Теорія ймовір. та матем. статист.
Вип. 89, 2013, стор. 130–139

Teor. Ĭmovir. ta Matem. Statyst.
No. 89, 2013, pp. 130–139

PROPERTIES OF INTEGRALS WITH RESPECT TO FRACTIONAL POISSON PROCESS WITH THE COMPACT KERNEL

UDC 519.21

Y. MISHURA AND V. ZUBCHENKO

Abstract. We study the properties of the fractional Poisson process with the Molchan–Golosov kernel. The kernel can be characterized as a compact since it is non-zero on compact interval. The integral of nonrandom function with respect to the centered and non-centered fractional Poisson processes with the Molchan–Golosov kernel is defined. The second moments of these integrals in terms of the norm of the integrand in $L_{1/H}([0,T])$ space are obtained. Moment estimates for the higher moments of these integrals are established via the Bichteler–Jacod inequality.

Анотація. Вивчено властивості дробово-пуассонівських процесів з ядром Молчана–Голосова, яке можна охарактеризувати як компактне, тому що воно ненульове лише на компактному інтервалі. Визначено інтеграли від невипадкових функцій за центрованим та нецентрованим дробово-пуассонівськими процесами з ядром Молчана–Голосова. Оцінено другі моменти цих інтегралів в термінах норми підінтегральної функції в просторі $L_{1/H}([0,T])$ та одержано моментні оцінки за допомогою нерівності Біхтелера–Жакода.

Аннотация. Изучены свойства дробно-пуассоновских процессов с ядром Молчана–Голосова, которое можно охарактеризовать как компактное, поскольку оно ненулевое на компактном интервале. Определены интегралы от неслучайной функции по центрированному и нецентрированному дробно-пуассоновским процессам с ядром Молчана–Голосова. Оценены вторые моменты этих интегралов в терминах нормы подинтегральной функции в пространстве $L_{1/H}([0,T])$ и получены оценки для моментов высшего порядка с помощью неравенства Бихтелера–Жакода.

1. Introduction

Models based on a fractional Brownian motion are an important tool for the study of many theoretical and applied problems. Due to the structure of its covariance function, the fractional Brownian motion that is the process parametrized by its Hurst index, allows to model the dependence on the past history of the process. It is known that for Hurst parameter $H > 1/2$ the fractional Brownian motion has so-called long-range dependence property, for $H \in (0, 1/2)$ it is a process with short memory, and for $H = 1/2$ we have the standard Brownian motion.

At the same time, many natural, technical and economic phenomena are characterized by the instantaneous change in the dynamics of the studied characteristics that cannot be described with the help of the fractional Brownian motion. In particular, such dynamics is typically seen in “jumps” of interest rates, exchange rates, financial indices. Models with jumps can be described with the help of stochastic differential equations that include Poisson measure (see, e.g., [17] and references therein). However, current dynamics of these processes depends essentially on their past history. So construction of models which are able to reflect effectively such features of the process is relevant. Particularly, it is significant for estimation and forecasting of future dynamics of complex financial instruments based on interest rates and financial indices. That’s why we are interested in

2000 *Mathematics Subject Classification.* Primary 60G22; Secondary 60G51.

Key words and phrases. Fractional Poisson process; integral representation of the fractional Poisson process; Mandelbrot – van Ness kernel; Molchan–Golosov kernel; integral with respect to the fractional Poisson process; Bichteler–Jacod inequality.

study of the processes that can combine dependence on the past with an instant change or "jumping" change of the dynamics.

Combining randomness with independence on the past history of the process, the property of long memory and "jumping" change of the dynamics of characteristics under investigation can be expressed mathematically correspondingly by the standard Brownian motion, the fractional Brownian motion and by the Lévy processes.

Mathematical model combining dependence on the past and possibility of instantaneous change of characteristics can be expressed, in particular, by the fractional Poisson process.

There are different approaches to the definition of the fractional Poisson process. In the paper [1] several methods of the fractional Poisson process construction are proposed. One of them is the following: it is assumed that for the fractional Poisson process $N_\nu(t)$, $t > 0$, its distribution $p_k = \mathsf{P}\{N_\nu(t) = k\}$, $k \geq 0$, solves the following equation

$$\frac{d^\nu p_k}{dt^\nu} = -\lambda p_k + \lambda p_{k-1}, k \geq 0,$$

where $p_{-1}(t) = 0$ and $p_k(0) = \mathbb{1}_{\{k=0\}}$ and for $m \in \mathbb{N}$

$$\frac{d^\nu u(t)}{dt^\nu} = \begin{cases} \frac{1}{\Gamma(m-\nu)} \int_0^t \frac{1}{(t-s)^{1+\nu-m}} \frac{d^m}{ds^m} u(s)\, ds, & \text{for } m-1 < \nu < m, \\ \frac{d^m}{dt^m} u(t), \quad s \in [0,T], & \text{for } \nu = m, \end{cases}$$

is the fractional derivative in the sense of Dzherbashyan–Caputo.

Another method is to replace the factorial functions in the distribution of the Poisson process by the Gamma functions. In works [4, 5, 14] the so-called "renewal" approach is used. In contrast to classical characterization of the usual Poisson process as a renewal process, which is constructed as the sum of non-negative independent random variables with exponential distribution, it is assumed that these random variables have Mittag-Leffler distribution. One more approach to the fractional Poisson process construction is the use of so-called "inverse subordinator" method [8].

In order to introduce our approach, we perform certain analogy with a fractional Brownian motion, see, e.g. [10]. Besides the definition of the latter as a Gaussian process with some covariance structure, the fractional Brownian motion can be represented as the integral of a nonrandom kernel with respect to the standard Brownian motion. Examples of kernels used for such representation are the Mandelbrot – van Ness with infinite support and the compactly supported Molchan–Golosov kernel.

Using such representation, it is natural to define the fractional Poisson process as the integral of one of such kernels with respect to the Poisson process (Lévy process). The fractional Lévy processes was first defined using Mandelbrot – van Ness kernel in the work [2], the theory was developed in the paper [7]. The general definition of the fractional Lévy process by using the Molchan–Golosov kernel is given in the work [16].

In this paper we conduct further research of the fractional Poisson processes with the Molchan–Golosov kernel. The integral of a nonrandom function with respect to the centered and non-centered fractional Poisson processes with the Molchan–Golosov kernel is defined. We estimate second moments of such integrals in terms of the norm of the integrand in $L_{1/H}([0,T])$ space. Moment estimates for the higher moments of these integrals via the Bichteler–Jacod inequality are established.

2. Main definitions

The fractional Brownian motion $B^H = \{B_t^H, t \in \mathbb{R}\}$ with Hurst index $H \in (0,1)$ is a Gaussian process with zero mean and the covariance

$$\mathsf{E}\, B_t^H B_s^H = \frac{1}{2}\left(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right).$$

In what follows we consider $H \in (1/2, 1)$. In order to represent a fractional Brownian motion via a Brownian motion we can use both the Mandelbrot – van Ness and the Molchan–Golosov kernel.

The Mandelbrot – van Ness kernel $f_H(t,s)$ is given by

$$f_H(t,s) = c_H \left((t-s)_+^{H-1/2} - (-s)_+^{H-1/2} \right), \qquad s,t \in \mathbb{R},$$

where

$$c_H = \left(\int_0^\infty \left((1+s)^{H-1/2} - s^{H-1/2} \right)^2 ds + \frac{1}{2H} \right)^{-1/2} = \frac{(2H \sin \pi H \Gamma(2H))^{1/2}}{\Gamma(H+1/2)}.$$

The Molchan–Golosov kernel $z_H(t,s)$ is given by

$$z_H(t,s) = \frac{C_H}{\Gamma(H-1/2)} s^{1/2-H} \int_s^t u^{H-1/2} (u-s)^{H-3/2} \, du, \qquad 0 < s < t.$$

In the work [16] it is proved that actually $c_H = C_H$.

The dynamics of $z_H(t,\cdot)$ is equivalent to the dynamics of $\cdot^{1/2-H}$ in the neighborhood of zero and to the dynamics $(t-\cdot)^{H-1/2}$ in the neighborhood of t, see, e.g. [3]. In particular, $z_H(t,\cdot)$ is locally square integrable on $(0,t)$ for every $t \in (0,\infty)$. Also, for $H > 1/2$ the kernel $z_H(t,\cdot)$ is continuous when $s \neq 0$ and has a continuous derivative on $(0,t)$.

The fractional Brownian motion can be represented by integration of the nonrandom kernel with respect to a Brownian motion, in particular:

– by integration over an infinite interval of the Mandelbrot – van Ness kernel:

$$\left(B_t^H \right)_{t \in \mathbb{R}} = \left(\int_{-\infty}^t f_H(t,s) \, dW_s \right)_{t \in \mathbb{R}}$$

– or by integration over a compact interval of the Molchan–Golosov kernel:

$$\left(B_t^H \right)_{t \geq 0} = \left(\int_0^t z_H(t,s) \, dW_s \right)_{t \geq 0}. \tag{1}$$

The right-sided Riemann–Liouville fractional integral operator $I_{T-}^\alpha f$ of order α on $[0,T]$ is defined as

$$(I_{T-}^\alpha f)(s) := \begin{cases} \frac{1}{\Gamma(\alpha)} \int_s^T f(u)(u-s)^{\alpha-1} \, du, & s \in [0,T], \text{ for } \alpha > 0, \\ f(s), & s \in [0,T], \text{ for } \alpha = 0, \end{cases}$$

$$I_{T-}^{-\alpha} f := D_{T-}^\alpha f, \qquad \alpha \in (0,1),$$

where $D_{T-}^\alpha f$ is *the right-sided Riemann–Liouville fractional derivative operator* of order α on $[0,T]$, which is defined as

$$(D_{T-}^\alpha f)(s) := \begin{cases} -\frac{d}{ds}(I_{T-}^{1-\alpha} f)(s), & s \in (0,T), \text{ for } \alpha \in (0,1), \\ -\frac{d}{ds} f(s), & s \in (0,T), \text{ for } \alpha = 1, \\ f(s), & s \in (0,T), \text{ for } \alpha = 0. \end{cases}$$

The right-sided Riemann–Liouville fractional integral operator of order α on $\mathbb{R}$ is defined as

$$(I_-^\alpha f)(s) := \begin{cases} \frac{1}{\Gamma(\alpha)} \int_s^\infty f(u)(u-s)^{\alpha-1} du, & s \in \mathbb{R}, \text{ for } \alpha > 0, \\ f(s), \quad s \in \mathbb{R}, \text{ for } \alpha = 0. \end{cases}$$

$$I_-^{-\alpha} f := D_-^\alpha f, \quad \alpha \in (0,1),$$

where $D^{\alpha}_{-}f$ is *the right-sided Riemann–Liouville fractional derivative operator* of order α on $\mathbb{R}$:

$$(D^{\alpha}_{-}f)(s) := \begin{cases} -\frac{d}{ds}\left(I^{1-\alpha}_{-}f\right)(s) = -\frac{1}{\Gamma(1-\alpha)}\frac{d}{ds}\int_s^{\infty} f(u)(u-s)^{-\alpha}\,du, & \\ & s\in\mathbb{R},\ \text{for } \alpha\in(0,1), \\ -\frac{d}{ds}f(s), & s\in\mathbb{R},\ \text{for } \alpha=1, \\ f(s), & s\in\mathbb{R},\ \text{for } \alpha=0. \end{cases}$$

The centered fractional Poisson process with the Mandelbrot – van Ness kernel. Investigation of a fractional Poisson process with the Mandelbrot – van Ness kernel and the integral with respect to this process is carried out in the work [7]. Below we give an overview of the main results.

Definition 2.1. Two-sided centered Poisson process $(\tilde{\lambda}_t)_{t\in\mathbb{R}}$ is defined as follows: $\tilde{\lambda}_t = \tilde{\lambda}^{(1)}_t$, if $t\ge 0$ and $\tilde{\lambda}_t = -\tilde{\lambda}^{(2)}_{(-t)-} := -\lim_{\varepsilon\to 0+}\tilde{\lambda}^{(2)}_{(-t-\varepsilon)}$, if $t<0$, where $\tilde{\lambda}^{(1)}$ and $\tilde{\lambda}^{(2)}$ are independent and identically distributed centered Poisson processes.

Definition 2.2. Let $(\tilde{\lambda}_t)_{t\in\mathbb{R}}$ be a two-sided Poisson process on $\mathbb{R}$, $f_H(t,s)$ is the Mandelbrot – van Ness kernel. For $H\in(1/2,1)$ a stochastic process

$$X_t = \int_{-\infty}^{t} f_H(t,s)\,d\tilde{\lambda}_s,$$

is called a fractional Poisson process with the Mandelbrot – van Ness kernel. This integral exists in L^2-sense (as the limit in L^2 of integrals of a sequence of approximating $f_H(t,s)$ step functions; the limit does not depend on the choice of the sequence of approximating functions).

The fractional Poisson process X_t can be represented as follows:

$$X_t = \int_{\mathbb{R}} \left(I^{H-1/2}_{-}\mathbb{1}_{(0,t)}\right)(s)\,d\tilde{\lambda}_s,$$

where I_{-} is the right-sided Riemann-Liouville fractional integral operator on $\mathbb{R}$.

Define the space $\mathcal{H}$ as the completion of $L^1(\mathbb{R})\cap L^2(\mathbb{R})$ with respect to the norm

$$\|g\|_{\mathcal{H}} := \left(\lambda\int_{\mathbb{R}}\left(I^{H-1/2}_{-}f\right)^2(s)\,ds\right)^{1/2}.$$

It is known from [7] that for the functions $f\in L^1(\mathbb{R})\cap L^2(\mathbb{R})$

$$\int_{\mathbb{R}}\left(I^{H-1/2}_{-}f\right)^2(s)\,ds<\infty.$$

Let $\phi\colon\mathbb{R}\to\mathbb{R}$ be a simple function:

$$\phi(s) = \sum_{i=1}^{n-1} a_i\mathbb{1}_{[s_i,s_{i+1})}(s),$$

where $a_i\in\mathbb{R}$, $i=1,\dots,n$ and $-\infty<s_1<s_2<\dots<s_n<\infty$. Notice that simple functions belong to the space $\mathcal{H}$.

The integral with respect to the fractional Poisson process with the Mandelbrot – van Ness kernel is defined for simple functions at first. Let ϕ be a simple function. Then

$$\int_{\mathbb{R}}\phi(s)\,dX_s = \int_{\mathbb{R}}\left(I^{H-1/2}_{-}\phi\right)d\tilde{\lambda}_s.$$

Also from [7] we have the following L^2-isometry:

$$\mathsf{E}\left(\int_{\mathbb{R}}\phi(s)\,dX_s\right)^2 = \mathsf{E}\left(\int_{\mathbb{R}}\left(I^{H-1/2}_{-}\phi\right)d\tilde{\lambda}_s\right)^2 = \lambda\int_{\mathbb{R}}\left(I^{H-1/2}_{-}\phi\right)^2(s)\,ds = \|\phi\|^2_{\mathcal{H}}.$$

We can extend the definition of the integral with respect to the fractional Poisson process for the class of functions $f \in \mathcal{H}$. Namely,

$$\int_{\mathbb{R}} f(s)\,dX_s = \int_{\mathbb{R}} \left(I_-^{H-1/2} f\right)(s)\,d\tilde{\lambda}_s$$

with equality in L^2-sense.

The noncentered fractional Poisson process Y_t with the Molchan–Golosov kernel is defined as follows:

$$Y_t = \int_0^t z_H(t,s)\,d\lambda_s,$$

where λ_s is the simple Poisson process with intensity λ, $z_H(t,s)$ is the Molchan–Golosov kernel, and the integral exists in the pathwise sense due to step structure of the Poisson process and smooth properties of $z_H(t,s)$, mentioned above.

The centered fractional Poisson process $\tilde{Y}_t$ with the Molchan–Golosov kernel is defined as follows:

$$\tilde{Y}_t = \int_0^t z_H(t,s)\,d\tilde{\lambda}_s,$$

where $\tilde{\lambda}_s = \lambda_s - \lambda s$ is the centered Poisson process. $\tilde{Y}_t$ is defined as the integral with respect to the square integrable martingale. So the centered fractional Poisson process exists as the integral in L^2 sense.

Later on we shall consider both integrals with respect to the non-centered and centered fractional Poisson process.

3. Distribution characteristics of the fractional Poisson process with the Molchan–Golosov kernel

Using well-known formulas for the integrals with respect to the Poisson process, we obtain the following first and second noncentral moments for the noncentered fractional Poisson process with the Molchan–Golosov kernel:

$$\begin{aligned} m_1 &= \lambda \int_0^t z_H(t,s)\,ds = \lambda C_H \int_0^t u^{H-1/2} \int_0^u s^{1/2-H}(u-s)^{H-3/2}\,ds\,du \\ &= \lambda C_H \frac{\pi}{\sin(\pi(3/2-H))} \int_0^t u^{H-1/2}\,du = \lambda C_H \frac{\pi}{\sin(\pi(3/2-H))} \frac{t^{H+1/2}}{H+1/2}, \end{aligned} \tag{2}$$

$$\begin{aligned} m_2 &= \lambda^2 \left(\int_0^t z_H(t,s)\,ds\right)^2 + \lambda \int_0^t z_H^2(t,s)\,ds \\ &= \lambda^2 \left(C_H \frac{\pi}{\sin(\pi(3/2-H))} \frac{1}{H+1/2}\right)^2 t^{2H+1} + \lambda t^{2H}. \end{aligned}$$

Here we have used the equality

$$\int_0^t z_H^2(t,s)\,ds = t^{2H}$$

that follows from the representation (1) of the fractional Brownian motion and the form of its covariance function.

We know that the fractional Brownian motion has stationary increments. Now we investigate whether the property of stationarity of increments holds for the fractional Poisson process with the Molchan–Golosov kernel.

Lemma 3.1. *Both for centered and noncentered fractional Poisson process with the Molchan–Golosov kernel the property of stationarity of increments in general does not hold.*

Proof. Consider the noncentered process, for the centered one the proof is similar. We investigate whether the characteristic function of the fractional Poisson process

$$\mathsf{E}\exp\{iuY_t\}, \qquad u \in \mathbb{R},\ 0 < t < \infty,$$

and that of its increment

$$\mathsf{E}\exp\{iu(Y_{t+t_1} - Y_{t_1})\}, \qquad u \in \mathbb{R},\ 0 < t, t_1 < \infty,$$

are equal.

We use propositions 2.4, 2.6 [13] and results by [7]. Thus if some process Z allows the representation $Z_t = \int_{\mathbb{R}} f(t,s)\,dL_s$, where L is Lévy process with characteristic triplet $(0,0,\nu)$ without Gaussian component, such that $\mathsf{E}\,L_1 = 0$, $\mathsf{E}\,L_1^2 < \infty$, then

$$\mathsf{E}(\exp(iuZ_t)) = \exp\left(\int_{\mathbb{R}}\int_{\mathbb{R}} \left(e^{if(t,s)ux} - 1 - if(t,s)ux\right)\nu(dx)\,ds\right).$$

Therefore for fractional Poisson process Y_t with the Molchan–Golosov kernel we obtain the following characteristic function:

$$\mathsf{E}\exp[iuY_t] = \exp\left\{\int_{\mathbb{R}} \lambda\left(\exp\{iuz_H(t,s)\mathbb{1}_{[0,t]}(s)\} - 1\right)\,ds\right\}. \tag{3}$$

Further,

$$\begin{aligned} Y_{t+t_1} - Y_{t_1} &= \int_0^{t+t_1} z_H(t+t_1,s)\,d\lambda_s - \int_0^{t_1} z_H(t_1,s)\,d\lambda_s \\ &= \int_0^{t+t_1} \left(z_H(t+t_1,s) - z_H(t_1,s)\right)\,d\lambda_s, \end{aligned}$$

where in the last equality we use that according to definition we have $z_H(t,s) = 0$ if condition $0 < s < t$ does not hold. So

$$\begin{aligned} &\mathsf{E}\exp\{iu(Y_{t+t_1} - Y_{t_1})\} \\ &\quad = \exp\left\{\int_{\mathbb{R}} \lambda\big(\exp\{iu(z_H(t+t_1,s) - z_H(t_1,s))\mathbb{1}_{[0,t+t_1]}(s)\} - 1\big)\,ds\right\}. \end{aligned} \tag{4}$$

We compare (3) and (4). It is sufficient to compare

$$z_H(t,s)\cdot \mathbb{1}_{[0,t]}(s) = c_H s^{1/2-H}\int_s^t u^{H-1/2}(u-s)^{H-3/2}\,du \cdot \mathbb{1}_{[0,t]}(s), \tag{5}$$

and

$$\begin{aligned} &(z_H(t+t_1,s) - z_H(t_1,s))\cdot \mathbb{1}_{[0,t+t_1]}(s) \\ &\quad = c_H s^{1/2-H}\int_{t_1}^{t+t_1} u^{H-1/2}(u-s)^{H-3/2}\,du \cdot \mathbb{1}_{[0,t+t_1]}(s). \end{aligned} \tag{6}$$

As (5) and (6) are not equal, for the noncentered fractional Poisson process with the Molchan–Golosov kernel the property of stationarity of increments in general does not hold. □

4. Integral with respect to the fractional Poisson process with the Molchan–Golosov kernel and estimate of its second moment in terms of the norm of the integrand in $L_{1/H}([0,T])$ space

Consider the noncentered fractional Poisson process Y_t with the Molchan–Golosov kernel. Let a function f be defined on $[0,T]$, $H \in (\frac{1}{2},1)$. Define the following operator:

$$\left(K_T^H f\right)(s) = C_H s^{1/2-H}\left(I_{T-}^{H-1/2}(\cdot)^{H-1/2} f\right)(s), \qquad s \in (0,T),$$

where $I_{T-}^{H-1/2}$ is the right-sided Riemann–Liouville fractional operator defined in Section 2.

Introduce the spaces

$$L^2_{H,Pois}([0,T]) = \{f\colon [0,T] \to \mathbb{R} \mid K^H_T f \in L^2([0,T])\}$$

with the norm

$$\|f\|_{L^2_{H,Pois}([0,T])} = \left\|K^H_T f\right\|_{L^2([0,T])},$$

and

$$\tilde{L}^2_{H,Pois}([0,T]) = \left\{ f \in L^2_{H,Pois}([0,T]) \text{ and } (\cdot)^{H-1/2} f(\cdot) \in L_p([0,T]) \text{ for some } p > \frac{1}{H-1/2} \right\}$$

with the same norm.

Define for $f \in L^2_{H,Pois}([0,T])$ the integral with respect to the fractional Poisson processes in the following way:

$$\int_0^T f(s)\, dY_s = \int_0^T \left(K^H_T f\right)(s)\, d\lambda_s \tag{7}$$

and

$$\int_0^T f(s)\, d\tilde{Y}_s = \int_0^T \left(K^H_T f\right)(s)\, d\tilde{\lambda}_s. \tag{8}$$

Thus, we have the analogy with the construction of the integral with respect to the fractional Brownian motion. Note that from (2)

$$\tilde{Y}_t = \int_0^t z_H(t,s)\, d\tilde{\lambda}_s = Y_t - \lambda \int_0^t z_H(t,s)\, ds = Y_t - EY_t,$$

and

$$\int_0^T f(s)\, d\tilde{Y}_s := \int_0^T f(s)\, dY_s - \int_0^T f(s)\, d(EY_s),$$

where both integrals exist in L^2-sense.

Lemma 4.1. 1. *For $f \in L^2_{H,Pois}([0,T])$ both integrals* (7) *and* (8) *exist in L^2 sense.*
2. *For $f \in \tilde{L}^2_{H,Pois}([0,T])$ integral* (7) *exists in the pathwise sense.*

Proof. 1. Consider the noncentered case, the centered one is considered similarly. It holds [11] that $(K^H_T \mathbb{1}_{[0,t)})(s) = z_H(t,s)$. Using properties of integrals with respect to the Poisson process for step functions we have:

$$\begin{aligned} \mathsf{E}\left(\int_0^T \left(K^H_T f\right)(s)\, d\lambda_s\right)^2 &= \lambda^2 \left(\int_0^T (K^H_T f)(s)\, ds\right)^2 + \lambda \int_0^T \left(K^H_T f\right)^2(s)\, ds \\ &\le (\lambda^2 T + \lambda)\|f\|^2_{L^2_{H,Pois}([0,T])}, \end{aligned} \tag{9}$$

$$\mathsf{E}\left(\int_0^T (K^H_T f)(s) d\tilde{\lambda}_s\right)^2 = \lambda \int_0^T \left(K^H_T f\right)^2(s)\, ds = \lambda \|f\|^2_{L^2_{H,Pois}([0,T])},$$

where λ is the intensity of the Poisson process. Note that according to [12] step functions are dense in $L^2_{H,Pois}([0,T])$. Therefore, we can approximate the function $f \in L^2_{H,Pois}([0,T])$ by step functions f_n in $L^2_{H,Pois}([0,T])$ and to define the integral of the function f with respect to the fractional Poisson process using as follows:

$$\int_0^T f(s)\, dY_s = \lim_{n\to\infty} \int_0^T f_n(s)\, dY_s \quad \text{— convergence in } L^2(\mathsf{P}).$$

2. Consider the integrand of the right side of equality (7):

$$\left(K^H_T f\right)(s) = C_H s^{1/2-H} \left(I^{H-1/2}_{T-} \cdot^{H-1/2} f\right)(s).$$

Functions belonging to the space $\tilde{L}^2_{H,Pois}([0,T])$ satisfies the conditions of the Theorem 3.6 [15]. According to it the function $\left(I^{H-1/2}_{T-} \cdot^{H-1/2} f\right)(s)$ is Hölder of order $H-\frac{1}{2}-\frac{1}{p}$ on $(0,T)$.

In the right side of equality (7) the integration is with respect to the Poisson process, which is a process of bounded variation on $[0,T]$. Also, according to the properties of the Poisson process a.s. there exists such $\varepsilon(\omega)>0$, that $\lambda_s = 0$ for all $s \in [0,\varepsilon(\omega)]$. Thus, on $(\varepsilon(\omega),T)$ a.s. we have the continuous function, that can be integrated with respect to the process of bounded variation. Therefore, integral (7) exists in the pathwise sense. $\square$

Remark 4.1. To estimate the second moment of the integral with respect to the fractional Poisson process with the Molchan–Golosov kernel we need to estimate $\int_0^T (K^H_T f)^2(s)\,ds$. It can be done similarly to the fractional Brownian motion case [9] with the help of (9). Denote $\alpha = H - \frac{1}{2}$. Then

$$\begin{aligned}
\mathsf{E}\left(\int_0^T f(s)\,dY_s\right)^2 &= \mathsf{E}\left(\int_0^T \left(K^H_T f\right)(s)\,d\lambda_s\right)^2 \le \left(\lambda^2 T + \lambda\right)\int_0^T \left(K^H_T f\right)^2(s)\,ds \\
&= C\int_0^T s^{-2\alpha}\left(\int_s^T f(u)u^{\alpha}(u-s)^{\alpha-1}\,du\right)^2 ds \\
&\le CB(1-2\alpha,\alpha)\int_0^T\int_0^T f(u)f(v)|u-v|^{2\alpha-1}\,du\,dv \\
&\le C\|f\|^2_{L_{1/H}([0,T])}.
\end{aligned}$$

5. Estimate of higher moments of integral with respect to the fractional Poisson process with the Molchan–Golosov kernel

Let $f \in L^2_{H,Pois}([0,T])$. Recall that

$$\int_0^T f(s)\,dY_s = \int_0^T \left(K^H_T f\right)(s)\,d\lambda_s = \int_0^T \left(K^H_T f\right)(s)\,d\tilde{\lambda}_s + \int_0^T \left(K^H_T f\right)(s)\lambda\,ds, \qquad (10)$$

and the first integral in the right-hand side of (10) exists as the integral with respect to the square-integrable martingale $\tilde{\lambda}_s = \lambda_s - \lambda s$.

Now we are in the position to establish moment inequalities for integral with respect to the noncentered fractional Poisson process with the Molchan–Golosov kernel. For the centered process the similar bounds hold with obvious modification.

Theorem 5.1. *Let $f \in \tilde{L}^2_{H,Pois}([0,T])$, $H \in (\frac{1}{2},1)$. Then for any k such that $0<k<\frac{1}{2H-1}$ there exists the constant $C_k = C(H,k)$, such that for any $T>0$*

$$\mathsf{E}\left|\int_0^T f(s)\,dY_s\right|^{2k} \le C_k\left\|K^H_T f\right\|^{2k}_{L^{2k}_{[0,T]}} + C_k\lambda^{2k}\left(\int_0^T u^{H-1/2}|f(u)|\,du\right)^{2k}.$$

Proof. We consider moments of the order $2k$:

$$\begin{aligned}
\mathsf{E}\left|\int_0^T f(s)\,dY_s\right|^{2k} &= \mathsf{E}\left(\left|\int_0^T \left(K^H_T f\right)(s)\,d\tilde{\lambda}_s + \int_0^T \left(K^H_T f\right)(s)\lambda\,ds\right|\right)^{2k} \\
&\le 2^{2k}\,\mathsf{E}\left(\left|\int_0^T \left(K^H_T f\right)(s)\,d\tilde{\lambda}_s\right|\right)^{2k} + 2^{2k}\left(\int_0^T |(K^H_T f)(s)|\lambda\,ds\right)^{2k} \\
&:= I_1 + I_2.
\end{aligned}$$

To bound the first integral, we use the Bichteler–Jacod inequality (see, e.g., [6]):

$$I_1 \le C\int_0^T \left(\left(K_T^H f\right)^{2k}\lambda + \left(\left(K_T^H f\right)^2(s)\lambda\right)^k\right) ds \le C_k \int_0^T \left(K_T^H f\right)^{2k}(s)\, ds$$
$$= C_k \|K_T^H f\|^{2k}_{L^{2k}_{[0,T]}}.$$

Establish whether the last integral exists:

$$\tilde{I}_1 := \int_0^T \left(K_T^H f\right)^{2k}(s)\, ds = C_H^{2k}\int_0^T s^{(1/2-H)2k}\left(I_{T-}^{H-1/2}\cdot^{H-1/2} f\right)^{2k}(s)\, ds.$$

Remind that according to the definition of the space $\tilde{L}^2_{H,Pois}([0,T])$ there exists some $p > \frac{1}{H-1/2}$: $(\cdot)^{H-1/2}f(\cdot) \in L_p([0,T]$. So the same way as in the proof of the Lemma 4.1 we can establish that the function $\left(I_{T-}^{H-1/2}\cdot^{H-1/2} f\right)(s)$ is Hölder of order $H-\frac{1}{2}-\frac{1}{p}$ on $(0,T)$. Thus $\tilde{I}_1$ is finite if and only if

$$\int_0^T s^{(1/2-H)2k}\, ds < \infty,$$

and due to the condition $k < \frac{1}{2H-1}$ the integral $\tilde{I}_1$ is finite.

For estimation of I_2 we use the equality

$$\begin{aligned}\int_0^T \left(K_T^H f\right)(s)\, ds &= C_H\int_0^T s^{1/2-H}\int_s^T u^{H-1/2}f(u)(u-s)^{H-3/2}\, du\, ds\\ &= C_H\int_0^T u^{H-1/2}f(u)\int_0^u s^{1/2-H}(u-s)^{H-3/2}\, ds\, du\\ &= C_H\frac{\pi}{\sin(\pi(3/2-H))}\int_0^T u^{H-1/2}f(u)\, du.\end{aligned}$$

Therefore

$$\int_0^T \left|\left(K_T^H f\right)(s)\right|\, ds \le C_H\int_0^T u^{H-1/2}|f(u)|\, du,$$

and

$$I_2 \le C_k\lambda^{2k}\left(\int_0^T u^{H-1/2}|f(u)|\, du\right)^{2k}. \qquad \square$$

References

1. L. Beghin and E. Orsingher, *Fractional Poisson process and related planar random motions*, Bernoulli **12** (2006), no. 6, 1099–1126.
2. A. Benassi, S. Cohen, and J. Istas, *Identification and properties of real harmonizable fractional Lévy motions*, Bernoulli **8** (2002), 97–115.
3. C. Jost, *Transformation formulas for fractional Brownian motion*, Stochastic Processes and their Applications **116** (2006), 1341–1357.
4. F. Mainardi, R. Gorenflo, and E. Scalas, *A fractional generalization of the Poisson processes*, Vietnam J. Math. **32** (2004), 53–64.
5. F. Mainardi, R. Gorenflo, and A. Vivoli, *Renewal processes of Mittag–Leffler and Wright type*, Fract. Calc. Appl. Anal. **8** (2005), no. 1, 7–38.
6. C. Marinelli, C. Prévôt, and M. Röckner, *Regular dependence on initial data for stochastic evolution equations with multiplicative Poisson noise*, Journal of Functional Analysis **258** (2010), no. 2, 616–649.
7. T. Marquardt, *Fractional Lévy process with an application to long memory moving average processes*, Bernoulli **12** (2006), no. 6, 1099–1126.
8. M. M. Meerschaert, E. Nane, and P. Vellaisamy, *The fractional Poisson process and the inverse stable subordinator*, Electron. J. Probab. **16** (2011), no. 59, 1600–1620.
9. J. Memin, Yu. Mishura, and E. Valkeila, *Inequalities for the moments of Wiener integrals with respect to fractional Brownian motions*, Stat. Prob. Lett. **51** (2001), 197–206.

10. Yu. Mishura, *Stochastic calculus for fractional Brownian motion and related processes*, Springer, Berlin, 2008.
11. G. Molchan and J. Golosov, *Gaussian stationary processes with asymptotic power spectrum*, Soviet Mathematics Doklady **10** (1969), no. 1, 134–137.
12. V. Pipiras and M. Taqqu, *Are classes of deterministic integrands for fractional Brownian motion on an interval complete?*, Bernoulli **7** (2001), no. 6, 873–897.
13. B. S. Rajput and J. Rosinski, *Spectral representaion of infinitely divisible processes*, Probab. Th. Rel. Fields **82** (1989), 451–487.
14. O. N. Repin and A. I. Saichev, *Fractional Poisson law*, Radiophysics and Quantum Electronics **43** (2000), no. 9, 738–741.
15. S. G. Samko, A. A. Kilbas, and O. I. Marichev, *Fractional integrals and derivatives. Theory and applications*, Gordon and Breach Science Publishers, 1993.
16. H. Tikanmaki and Yu. Mishura, *Fractional Lévy process as a result of compact interval intergal transformation*, Stochastic Analysis and Applications **29** (2011), 1081–1101.
17. V. P. Zubchenko, *Properties of solutions of stochastic differential equations with random coefficients, non-Lipschitzian diffusion, and Poisson measures*, Theor. Probability and Math. Statist. **82** (2011), 11–26.

Department of probability theory, statistics and actuarial mathematics, Mechanics and mathematics faculty, Kyiv National Taras Shevchenko University, Volodymyrska st., 64, Kyiv, 01601, Ukraine
E-mail address: myus@univ.kiev.ua

Department of probability theory, statistics and actuarial mathematics, Mechanics and mathematics faculty, Kyiv National Taras Shevchenko University, Volodymyrska st., 64, Kyiv, 01601, Ukraine
E-mail address: v_zubchenko@ukr.net

Received 26/02/2013