Теорія ймовір. та матем. статист. Вип. 89, 2013, стор. 91–103

Teor. Ĭmovir. ta Matem. Statyst. No. 89, 2013, pp. 91–103

ASYMPTOTIC BEHAVIOR OF THE INTEGRAL FUNCTIONALS FOR UNSTABLE SOLUTIONS OF ONE-DIMENSIONAL ITÔ STOCHASTIC DIFFERENTIAL EQUATIONS

UDC 519.21

G. L. KULINICH, S. V. KUSHNIRENKO, AND Y. S. MISHURA

Abstract. We consider the stochastic one-dimensional differential equations with homogeneous drift and unit diffusion. The drift satisfies conditions supplying the unstable property of the unique strong solution. The explicit form of normalizing factor for certain integral functionals of unstable solution is established to provide the weak convergence to the limiting process. As a result we get the new class of limiting processes that are the functionals of Bessel diffusion processes.

Анотація. Розглядається одновимірне стохастичне диференціальне рівняння з однорідним коефіцієнтом переносу та одиничною дифузією. Коефіцієнт переносу задовольняє умови, при яких єдиний сильний розв'язок даного рівняння є нестійким. Знайдено явний вигляд нормування для певних функціоналів інтегрального типу від нестійкого розв'язку, що забезпечує слабку збіжність до граничного процесу. Отримано новий клас граничних процесів, які є певними функціоналами від бесселівських дифузійних процесів.

Аннотация. Рассматривается одномерное стохастическое дифференциальное уравнение с однородным коэффициентом сноса и единичной диффузией. Коэффициент сноса удовлетворяет условиям, при которых единственное сильное решение данного уравнения является неустойчивым. Найден явный вид нормировки для определенных функционалов интегрального типа от неустойчивого решения, что обеспечивает слабую сходимость к предельному процессу. Получен новый класс предельных процессов, которые являются определенными функционалами от бесселевских диффузионных процессов.

1. Introduction

Let $(\Omega, \Im, \mathsf{P})$ be the complete probability space and $W = \{W(t), t \geq 0\}$ be one-dimensional Wiener process on this space. Let the function $a = a(x)\colon \mathbb{R} \to \mathbb{R}$ be measurable and bounded. It is well-known (see, e.g. [15] and [14], Theorem 4) that the stochastic differential equation with the homogeneous drift and the unit diffusion

$$d\xi(t) = a(\xi(t))\,dt + dW(t), \qquad t \geq 0, \tag{1}$$

has the unique strong solution $\xi = \{\xi(t), t \geq 0\}$ for any initial condition $\xi(0) = x_0 \in \mathbb{R}$.

Definition 1.1. Solution $\xi = \{\xi(t), t \geq 0\}$ of equation (1) is called unstable if for any constant $N > 0$

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t \mathsf{P}\{|\xi(s)| < N\}\,ds = 0.$$

Definition 1.2. Solution $\xi = \{\xi(t), t \geq 0\}$ of equation (1) has ergodic distribution $G(x)$ if for all $x \in \mathbb{R}$

$$\lim_{t\to\infty} \mathsf{P}\{\xi(t) < x\} = G(x).$$

2000 *Mathematics Subject Classification.* Primary 60H10.

Key words and phrases. Itô stochastic differential equation, unstable solution, asymptotic behavior of integral functionals.

Definition 1.3. The family $\{\zeta_T(t), t \ge 0\}$ of stochastic processes is said to converge weakly as $T \to \infty$ to the process $\{\zeta(t), t \ge 0\}$ if for any $L > 0$ measures $\mu_T[0, L]$ that correspond to the processes $\zeta_T(\cdot)$ on the interval $[0,\, L]$ converge weakly to the measure $\mu[0,\, L]$ that corresponds to the process $\zeta(\cdot)$.

Throughout the paper we suppose that the drift coefficient a satisfies assumption

(A_1) there exists such $C > 0$ that for any $x \in \mathbb{R}$

$$|xa(x)| \le C.$$

In this case we can say that the class of equations (1) is located on the border between the equations whose solutions have ergodic distribution, and the equations with unstable solutions. To illustrate this observation, consider the drift coefficient of the form $a(x) = \frac{ax}{1+x^2}$ and introduce the function

$$f(x) = \exp\left\{-2\int_0^x a(v)\,dv\right\}. \tag{2}$$

Note that in our case $f(x) = (1+x^2)^{-a}$. In the paper [11] two cases were considered, namely, $a < -\frac{1}{2}$, $a > -\frac{1}{2}$. It was proved that in the case $a < -\frac{1}{2}$ the solution ξ of equation (1) has ergodic distribution, is transient and moreover

$$\lim_{t\to\infty} \mathsf{P}\{\xi(t) < x\} = \left[\int_{\mathbb{R}} \frac{dv}{f(v)}\right]^{-1} \left[\int_{-\infty}^x \frac{dv}{f(v)}\right] = \left[\int_{\mathbb{R}} (1+v^2)^a dv\right]^{-1} \left[\int_{-\infty}^x (1+v^2)^a dv\right]. \tag{3}$$

At the same time in the case $a > -\frac{1}{2}$ the solution ξ of equation (1) is unstable and recurrent and furthermore the process $r_T(t) = \frac{|\xi(tT)|}{\sqrt{T}}$ with normalizing factor $\frac{1}{\sqrt{T}}$ weakly converges as $T \to \infty$ to the Bessel process $r(t)$ that is the solution of the Itô's equation

$$dr^2(t) = (2a+1)\,dt + 2r(t)\,d\widehat{W}(t) \tag{4}$$

with some Wiener process $\{\widehat{W} = \widehat{W}(t), t \ge 0\}$. Here the weak convergence is considered in the uniform topology on the space of continuous functions. The case $a = -\frac{1}{2}$ is critical in the sense that for $a = -\frac{1}{2}$ the process is recurrent, $\mathsf{P}\{\overline{\lim}_{t\to\infty} \xi(t) = +\infty\} = \mathsf{P}\{\underline{\lim}_{t\to\infty} \xi(t) = -\infty\} = 1$, however, we do not know the normalizing factor that supplies the weak convergence.

The assertion that value $a = -\frac{1}{2}$ is critical can be illustrated by the following examples:

1) If $a(x) = -\frac{1}{2}\frac{x}{1+x^2} - 2\frac{x}{(1+x^2)\ln(1+x^2)}$ then the solution ξ of equation (1) has the ergodic distribution and moreover, we have in equality (3) $f(x) = \sqrt{1+x^2}\left[\ln(1+x^2)\right]^2$.

2) If $a(x) = -\frac{1}{2}\frac{x}{1+x^2} + \frac{x}{(1+x^2)\ln(1+x^2)}$ then the solution ξ of equation (1) is unstable, and stochastic process $\frac{\xi(tT)}{\sqrt{T}}$ converges to degenerate process $r(t) \equiv 0$ as $T \to \infty$.

The present paper is devoted to the asymptotic behavior of the integral functionals $\beta(t) = \int_0^t g(\xi(s))\,ds$ as $t \to \infty$. We suppose that $g = g(x)\colon \mathbb{R} \to \mathbb{R}$ is locally integrable function, ξ is the solution of equation (1). Also, introduce some additional notations. Denote Ψ the class of functions $\psi = \psi(r) > 0$, $r \ge 0$, that are non-decreasing and regularly varying (at infinity) with index $\alpha > 0$, i.e., $\lim_{T\to\infty} \frac{\psi(rT)}{\psi(T)} = r^\alpha$ for all $r > 0$.

Now, take function f that is defined via the relation (2), some constant $b \in \mathbb{R}$ and denote

$$q(x) = \frac{f(x)}{\psi(|x|)}\int_0^x \frac{g(u)}{f(u)}\,du - \bar{b}(x), \qquad \bar{b}(x) = b\,\mathrm{sign}\,x. \tag{5}$$

Suppose additionally that the drift coefficient a and function g satisfy assumption (A_2) (i) with one of the additional restrictions (ii), (iii) or (iv) and also one of the assumptions (A_3) and (A_4):

(A_2) (i) There exist the constants c_i, $i = 1, 2$ such that

$$\lim_{|x|\to\infty}\left[\frac{1}{x}\int_0^x va(v)\,dv - \bar{c}(x)\right] = 0, \tag{6}$$

where

$$\bar{c}(x) = \begin{cases} c_1, & x > 0, \\ c_2, & x < 0, \end{cases}$$

and moreover, one of the following restrictions on the coefficients hold:

(ii) $c_1 = c_2 = c_0 > -\frac{1}{2}$;

(iii) $c_1 > \frac{1}{2}$, $c_2 < \frac{1}{2}$;

(iv) $c_1 < \frac{1}{2}$, $c_2 > \frac{1}{2}$.

(A_3) (i) there exists a constant $C > 0$ such that $f(x) \le C$ for any $x \in \mathbb{R}$ and
(ii) there exist such $b \in \mathbb{R}$ and function $\psi \in \Psi$ that

$$\lim_{|x|\to\infty}\frac{1}{x}\int_0^x \frac{q^2(u)}{f(u)}\,du = 0; \tag{7}$$

(A_4) (i) there exists a constant $\delta > 0$ such that $0 < \delta \le f(x)$ for any $x \in \mathbb{R}$ and
(ii) there exist such $b \in \mathbb{R}$ and function $\psi \in \Psi$ that

$$\lim_{|x|\to\infty}\frac{f(x)}{x}\int_0^x q^2(u)\,du = 0. \tag{8}$$

In the present paper in order to proof that under the conditions (A_1), (A_2) and one of the conditions (A_3) and (A_4) random variable $\frac{\beta(t)}{\sqrt{t}\psi(\sqrt{t})}$ with normalizing factor $\frac{1}{\sqrt{t}\psi(\sqrt{t})}$ has the limit distribution as $t \to \infty$, we study the limit behavior as $T \to \infty$ of the process

$$\beta_T(t) = \frac{1}{\sqrt{T}\psi(\sqrt{T})}\int_0^{tT} g(\xi(s))\,ds,$$

with parameter $T > 0$. Theorems 2.1 and 2.2 describe the limit behavior mentioned above.

Remark 1.1. It is very easy to see that any of conditions (A_3) and (A_4) supply the convergence

$$\lim_{|x|\to\infty}\frac{1}{x}\int_0^x q^2(u)\,du = 0. \tag{9}$$

If condition (A_3) holds then

$$\frac{1}{x}\int_0^x q^2(u)\,du \le C\frac{1}{x}\int_0^x \frac{q^2(u)}{f(u)}\,du \to 0 \quad \text{as } |x| \to \infty.$$

If condition (A_4) holds then

$$\frac{1}{x}\int_0^x q^2(u)\,du \le \frac{1}{\delta}\,\frac{f(x)}{x}\int_0^x q^2(u)\,du \to 0 \quad \text{as } |x| \to \infty.$$

Moreover, if $0 < \delta \le f(x) \le C$ then (9) is equivalent both to (A_3), (ii) and (A_4), (ii). However, neither (A_3), (ii) and (A_4), (ii) nor (9) do not supply convergence $q(x) \to 0$ as $|x| \to \infty$. In other words, under any of these conditions function q can admit “explosions”.

Remark 1.2. The function $q(x)$ (see Example 2.1) satisfies the condition (9). Obviously, $q(x) \nrightarrow 0$ as $|x| \to \infty$.

As to previous results in this direction, it was proved in [11] that under the condition (A_2) solution ξ of equation (1) is unstable. Moreover, in the case when (A_2), (ii) holds then $\frac{|\xi(tT)|}{\sqrt{T}}$ weakly converges as $T \to \infty$ to process r that is the solution of equation (4) with $a = c_0$. In the case when (A_2), (iii) holds then $\frac{\xi(tT)}{\sqrt{T}}$ weakly converges to process r with $a = c_1$, and in the case when (A_2), (iv) holds then $\frac{-\xi(tT)}{\sqrt{T}}$ weakly converges to process r with $a = c_2$. Asymptotic behavior of the process $\beta_T(t)$ in the case when conditions (A_2), (i) and (ii) hold and additionally $q(x) \to 0$ as $|x| \to \infty$ were considered in the papers [5] and [12]. The results of the paper [5] are generalized in the present paper to the case of the functions $q = q(x)$ with possible "explosions" (conditions (A_3) and (A_4)) and are extended to the cases when (A_2), (i) and (iii) or (A_2), (i) and (iv) hold. Moreover, the proofs from [5] are essentially simplified in the present paper due to the representation (12). The paper [12] contains similar result for the functional $\beta_T(t)$ of the solution ξ of equation (1) on the half-axis $(0, +\infty)$ with the instant reflection of the solution at zero point, and in this case it was supposed that $\psi(|x|) = |x|^\alpha$, $\alpha \geq 0$, $q(x) \to 0$ as $x \to \infty$.

The most complete results concerning the asymptotic behavior of the functionals $\beta_T(t)$ are proved for the equations (1) with more restrictive assumption on the drift coefficient, namely, $\left|\int_0^x a(u)\,du\right| \leq C$ (see [8] – [10]). The paper [8] contains the weak convergence of distributions of $\beta_T(t)$ in the case when $q(x) \to 0$ as $|x| \to \infty$. In the paper [9] the weak convergence of distributions of $\beta_T(t)$ was obtained under assumption (9) on function $q = q(x)$. In the paper [10] the necessary and sufficient conditions of weak convergence were obtained that are connected, in some sense, to (9).

The asymptotic behavior of the integral functionals of the form $\int_0^t g_T(\xi_T(s))\,d\mu_T(s)$, where $\xi_T(t)$ are the solutions of stochastic differential equations and $\mu_T(t)$ is the family of martingales that converge in probability, was considered in the paper [3, §5, Chapter IX] under the assumption of locally uniform convergence of the coefficients of the equation.

The paper is organized as follows: principal results are proved in Section 2 while an auxiliary lemma is relegated to Section 3. Section 4 concludes.

2. The main results

In what follows we denote C or C with some subscripts constants whose values are not so important and can change from line to line.

Theorem 2.1. *Let ξ be the solution of equation* (1) *with the drift coefficient a satisfying assumptions (A_1), (A_2), (i) and one of the assumptions (A_2), (ii), (iii) or (iv).*

Then the stochastic process

$$\beta_T(t) = \frac{1}{\sqrt{T}\psi(\sqrt{T})} \int_0^{tT} g(\xi(s))\,ds$$

converges as $T \to \infty$ weakly in the unform topology of the space of continuous functions to the process

$$\beta(t) = 2b\left[\frac{r^{\alpha+1}(t)}{\alpha+1} - \int_0^t r^\alpha(s)\,d\widehat{W}(s)\right],$$

where $r(t) \geq 0$ is the solution of stochastic differential equation

$$dr^2(t) = (2a+1)\,dt + 2r(t)\,d\widehat{W}(t).$$

Here $a = c_0$ in the case when assumption (A_2), (ii) is satisfied, $a = c_1$ in the case when assumption (A_2), (iii) is satisfied and $a = c_2$ in the case when assumption (A_2), (iv) is satisfied.

Proof. Introduce parameter $T > 0$ and set

$$r_T(t) = \frac{|\xi(tT)|}{\sqrt{T}}, \qquad W_T(t) = \frac{W(tT)}{\sqrt{T}}, \qquad \widehat{W}_T(t) = \int_0^t \operatorname{sign} \xi(sT)\, dW_T(s),$$
$$P_N = \mathsf{P}\left\{ \sup_{0\le t\le L} r_T(t) > N \right\}, \qquad \alpha_T(t) = \frac{1}{T}\int_0^{tT} \left[\xi(s)a(\xi(s)) - \bar{c}(\xi(s))\right] ds,$$

where L and N are arbitrary positive constants. Evidently, for any fixed $T > 0$ process $W_T = \{W_T(t), t \ge 0\}$ is a Wiener process. Furthermore, it follows, e.g., from [2, Chapter 6, §3, Lemma 5] that

$$\int_0^t \mathsf{P}\{\xi(s) = 0\}\, ds = 0$$

for any $t > 0$. Therefore $\widehat{W}_T = \{\widehat{W}_T(t), t \ge 0\}$ for any $T > 0$ is continuous with probability 1 square integrable martingale with the quadratic characteristics $\langle \widehat{W}_T\rangle(t) = t$. It immediately follows from the Doob's theorem that $\widehat{W}_T$ is a Wiener process for any $T > 0$. Applying Itô's formula to the process r_T^2, we get

$$r_T^2(t) = \frac{x_0^2}{T} + \int_0^t \left[2\bar{c}(\xi(sT)) + 1\right] ds + 2\int_0^t r_T(s)\, d\widehat{W}_T(s) + 2\alpha_T(t).$$

Consider the function

$$F(x) = 2\int_0^x f(u)\left(\int_0^u \frac{g(v)}{f(v)}\, dv\right) du.$$

Obviously, function F has a continuous derivative F' and a.e. w.r.t. to the Lebesgue measure on $\mathbb{R}$ has a second derivative F'' that is locally integrable. Therefore we can apply an Itô's formula from [6, Chapter 2, §10] to $F(\xi(t))$ and get the equality

$$F(\xi(t)) - F(x_0) = \int_0^t \left[F'(\xi(s))a(\xi(s)) + \frac{1}{2}F''(\xi(s))\right] ds + \int_0^t F'(\xi(s))\, dW(s) \qquad (10)$$

with probability 1 for any $t \ge 0$. It is easy to see that a.e. w.r.t. to the Lebesgue measure on $\mathbb{R}$ the following equality holds

$$F'(x)a(x) + \frac{1}{2}F''(x) = g(x). \qquad (11)$$

Applying (11) to (10) we get that

$$F(\xi(t)) - F(x_0) = \int_0^t g(\xi(s))\, ds + \int_0^t F'(\xi(s))\, dW(s)$$

with probability 1 for any $t \ge 0$. After some evident transformations we get from the last equality that

$$\beta_T(t) = \frac{1}{\sqrt{T}\psi(\sqrt{T})}\left[F(\xi(tT)) - F(x_0) - \int_0^{tT} F'(\xi(s))\, dW(s)\right].$$

Let us consider the first term

$$\begin{aligned}
\frac{F(\xi(tT))}{\sqrt{T}\psi(\sqrt{T})} &= \frac{2}{\sqrt{T}\psi(\sqrt{T})}\int_0^{\xi(tT)} f(u)\left(\int_0^u \frac{g(v)}{f(v)}\,dv\right)\,du \\
&= \frac{2}{\sqrt{T}\psi(\sqrt{T})}\int_0^{\xi(tT)} \left(\frac{f(u)}{\psi(|u|)}\int_0^u \frac{g(v)}{f(v)}\,dv \pm \bar{b}(u)\right)\psi(|u|)\,du \\
&= \frac{2}{\sqrt{T}\psi(\sqrt{T})}\left(\int_0^{\xi(tT)} \bar{b}(u)\psi(|u|)\,du + \int_0^{\xi(tT)} q(u)\psi(|u|)\,du\right) \\
&= 2b\int_0^{\frac{|\xi(tT)|}{\sqrt{T}}} \frac{\psi\left(|u|\sqrt{T}\right)}{\psi(\sqrt{T})}\,du + \frac{2}{\sqrt{T}\psi(\sqrt{T})}\int_0^{\xi(tT)} q(u)\psi\left(|u|\right)\,du \\
&= 2b\int_0^{\frac{|\xi(tT)|}{\sqrt{T}}} |u|^\alpha\,du + 2b\int_0^{\frac{|\xi(tT)|}{\sqrt{T}}} \left(\frac{\psi\left(|u|\sqrt{T}\right)}{\psi(\sqrt{T})} - |u|^\alpha\right)\,du \\
&\quad + \frac{2}{\sqrt{T}\psi(\sqrt{T})}\int_0^{\xi(tT)} q(u)\psi\left(|u|\right)\,du,
\end{aligned}$$

and transform the last term

$$\begin{aligned}
\frac{1}{\sqrt{T}\psi(\sqrt{T})}\int_0^{tT} F'(\xi(s))\,dW(s) &= \frac{2}{\sqrt{T}\psi(\sqrt{T})}\int_0^{tT} f(\xi(s))\left(\int_0^{\xi(s)} \frac{g(u)}{f(u)}\,du\right)\,dW(s) \\
&= \frac{2}{\sqrt{T}\psi(\sqrt{T})}\left[\int_0^{tT} \bar{b}(\xi(s))\psi(|\xi(s)|)\,dW(s) + \int_0^{tT} q(\xi(s))\psi(|\xi(s)|)\,dW(s)\right] \\
&= \frac{2}{\psi(\sqrt{T})}\int_0^t \bar{b}(\xi(sT))\psi(|\xi(sT)|)\,\frac{dW(sT)}{\sqrt{T}} + 2\int_0^t q(\xi(sT))\frac{\psi(|\xi(sT)|)}{\psi(\sqrt{T})}\,dW_T(s) \\
&= 2b\int_0^t \frac{\psi(|\xi(sT)|)}{\psi(\sqrt{T})}\,d\widehat{W}_T(s) + 2\int_0^t q(\xi(sT))\frac{\psi(|\xi(sT)|)}{\psi(\sqrt{T})}\,dW_T(s) \\
&= 2b\int_0^t r_T^\alpha(s)\,d\widehat{W}_T(s) + 2b\int_0^t \left[\frac{\psi(r_T(s)\sqrt{T})}{\psi(\sqrt{T})} - r_T^\alpha(s)\right]\,d\widehat{W}_T(s) \\
&\quad + 2\int_0^t q(\xi(sT))\frac{\psi(r_T(s)\sqrt{T})}{\psi(\sqrt{T})}\,dW_T(s).
\end{aligned}$$

Therefore

$$\beta_T(t) = -\frac{F(x_0)}{\sqrt{T}\psi(\sqrt{T})} + 2b\int_0^{r_T(t)} u^\alpha du - 2b\int_0^t r_T^\alpha(s)\,d\widehat{W}_T(s) + 2\sum_{k=1}^4 S_T^{(k)}(t), \tag{12}$$

where

$$\begin{aligned}
S_T^{(1)}(t) &= b\int_0^{r_T(t)} \left[\frac{\psi(u\sqrt{T})}{\psi(\sqrt{T})} - u^\alpha\right]\,du, \\
S_T^{(2)}(t) &= \frac{1}{\sqrt{T}\psi(\sqrt{T})}\int_0^{\xi(tT)} q(u)\psi(|u|)\,du, \\
S_T^{(3)}(t) &= -b\int_0^t \left[\frac{\psi(r_T(s)\sqrt{T})}{\psi(\sqrt{T})} - r_T^\alpha(s)\right]\,d\widehat{W}_T(s), \\
S_T^{(4)}(t) &= -\int_0^t q(\xi(sT))\frac{\psi(r_T(s)\sqrt{T})}{\psi(\sqrt{T})}\,dW_T(s).
\end{aligned}$$

It is known from [11] that under condition (A_1) the process $\{r_T(t), t \geq 0\}$ converges weakly as $T \to \infty$ to the process $\{r(t), t \geq 0\}$ that is the solution of equation (4) with $a = c_0$ in the case (A_2),(ii), with $a = c_1$ in the case (A_2), (iii) and with $a = c_2$ in the case (A_2), (iv). Furthermore, for any $L > 0$ and $\varepsilon > 0$ we have that

$$\lim_{N\to\infty} \overline{\lim_{T\to\infty}} P_N = 0,$$

$$\lim_{h\to 0} \overline{\lim_{T\to\infty}} \sup_{|t_1-t_2|\leq h;\, t_i\leq L} \mathsf{P}\left\{|r_T(t_2) - r_T(t_1)| > \varepsilon\right\} = 0. \tag{13}$$

Now we are in position to establish that $S_T^{(k)}$, $k = 1, \ldots, 4$, uniformly converge to zero in probability. In particular, it means that they satisfy equalities (13) as well. To start with, note that it follows from Lemma 3.1, evident inequalities

$$\mathsf{P}\left\{|\eta + \zeta| > \varepsilon\right\} \leq \mathsf{P}\left\{|\eta| > \frac{\varepsilon}{2}\right\} + \mathsf{P}\left\{|\zeta| > \frac{\varepsilon}{2}\right\}, \qquad \mathsf{P}\left\{|\eta| > \varepsilon\right\} \leq \frac{\mathsf{E}\, h(|\eta|)}{h(\varepsilon)}$$

with $h = x$ and $h = x^2$ and the properties of Itô's integrals that for any $\varepsilon > 0$, $L > 0$ and $T \geq T_N$, where T_N are introduced in Lemma 3.1, the following inequalities hold true:

$$\begin{aligned} \mathsf{P}\left\{\sup_{0\leq t\leq L} |S_T^{(1)}(t)| > \varepsilon\right\} &\leq P_N + \frac{2}{\varepsilon}\, \mathsf{E} \sup_{0\leq t\leq L} |S_T^{(1)}(t)| \chi_{\{r_T(t)\leq N\}} \\ &\leq P_N + \frac{2}{\varepsilon}\, |b| \int_0^N \left| \frac{\psi(u\sqrt{T})}{\psi(\sqrt{T})} - u^{\alpha} \right| du, \end{aligned} \tag{14}$$

$$\begin{aligned} \mathsf{P}\left\{\sup_{0\leq t\leq L} \left|S_T^{(2)}(t)\right| > \varepsilon\right\} &\leq \mathsf{P}\left\{\sup_{0\leq t\leq L} \left| \int_0^{\frac{\xi(tT)}{\sqrt{T}}} q(u\sqrt{T}) \frac{\psi(|u|\sqrt{T})}{\psi(\sqrt{T})}\, du \right| > \varepsilon\right\} \\ &\leq P_N + \frac{2}{\varepsilon}\, \mathsf{E} \sup_{0\leq t\leq L} \left| \int_0^{\frac{\xi(tT)}{\sqrt{T}}} q(u\sqrt{T}) \frac{\psi(|u|\sqrt{T})}{\psi(\sqrt{T})}\, du \right| \chi_{\{r_T(t)\leq N\}} \\ &\leq P_N + \frac{2}{\varepsilon}\, C_N \int_{-N}^{N} |q(u\sqrt{T})|\, du \leq P_N + \frac{2}{\varepsilon}\, C_N (2N)^{\frac{1}{2}} \left(\frac{1}{\sqrt{T}} \int_{-N\sqrt{T}}^{N\sqrt{T}} q^2(u)\, du \right)^{\frac{1}{2}}, \end{aligned} \tag{15}$$

$$\mathsf{P}\left\{\sup_{0\leq t\leq L} |S_T^{(3)}(t)| > \varepsilon\right\} \leq P_N + 4\left(\frac{2}{\varepsilon}\right)^2 b^2\, \mathsf{E} \int_0^L \left| \frac{\psi(r_T(s)\sqrt{T})}{\psi(\sqrt{T})} - r_T^{\alpha}(s) \right|^2 \chi_{\{r_T(s)\leq N\}}\, ds, \tag{16}$$

$$\begin{aligned} \mathsf{P}\left\{\sup_{0\leq t\leq L} |S_T^{(4)}(t)| > \varepsilon\right\} &\leq P_N + 4\left(\frac{2}{\varepsilon}\right)^2 \mathsf{E} \int_0^L q^2(\xi(sT)) \left[\frac{\psi(r_T(s)\sqrt{T})}{\psi(\sqrt{T})}\right]^2 \chi_{\{r_T(s)\leq N\}}\, ds \\ &\leq P_N + 4\left(\frac{2}{\varepsilon}\right)^2 C_N^2\, \mathsf{E} \int_0^L q^2(\xi(sT)) \chi_{\{r_T(s)\leq N\}}\, ds. \end{aligned} \tag{17}$$

Taking into account the convergence $\frac{\psi(|x|\sqrt{T})}{\psi(\sqrt{T})} - |x|^{\alpha} \to 0$ as $T \to \infty$, boundedness on the interval $|x| \leq N$ and relation (9), we let in inequalities (14) and (15) $T \to \infty$ and after that $N \to \infty$ and get

$$\sup_{0\leq t\leq L} \left|S_T^{(k)}(t)\right| \xrightarrow{\mathsf{P}} 0 \tag{18}$$

as $T \to \infty$ and for $k = 1, 2$.

Now we shall establish similar convergence for $k = 3, 4$. It is known from [4] that for any $0 < \delta < N < \infty$ the following convergence holds:

$$\sup_{0<\delta\le|x|\le N}\left|\frac{\psi(|x|\sqrt{T})}{\psi(\sqrt{T})} - |x|^\alpha\right| \to 0$$

as $T \to \infty$. Therefore, taking into account monotonicity of function $\psi(r)$, $r \ge 0$, we get the following convergence for any $0 < \delta < N$:

$$\begin{aligned}
&\mathsf{E}\int_0^L \left[\frac{\psi(r_T(s)\sqrt{T})}{\psi(\sqrt{T})} - r_T(s)\right]^2 \chi_{\{r_T(s)\le N\}}\,ds\\
&\quad\le L \sup_{\delta\le|x|\le N}\left|\frac{\psi(|x|\sqrt{T})}{\psi(\sqrt{T})} - |x|^\alpha\right|^2 + 2\int_0^L \left(\left[\frac{\psi(\delta\sqrt{T})}{\psi(\sqrt{T})}\right]^2 + \delta^2\right) ds \to 0,
\end{aligned}$$

if to tend at first $T \to \infty$ and after that $\delta \to 0$.

So, taking into account inequality (16) we get that convergence (18) holds for $S_T^{(3)}(t)$ as well. At last, in order to prove convergence (18) for $S_T^{(4)}(t)$, we apply Itô formula and get

$$\mathsf{E}\int_0^L q^2(\xi(sT))\chi_{\{|\xi(sT)|\le N\sqrt{T}\}}\,ds = \mathsf{E}\left[\Phi_T(\xi(LT)) - \Phi_T(x_0)\right],$$

where

$$\Phi_T(x) = \frac{1}{T}\int_0^x f(u)\left(\int_0^u \frac{q^2(v)}{f(v)}\chi_{\{|v|\le N\sqrt{T}\}}\,dv\right) du.$$

Now we consider separately conditions (A_3) and (A_4). It is easy to see that under condition (A_3) we have the following relations

$$\begin{aligned}
\frac{1}{x^2}\left|\int_0^x f(u)\left(\int_0^u \frac{q^2(v)}{f(v)}\,dv\right) du\right| &\le \frac{C}{x^2}\left|\int_0^x \left(\int_0^u \frac{q^2(v)}{f(v)}\,dv\right) du\right|\\
&= \frac{C}{x^2}\left|\int_0^x u\left(\frac{1}{u}\int_0^u \frac{q^2(v)}{f(v)}\,dv\right) du\right| \to 0 \quad \text{as } |x| \to \infty.
\end{aligned}$$

In turn, under condition (A_4) we have the following relations

$$\begin{aligned}
\frac{1}{x^2}\left|\int_0^x f(u)\left(\int_0^u \frac{q^2(v)}{f(v)}\,dv\right) du\right| &\le \frac{1}{\delta}\frac{1}{x^2}\left|\int_0^x f(u)\left(\int_0^u q^2(v)\,dv\right) du\right|\\
&= \frac{1}{\delta}\frac{1}{x^2}\left|\int_0^x u\left(\frac{f(u)}{u}\int_0^u q^2(v)\,dv\right) du\right| \to 0 \quad \text{as } |x| \to \infty.
\end{aligned}$$

Therefore, any of conditions (A_3) and (A_4) supply the following convergence

$$\frac{1}{x^2}\int_0^x f(u)\left(\int_0^u \frac{q^2(v)}{f(v)}\,dv\right) du \to 0$$

as $|x| \to \infty$. Therefore for any $\varepsilon > 0$ there exists such L_ε that for $|x| > L_\varepsilon$ we have inequality

$$\frac{1}{x^2}\left|\int_0^x f(u)\left(\int_0^u \frac{q^2(v)}{f(v)}\,dv\right) du\right| < \varepsilon. \tag{19}$$

Furthermore, since function $\frac{1}{x^2}\int_0^x f(u)\left(\int_0^u \frac{q^2(v)}{f(v)}\,dv\right) du$ is bounded at zero, there exists such $C_\varepsilon > 0$ that

$$\sup_{|x|\le L_\varepsilon}\frac{1}{x^2}\left|\int_0^x f(u)\left(\int_0^u \frac{q^2(v)}{f(v)}\,dv\right) du\right| \le C_\varepsilon. \tag{20}$$

Besides this,

$$\mathsf{E}\,\frac{|\xi(tT)|^2}{T} \le C + C_1 t. \tag{21}$$

Relations (19) and (20) together with (21) provide that

$$\begin{aligned}\mathsf{E}\,|\Phi_T(\xi(LT))| &\le \mathsf{E}\,\frac{|\xi(tT)|^2}{T}\cdot\frac{1}{|\xi(LT)|^2}\left|\int_0^{\xi(LT)} f(u)\left(\int_0^u \frac{q^2(v)}{f(v)}\,dv\right)du\right| \\ &\le \frac{C_\varepsilon}{T} + \varepsilon(C + C_1 t),\end{aligned}$$

whence $\mathsf{E}\,|\Phi_T(\xi(LT))| \to 0$ as $T\to\infty$. Evidently, $|\Phi_T(x_0)| \le \frac{C}{T}$. Therefore,

$$\mathsf{E}\int_0^L q^2(\xi(sT))\chi_{\{|\xi(sT)|\le N\sqrt{T}\}}\,ds \to 0$$

as $T\to\infty$. Together with (17) it means that the convergence (18) holds for $S_T^{(4)}(t)$ as well. Evidently, relation (13) holds for processes $\widehat{W}_T(t)$.

It means that we can apply Skorokhod representation theorem [13] and for any sequence $T_n\to\infty$ to choose the subsequence $T'_n\to\infty$, probability space $(\tilde\Omega, \tilde{\Im}, \tilde{\mathsf{P}})$ and processes $(\tilde r_{T'_n}(t), \tilde W_{T'_n}(t), \tilde S^{(i)}_{T'_n}(t), i=1,\dots,4)$ on this space so that the couple of processes will be stochastically equivalent to the process $(r_{T'_n}(t), \widehat{W}_{T'_n}(t), S^{(i)}_{T'_n}(t), i=1,\dots,4)$ and moreover,

$$\tilde r_{T'_n}(t) \xrightarrow{\tilde{\mathsf{P}}} \tilde r(t), \qquad \tilde W_{T'_n}(t) \xrightarrow{\tilde{\mathsf{P}}} \tilde W(t), \qquad \tilde S^{(i)}_{T'_n}(t) \xrightarrow{\tilde{\mathsf{P}}} \tilde S^{(i)}(t), \qquad i=1,\dots,4,$$

as $T'_n\to\infty$. In our case, according to (18), $\tilde S^{(i)}(t)=0$, $i=1,\dots,4$, and the processes $\tilde r(t)$, $\tilde W(t)$ satisfy equations (4) with $a=c_0$ in the case (A_2), (ii), $a=c_1$ in the case (A_2), (iii) and $a=c_2$ in the case (A_2), (iv), see [11].

According to equality (12) we have that the functional $\beta_{T'_n}(t)$ is stochastically equivalent to the functional $\tilde\beta_{T'_n}(t)$ for which we have similar equality

$$\tilde\beta_{T'_n}(t) = -\frac{F(x_0)}{\sqrt{T'_n}\psi(\sqrt{T'_n})} + 2b\int_0^{\tilde r_{T'_n}(t)} u^\alpha\,du - 2b\int_0^t \tilde r^\alpha_{T'_n}(s)\,d\tilde W_{T'_n}(s) + 2\sum_{i=1}^4 \tilde S^{(i)}_{T'_n}(t). \tag{22}$$

It is possible to get the limit as $T'_n\to\infty$ [13] in this equality and get that $\tilde\beta_{T'_n}(t) \xrightarrow{\tilde{\mathsf{P}}} \tilde\beta(t)$, where

$$\tilde\beta(t) = 2b\left[\int_0^{\tilde r(t)} u^\alpha du - \int_0^t \tilde r^\alpha(s)\,d\tilde W(s)\right]. \tag{23}$$

It follows from the strong uniqueness of the solution of equation (4) (see, e.g., [7]) that the distributions of the limit process $\tilde\beta(t)$ are unique as well. Therefore, it follows from arbitrary choice of $T_n\to\infty$ that the finite-dimensional distributions of the processes $\beta_T(t)$ tend as $T\to\infty$ to the corresponding distributions of the process $\tilde\beta(t)$ that is defined by equality (23). In order to establish the weak convergence of the processes $\beta_T(t)$ to the process $\tilde\beta(t)$, it is sufficient to prove tightness, i.e., to prove that for any $L>0$

$$\lim_{h\to 0}\overline{\lim_{T\to\infty}}\,\mathsf{P}\left\{\sup_{|t_1-t_2|\le h;\, t_i\le L} |\beta_T(t_2)-\beta_T(t_1)| > \varepsilon\right\} = 0. \tag{24}$$

Tightness of the processes $r_T(t)$ was established in [11] and it was mentioned that tightness of $S_T^{(i)}(t)=0$, $i=1,\dots,4$, follows from (18). Furthermore, taking into account the properties of stochastic Itô integrals, we get the following bounds for any $\varepsilon > 0$, $L > 0$ and $N > 0$:

$$\begin{aligned} &\mathsf{P}\left\{\sup_{|t_1-t_2|\le h; t_i\le L}\left|\int_0^{r_T(t_2)} u^\alpha du - \int_0^{r_T(t_1)} u^\alpha du\right| > \varepsilon\right\} \\ &\qquad \le P_N + \mathsf{P}\left\{N^\alpha \sup_{|t_1-t_2|\le h;\, t_i\le L} |r_T(t_2) - r_T(t_1)| > \frac{\varepsilon}{2}\right\} \end{aligned} \tag{25}$$

and

$$\begin{aligned} &\mathsf{P}\left\{\sup_{|t_1-t_2|\le h; t_i\le L}\left|\int_{t_1}^{t_2} r_T^\alpha(s)\, d\widehat{W}_T(s)\right| > \varepsilon\right\} \\ &\quad \le P_N + \mathsf{P}\left\{4 \sup_{kh\le L}\ \sup_{kh\le t\le (k+1)h}\left|\int_{kh}^t r_T^\alpha(s)\chi_{\{r_T(s)\le N\}}\, d\widehat{W}_T(s)\right| > \frac{\varepsilon}{2}\right\} \\ &\quad \le P_N + \sum_{kh<L} \mathsf{P}\left\{\sup_{kh\le t\le (k+1)h}\left|\int_{kh}^t r_T^\alpha(s)\chi_{\{r_T(s)\le N\}}\, d\widehat{W}_T(s)\right| > \frac{\varepsilon}{8}\right\} \\ &\quad \le P_N + \sum_{kh<L}\left(\frac{8}{\varepsilon}\right)^4 \mathsf{E} \sup_{kh\le t\le (k+1)h}\left[\int_{kh}^t r_T^\alpha(s)\chi_{\{r_T(s)\le N\}}\, d\widehat{W}_T(s)\right]^4 \\ &\quad \le P_N + \sum_{kh\le L}\left(\frac{8}{\varepsilon}\right)^4\left(\frac{4}{3}\right)^4 \mathsf{E}\left[\int_{kh}^{(k+1)h} r_T^\alpha(s)\chi_{\{r_T(s)\le N\}}\, d\widehat{W}_T(s)\right]^4 \\ &\quad \le P_N + \left(\frac{8}{\varepsilon}\right)^4\left(\frac{4}{3}\right)^4 \cdot 36 N^{4\alpha}\sum_{kh\le L} h^2 \\ &\quad \le P_N + \frac{ChN^{4\alpha}}{\varepsilon^4}. \end{aligned} \tag{26}$$

In the last inequality the following upper bound for the fourth moment of the Itô's integral w.r.t. the Wiener process from [1] or [13] was used:

$$\mathsf{E}\left(\int_a^b f(t)\, dW(t)\right)^4 \le 36(b-a)\int_a^b \mathsf{E}\,|f(t)|^4\, dt.$$

It follows from (25) and (26) that the right-hand side of (12) is tight, i.e., satisfies (24). So, we have tightness (24) and consequently $\beta_T(t)$ weakly converges as $T \to \infty$ to the process $\beta(t)$ whence the proof follows. □

Example 2.1. Consider equation (1) with the drift coefficient of the form $a(x) = \frac{x}{1+x^2}$. In this case $f(x) = (1+x^2)^{-1}$ and the function $q(x)$ from (5) can be rewritten as

$$q(x) = \frac{1}{\psi(|x|)(1+x^2)}\int_0^x g(u)\left(1+u^2\right) du - b \operatorname{sign} x.$$

Let $\psi(|x|) = |x|$ is slowly varying (at infinity) function ($\alpha = 1$), then

$$\int_0^x g(u)\left(1+u^2\right) du = bx\left(1+x^2\right) + q(x)|x|\left(1+x^2\right) = x\left(1+x^2\right)\left[b + q(x)\operatorname{sign} x\right],$$

whence

$$g(x) = \frac{1}{1+x^2}\left[x\left(1+x^2\right)\left(b+q(x)\operatorname{sign} x\right)\right]'$$

a.e. w.r.t. to the Lebesgue measure on $\mathbb{R}$.

Consider the continuous function with "explosions"

$$q(x) = \begin{cases} q_1(x), & x \in \Delta_n, \\ 0, & x \notin \Delta_n, \end{cases}$$

where $q_1(x) > 0$, $\max_{x\in\Delta_n} q_1(x) = 1$, $\Delta_n = \left(n; n + \frac{1}{n^3}\right)$, $n \in \mathbb{N}$. Continuing $q(x)$ in a symmetric way to $(-\infty, 0)$, we obtain that the function $q(x)$, $x \in \mathbb{R}$, satisfies the condition (7) with the function $f(x) = (1+x^2)^{-1} \le C$.

If we put $q(x)$ in the last allocated equality we get $g(x)$ such that the stochastic process

$$\beta_T(t) = \frac{1}{\sqrt{T}\psi(\sqrt{T})} \int_0^{tT} g(\xi(s))\, ds = \frac{1}{T} \int_0^{tT} g(\xi(s))\, ds$$

converges as $T \to \infty$ weakly to the process

$$\beta(t) = 2b \left[\frac{r^2(t)}{2} - \int_0^t r(s)\, d\widehat{W}(s) \right]$$

where $r(t) \ge 0$ is the solution of stochastic differential equation

$$dr^2(t) = 3\, dt + 2r(t)\, d\widehat{W}(t).$$

In this case $\beta(t) = 3bt$.

Remark 2.1. Analyzing the proof of Theorem 2.1 it is easy to see that it is true even in the case when we establish just the weak convergence of the processes $r_T(t)$ to the process $r(t)$ and the representation (12) in which $\sup_{0\le t\le L}\left|S_T^{(k)}(t)\right| \xrightarrow{\mathsf{P}} 0$, $k = 1, \dots, 4$ as $T \to \infty$ for any $L > 0$.

In this connection, we can deduce the following statement as a corollary of Theorem 2.1.

Theorem 2.2. *Let ξ be a solution of equation* (1) *and let convergence relation* (6) *holds. Also, let locally integrable real-valued function g is such that there exists non-decreasing function $\psi(r)$, $r \ge 0$ that is regularly varying at infinity of order $\alpha > 0$ and $q(x) \to 0$ as $|x| \to \infty$. Here q is defined in* (5). *Then Theorem 2.1 holds.*

Proof. Indeed, apply the representation (12). Similarly to proof of Theorem 2.1 we get that $\sup_{0\le t\le L}\left|S_T^{(k)}(t)\right| \xrightarrow{\mathsf{P}} 0$ as $T \to \infty$, $k = 1, 2, 3$. Convergence $\sup_{0\le t\le L}\left|S_T^{(4)}(t)\right| \xrightarrow{\mathsf{P}} 0$ as $T \to \infty$ follows directly from inequality (17) and convergence $q(x) \to 0$ as $|x| \to \infty$. In order to finish the proof of the present theorem, it is sufficient to apply Remark 2.1. □

Example 2.2. Consider the class of equations (1) with the drift coefficient of the form

$$a(x) = \frac{x\bar{c}(x)}{1+x^2},$$

where

$$\bar{c}(x) = \begin{cases} c_1, & x > 0, \\ c_2, & x < 0, \end{cases} \qquad c_1 = c_2 = c_0, \qquad 2c_0 + 1 > 0.$$

1) Let $c_0 = 1$. In this case $f(x) = (1+x^2)^{-1}$ and in order to satisfy the assumptions of Theorem 2.2 the function $q(x)$ can be rewritten as

$$q(x) = \frac{1}{|x|(1+x^2)} \int_0^x g_0(1+u^2)\, du - b \operatorname{sign} x.$$

If $b = \frac{g_0}{3}$, $g(x) = g_0$, $\psi(|x|) = |x|$ is slowly varying (at infinity) function ($\alpha = 1$), then $q(x) \to 0$ as $|x| \to \infty$ and the stochastic process $\beta_T(t) = \frac{1}{\sqrt{T}\psi(\sqrt{T})} \int_0^{tT} g_0\, ds = \frac{1}{T} g_0 \int_0^{tT} ds$ converges as $T \to \infty$ weakly to the process

$$\beta(t) = 2b \left[\frac{r^{\alpha+1}(t)}{\alpha+1} - \int_0^t r^\alpha(s)\, d\widehat{W}(s) \right] = \frac{2}{3} g_0 \left[\frac{r^2(t)}{2} - \int_0^t r(s)\, d\widehat{W}(s) \right]$$

where $r(t) \geq 0$ is the solution of stochastic differential equation

$$dr^2(t) = 3\, dt + 2r(t)\, d\widehat{W}(t).$$

In this case $\beta(t) = g_0 t$.

2) Let $c_0 = \frac{1}{2}$, so $f(x) = (1+x^2)^{\frac{-1}{2}}$. If $g(x) = g_0$, $\psi(|x|) = |x|$, $b = \frac{g_0}{2}$ and

$$q(x) = \frac{1}{|x|\sqrt{1+x^2}} g_0 \int_0^x \sqrt{1+u^2}\, du - \frac{g_0}{2} \operatorname{sign} x \to 0 \quad \text{as } |x| \to \infty,$$

then the stochastic process $\beta_T(t)$ converges as $T \to \infty$ weakly to the process $\beta(t) = g_0 t$.

3) If $c_0 = 1$, $g(x) = \sin^2 x$, $\psi(|x|) = |x|$, $b = \frac{1}{6}$, then

$$q(x) = \frac{1}{|x|(1+x^2)} \int_0^x \left(1+u^2\right) \sin^2 u\, du - \frac{1}{6} \operatorname{sign} x \to 0 \quad \text{as } |x| \to \infty.$$

The stochastic process $\beta_T(t) = \frac{1}{\sqrt{T}\psi(\sqrt{T})} \int_0^{tT} \sin^2(\xi(s))\, ds = \frac{1}{T} \int_0^{tT} \sin^2(\xi(s))\, ds$ converges as $T \to \infty$ weakly to the process

$$\beta(t) = \frac{1}{3} \left[\frac{r^2(t)}{2} - \int_0^t r(s)\, d\widehat{W}(s) \right] = \frac{t}{2}.$$

3. Auxiliary result

Now we prove an auxiliary result concerning regularly varying functions $\psi(r)$, $r \geq 0$, that was applied in the proof of Theorem 2.1.

Lemma 3.1. *Let the function $\psi(r)$, $r \geq 0$ be positive, non-decreasing and regularly varying (at infinity) with index $\alpha \geq 0$. Then for an arbitrary $N > 0$ there exist constants $C_N < \infty$, $0 < T_N < \infty$ such that uniformly on $T \geq T_N$*

$$\sup_{0 \leq r \leq N} \frac{\psi(r\sqrt{T})}{\psi(\sqrt{T})} \leq C_N.$$

Proof. It is clear that

$$\sup_{0 \leq r \leq N} \frac{\psi(r\sqrt{T})}{\psi(\sqrt{T})} \leq \frac{\psi(N\sqrt{T})}{\psi(\sqrt{T})}.$$

Since for regularly varying function $\psi(r)$ we have convergence

$$\frac{\psi(N\sqrt{T})}{\psi(\sqrt{T})} \to N^\alpha,$$

as $T \to \infty$, then for $\varepsilon = 1$ there exists a constant $T_N < \infty$ such that for all $T \geq T_N$ the following inequality holds true

$$\frac{\psi(N\sqrt{T})}{\psi(\sqrt{T})} \leq N^\alpha + 1.$$

Hence the statement of Lemma 3.1 is proved for $C_N = N^\alpha + 1$. □

References

1. I. I. Gikhman and A. V. Skorokhod, *Introduction to the Theory of Random Processes*, W. B. Saunders Co., Philadelphia, Pa.–London–Toronto, Ont., 1969.
2. I. I. Gikhman and A. V. Skorokhod, *Stochastic Differential Equations and their Applications*, "Naukova dumka", Kiev, 1982. (Russian)
3. J. Jacod and A. N. Shiryaev, *Limit Theorems for Stochastic Processes*, 2-nd ed., Springer, Berlin, 2003.
4. J. Karamata, *Sur un mode de croissance reguliere des fonctions*, Mathematica (Cluj) **4** (1930), 38–53.
5. E. Kas'kun, *On the asymptotic behaviour of functionals of integral type on diffusion processes*, Visn., Mat. Mekh., Kyiv. Univ. Im. Tarasa Shevchenka **1** (1998), 16–24. (Ukrainian)
6. N. V. Krylov, *Controlled Diffusion Processes*, Springer, 2008.
7. G. L. Kulinich, *On asymptotic behaviour of unstable solutions to a system of stochastic diffusion equations*, Proceedings of the School-Seminar on the Theory of Random Processes, Druskininkai, 25–30 November, 1974, Institute of Physics and Mathematics of LitSSR, Vilnius, 1975, pp. 169–201.
8. G. L. Kulinich, *Limit distributions for functionals of integral type of unstable diffusion processes*, Theor. Prob. and Math. Statist. **11** (1976), 82–86.
9. G. L. Kulinich, *On the asymptotic behavior of the solution of one dimensional stochastic diffusion equation*, Lecture Notes in Control and Information Sciencs, vol. 25, Springer-Verlag 1980, pp. 334–343.
10. G. L. Kulinich, *On necessary and sufficient conditions for convergence of homogeneous additive functionals of diffusion processes*, Proceedings of the second Ukrainian-Hungarian conference: New trends in probability and mathematical statistics (M. Arato and M. Yadrenko, eds.), vol. 2, "TViMS", Kyiv, 1995, pp. 381–390.
11. G. L. Kulinich and E. P. Kas'kun, *On the asymptotic behavior of solutions of a certain class of one-dimensional Ito stochastic differential equations*, Theory Probab. Math. Stat. **56** (1998), 97–105.
12. G. U. Minbaeva, *Limit theorems for integral functionals of a process with instantaneous reflection*, Theory Probab. Math. Stat. **47** (1993), 101–105.
13. A. V. Skorokhod, *Studies in the Theory of Random Processes*, Addison-Wesley Publ. Co., Inc., Reading. Mass., 1965.
14. A. Yu. Veretennikov, *On the strong solutions of stochastic differential equations*, Theory Probab. Appl. **24** (1979), no. 2, 354–366.
15. A. K. Zvonkin, *A transformation of the phase space of a diffusion process that removes the drift*, Mathematics of the USSR, Sbornik **93** (1974), no. 1, 129–149.

Department of General Mathematics, Faculty of Mechanics and Mathematics, Taras Shevchenko National University of Kyiv, 64/13, Volodymyrska Street, Kyiv, Ukraine, 01601
E-mail address: zag_mat@univ.kiev.ua

Department of General Mathematics, Faculty of Mechanics and Mathematics, Taras Shevchenko National University of Kyiv, 64/13, Volodymyrska Street, Kyiv, Ukraine, 01601
E-mail address: bksv@univ.kiev.ua

Department of Probability Theory, Statistics and Actuarial Mathematics, Faculty of Mechanics and Mathematics, Taras Shevchenko National University of Kyiv, 64/13 Volodymyrska Street, Kyiv, Ukraine, 01601
E-mail address: yumishura@gmail.com

Received 12/12/2012