=\sigma\end{cases}$$

Substituting now 19 into 3, we have the following representation

$$\begin{aligned}P_1(z)=&\frac{K}{A}\frac{A(\sigma)(\sigma-1)}{-\lambda N'(\lambda-\lambda\sigma)-1}\left(\frac{\sigma-z}{\sigma-1}\right)^{-a-b-1}\Lambda(z)\\&+\frac{K}{A}A(\sigma)\left(\frac{\sigma-z}{\sigma-1}\right)^{-a-b}S(z)\Lambda(z)\end{aligned} \tag{21}$$

Denotes respectively by $\sum_{n=0}^{\infty} p_n z^n$ and $\sum_{n=0}^{\infty} q_n z^n$ the series expansion of the two terms in 21. We obtain by the same method that

$$p_n = P(X = 1, Q = n) \propto c_0' n^{a+b-1} \sigma^{-n}, \qquad c_0' = A(\sigma)(\sigma - 1)c_0, \tag{22}$$

$$q_n \propto c_1 S(\sigma) n^{a+b} \sigma^{-n} \tag{23}$$

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