SEMI-MARKOV APPROACH TO THE PROBLEM OF DELAYED REFLECTION OF DIFFUSION MARKOV PROCESSES

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Abstract. An one-dimensional diffusion process with positive values, reflecting from zero, is considered. All the variants of reflecting with preservation of the semi-Markov property are described. This property is characterized by a family of Laplace images of times from the first hitting of zero up to the first hitting of a level r for any $r > 0$. The parameter $C(\lambda)$ of this family is used for construction of a time change, transforming a process with instantaneous reflection to the process with delayed reflection.

Анотація. Розглядається одновимірний дифузійний процес з додатними значеннями, що відбивається від точки нуль. Описано всі варіанти відбиття із збереженням напівмарковської властивості процесу, що характеризується сімейством перетворень Лапласа моментів від першого досягнення нуля до першого досягнення заданого рівня r для всіх $r > 0$. Параметр-функція $C(\lambda)$ цього сімейства використовується для побудови заміни часу, що перетворює процес із миттєвим відбиттям у процес із уповільненим відбиттям.

Аннотация. Рассматривается одномерный диффузионный процесс с положительными значениями, отражающийся от точки 0. Описываются все варианты отражения с сохранением полумарковского свойства процесса, которое характеризуется семейством преобразований Лапласа времен от первого достижения нуля до первого достижения заданного уровня r для всех $r > 0$. Параметр-функция $C(\lambda)$ этого семейства используется для вывода характеристик замены времени, превращающей процесс с мгновенным отражением в процесс с замедленным отражением.

1. Introduction

Apparently Gihman and Skorokhod were the first who investigated reflection with delaying of one-dimensional Markov diffusion processes ([1, p. 197]). They applied a method of stochastic integral equations which takes into account preserving the Markov property while reflecting. However there exist examples of interaction between a process and a boundary of its range of values, which can be interpreted like reflection, when the Markov property is being lost, although the property of continuous semi-Markov processes is preserved. Here is a simple example.

Let $w(t)$, $t \geq 0$, be Wiener process. Let us consider on the segment $[a, b]$, $a < w(0) < b$, the truncated process

$$\overline{w}(t) = \begin{cases} b, & w(t) \geq b \\ w(t), & a < w(t) < b \\ a, & w(t) \leq a \end{cases}$$

for all $t \geq 0$. It is clear that this process is not Markov. However it remains to be continuous semi-Markov [4]: the Markov property is fulfilled with respect to the first exit time from any open interval inside the segment, and also that from any one-sided neighborhood of any end of the segment.

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The semi-Markov approach to the problem of reflection consists in solution of the following task: to determine a semi-Markov transition function for the process at a boundary point for the process preserving its diffusion form inside its open range of values, i. e. that up to the first exit time from the region and any time when it leaves the boundary. A more specific task to find reflection, preserving a global Markov property, is reduced to a problem to find a subclass of Markov reflected processes in the class of all the semi-Markov ones. Tasks of such a kind are important for applications where one takes into account interaction of diffusion particles with a boundary of a container, leading to a dynamic equilibrium of the system (see, e. g. [7]).

In paper [3] all class of semi-Markov characteristics of reflection for a given locally Markov diffusion process is described. In paper [5] conditions for a semi-Markov characteristic to give a globally Markov process are found. In the present paper we continue to investigate processes with semi-Markov reflection. The aim of investigation is to find formulae, characterizing a time change, transforming a process with instantaneous reflection into the process with delaying reflection,

In paper [6] while analyzing a two-dimensional diffusion process in a neighborhood of a flat screen a time change in a tangential component of the process with respect to a normal component time run is factually treated. This splitting of the process on two components makes the situation easier to be understood, but at the same time it masks the true mechanism of transformation. In fact the time change could be learned on the initial stage of semi-Markov approach to the problem of reflection. In the present paper this shortcoming of our first paper on this theme is removed.

2. Semi-Markov transition function on a boundary

We will consider a diffusion process $X(t)$ on the half-line $t \ge 0$ with one boundary at zero. We assume that the process does not go to infinity and from any positive initial point it hits zero with probability one. For example, it could be a diffusion Markov process with a negative drift and bounded local variance. We had substantiated above why it is expedient to consider semi-Markov reflection. Semi-Markov approach permits to consider from unit point of view an operation of instantaneous reflection as well as an operation of truncation.

In frames of semi-Markov models of reflection it is natural to assume that $X(t)$ is a semi-Markov process of diffusion type. Let (P_x), $x \ge 0$, be a consistent family of measures of the process, depending on initial points of trajectories. On interval $(0, \infty)$ semi-Markov transition generating functions of the process

$$g_{(a,b)}(\lambda, x) := \mathbb{E}_x \left(e^{-\lambda \sigma_{(a,b)}}; X(\sigma_{(a,b)}) = a \right);$$
$$h_{(a,b)}(\lambda, x) := \mathbb{E}_x \left(e^{-\lambda \sigma_{(a,b)}}; X(\sigma_{(a,b)}) = b \right),$$

$a < x < b$, satisfy the differential equation

$$\frac{1}{2} f'' + A(x) f' - B(\lambda, x) f = 0,$$

with boundary conditions

$$g_{(a,b)}(\lambda, a+) = h_{(a,b)}(\lambda, b-) = 1, \qquad g_{(a,b)}(\lambda, b-) = h_{(a,b)}(\lambda, a+) = 0.$$

The coefficients of the equation are assumed to be piece-wise continuous functions of $x > 0$, and for any x function $B(\lambda, x)$ is non-negative and has completely monotone partial derivative with respect to λ. First of all reflection of the process from point $x = 0$ means addition of this point to the range of values of the process. Further all the semi-closed intervals $[0, r)$ are considered what the process can only exit from open boundary. Corresponding semi-Markov transition generating functions are denoted as $h_{[0,r)}(\lambda, x)$ with main distinction from exit from an open set $h_{[0,r)}(\lambda, 0) > 0$. Function

$K(\lambda,r) := h_{[0,r)}(\lambda,0)$ plays an important role for description of properties of reflected processes. Using semi-Markov properties of the process, we obtain

$$h_{[0,r)}(\lambda,x) = h_{(0,r)}(\lambda,x) + g_{(0,r)}(\lambda,x)K(\lambda,r),$$

and also

$$K(\lambda,r) = K(\lambda,r-\varepsilon)(h_{(0,r)}(\lambda,r-\varepsilon) + g_{(0,r)}(\lambda,r-\varepsilon)K(\lambda,r)).$$

Assuming that there exist derivatives with respect to the second argument we have

$$\begin{aligned} g_{(a,b)}(\lambda,x) &= 1 + g'_{(a,b)}(\lambda,a+)(x-a) + o(x-a),\\ g_{(a,b)}(\lambda,x) &= -g'_{(a,b)}(\lambda,b-)(b-x) + o(b-x),\\ h_{(a,b)}(\lambda,x) &= h'_{(a,b)}(\lambda,a+)(x-a) + o(x-a),\\ h_{(a,b)}(\lambda,x) &= 1 - h'_{(a,b)}(\lambda,b-)(b-x) + o(b-x), \end{aligned}$$

and obtain the differential equation

$$K'(\lambda,r) + K(\lambda,r)h'_{(0,r)}(\lambda,r-) + K^2(\lambda,r)g'_{(0,r)}(\lambda,r-) = 0.$$

Its general solution is

$$K(\lambda,r) = \frac{h'_{(0,r)}(\lambda,0+)}{C(\lambda) - g'_{(0,r)}(\lambda,0+)},$$

where arbitrary constant $C(\lambda)$ can depend on λ. In order for $K(\lambda,r)$ to be a Laplace transform it is sufficient that function $C(\lambda)$ to be non-decreasing, $C(0) = 0$, and its derivative to be a completely monotone function [5]. Under our assumptions it is fair

$$K(\lambda,r) = 1 - C(\lambda)r + o(r), \qquad r \to 0.$$

Our next task is to learn a time change in the process with instantaneous reflection which derives the process with delayed reflection.

3. Time change with respect to time run under instantaneous reflection

Let us denote θ_t the shift operator on the set of trajectories; σ_Δ the operator of the first exit time from set Δ. For any Markov times τ_1, τ_2 (with respect to the natural filtration) on set $\{\tau_1 < \infty\}$ let us determine the following operation

$$\tau_1 \dot{+} \tau_2 := \tau_1 + \tau_2 \circ \theta_{\tau_1}.$$

It is known [4], that for any open (in relative topology) sets Δ_1, Δ_2, if $\Delta_1 \subset \Delta_2$, then

$$\sigma_{\Delta_2} = \sigma_{\Delta_1} \dot{+} \sigma_{\Delta_2}.$$

In this case $\sigma_\Delta(\xi) = 0$, if $\xi(0) \notin \Delta$.

Let us introduce special denotations for some first exit times and their combinations, and that for random intervals as $\varepsilon > 0$

$$\alpha := \sigma_{[0,\varepsilon)}, \qquad \beta := \sigma_{(0,\infty)}, \qquad \gamma(0) := \beta,$$
$$\gamma := \alpha \dot{+} \beta, \qquad \gamma(n) := \gamma(n-1) \dot{+} \gamma, \qquad n \ge 1,$$
$$b(0) := [0,\beta), \qquad a(n) := \big[\gamma(n-1), \gamma(n-1)\dot{+}\alpha\big), \qquad b(n) = \big[\gamma(n-1)\dot{+}\alpha, \gamma_n\big).$$

The random times α, $\gamma(n)$, and intervals $a(n)$, $b(n)$, $n = 1, 2, \ldots$, depend on ε. In some cases we will denote this dependence by the lower index.

Let us remark that sequence $(\gamma(n))$ forms moments of jumps of a renewal process. Besides if $X(t) > 0$ then for any $t > 0$ there exist $\varepsilon > 0$, and $n \ge 1$ such that $t \in b_\varepsilon(n)$. It implies that for $\varepsilon \to 0$ random set $\bigcup_{k=1}^\infty b_\varepsilon(k)$ covers all the set of positive values of process X with probability one. On share of supplementary set (a limit of set $\bigcup_{k=1}^\infty a_\varepsilon(k)$) there remain possible intervals of constancy and also a discontinuum of points (closed set, equivalent to continuum, without any intervals, [2, p. 158]), consisted of zeros of process X. The linear measure of it can be more than or equal to 0. This measure is included as a component in a measure of delaying while reflecting.

It is known ([4, p. 111]) that continuous homogeneous semi-Markov process is a Markov process if and only if it does not contain intrinsic intervals of constancy (it can have an interval of terminal stopping). This does not imply that a process with delayed deflection cannot be globally Markov. Its delaying is exceptionally at the expense of the discontinuum. A process without intervals of constancy at zero, and with the linear measure of the discontinuum of zeros which equals to zero is said to be a process with instantaneous reflection.

We will construct a non-decreasing sequence of continuous non-decreasing functions $V_\varepsilon(t)$, $t \ge 0$, converging to some limit $V(t)$ as $\varepsilon \to 0$ uniformly on every bounded interval.

Let $X(0) > 0$, and $V_\varepsilon(t) = t$ on interval $b(0)$, and $V_\varepsilon(t) = \beta$ on interval $a(1)$. On interval $b(1)$ the process V_ε increases linearly with a coefficient 1. On interval $a(2)$ function V_ε is constant. Then it increases with coefficient 1 on interval $b(2)$, and so on, being constancy on intervals $a(k)$, increasing with coefficient 1 on intervals $b(k)$. Noting that if $\varepsilon_1 > \varepsilon_2$, for any iterval $a_{\varepsilon_2}(k)$ there exists n such that $a_{\varepsilon_2}(k) \subset a_{\varepsilon_1}(n)$, we convince ourself that the sequence of constructed functions does not decrease, bounded and consequently tends to a limit.

Let us define a process with instantaneous reflecting obtained from the original process X as a process, obtained after elimination of all its intervals of constancy at zero, and contraction of a linear measure of its discontinuum of zeros to zero. This process can be represented as a limit (in Skorokhod metric) of a sequence of processes $X_\varepsilon(t)$, determined for all t by formula

$$X_\varepsilon(t) = X\left(V_\varepsilon^{-1}(t)\right),$$

where $V_\varepsilon^{-1}(y)$ is defined as the first hitting time of the process $V_\varepsilon(t)$ to a level y. Hence $X_\varepsilon(t)$ has jumps of value ε at the first hitting time to zero and its iterations. Let us denote the process with instantaneous reflecting as $X_0(t)$, and the map $X \mapsto X_0$ as ϕ_V. Such a process is measurable (with respect to the original sigma-algebra of subsets) and continuous. Let $P_x^0 = P_x \circ \phi_V^{-1}$ be the induced measure of this process.

Then it is clear that V is an inverse time change transforming the process X_0 into the process X, i.e. $X = X_0 \circ V$. In this case for any open interval $\Delta = (a, b)$, $0 < a < b$, or $\Delta = [0, r)$, $r > 0$, it is fair

$$\sigma_\Delta(X_0 \circ V) = V^{-1}(\sigma_\Delta(X_0)).$$

The function V^{-1} we call a direct time change, which corresponds to every "intrinsic" Markov time of the original process (in given case $X_0(t)$) the analogous time of the transformed process.

Remark, that for $\varepsilon_1 > \varepsilon_2$ the set $\{\gamma_{\varepsilon_1}(n), n = 0, 1, 2, \dots\}$ is a subset of the set $\{\gamma_{\varepsilon_2}(n), n = 0, 1, 2, \dots\}$. That is why every Markov time $\gamma_\varepsilon(n)$ is a Markov regeneration time of the process V, what permits in principle to calculate finite-dimensional distributions of this process. On the other hand this process is synonymously characterized by its inverse, i.e. the process

$$V^{-1}(y) := \inf\{t \ge 0\colon V(t) \ge y\}, \qquad y > 0.$$

This process is more convenient to deal with because Laplace transform of its value at a point y can be found as a limit of a sequence of easy calculable Laplace images of values $V_\varepsilon^{-1}(y)$.

Theorem 1. *A direct time change $V^{-1}(y)$, mapping a process with instantaneous reflection into a process with delayed reflection satisfy the relation*

$$\mathbb{E}_0 \exp\left(-\lambda V^{-1}(y)\right) = \mathbb{E}_0 \exp(-\lambda y - C(\lambda) W(y)), \tag{1}$$

where $W^{-1}(t)$ is a non-decreasing process with independent increments for which

$$\mathbb{E}_0 \exp\left(-\lambda W^{-1}(t)\right) = \exp\left(g'_{(0,\infty)}(\lambda, 0+)t\right). \tag{2}$$

Short proof. Without loss of generality we suppose that $X(0) = 0$. Let $N_\varepsilon(t) = n$ if and only if

$$\sum_{k=1}^{n-1} |b(k)| < t \le \sum_{k=1}^{n} |b(k)|$$

($|a(k)|$ and $|b(k)|$ are lengths of intervals $a(k)$, $b(k)$). Then

$$\begin{aligned}
\mathbb{E}_0 \exp\left(-\lambda V^{-1}(y)\right) &= \lim_{\varepsilon\to 0} \mathbb{E}_0 \exp\left(-\lambda V_\varepsilon^{-1}(y)\right) \\
&= \lim_{\varepsilon\to 0} \mathbb{E}_0 \left(-\lambda y - \lambda \sum_{k=1}^{N_\varepsilon(y)} |a(k)|\right).
\end{aligned}$$

We have

$$\begin{aligned}
&\mathbb{E}_0 \exp\left(-\lambda \left(V_\varepsilon^{-1}(y) - y\right)\right) = \mathbb{E}_0 \exp\left(-\lambda \sum_{k=1}^{N_\varepsilon(y)} |a(k)|\right) \\
&\quad = \sum_{n=0}^{\infty} \mathbb{E}_0 \exp\left(-\lambda \sum_{k=1}^{n} \alpha\circ\theta_{\gamma(k-1)}; N_\varepsilon(t) = n\right) \\
&\quad = P_\varepsilon(\beta \ge y) + \sum_{n=1}^{\infty} \mathbb{E}_0\left(\exp\left(-\lambda \sum_{k=1}^{n} \alpha\circ\theta_{\gamma(k-1)}\right); \sum_{k=1}^{n-1} |b(k)| < y \le \sum_{k=1}^{n} |b(k)|\right) \\
&\quad = P_\varepsilon(\beta \ge y) \\
&\qquad + \sum_{n=1}^{\infty} \mathbb{E}_0\Bigg(\exp\left(-\lambda\alpha - \lambda \sum_{k=2}^{n} \alpha\circ\theta_{\gamma(k-1)}\right); \beta\circ\theta_\alpha + \sum_{k=2}^{n-1} \beta\circ\theta_\alpha\circ\theta_{\gamma(k-1)} \\
&\qquad\qquad < y \le \beta\circ\theta_\alpha + \sum_{k=2}^{n} \beta\circ\theta_\alpha\circ\theta_{\gamma(k-1)}\Bigg) \\
&\quad = P_\varepsilon(\beta \ge y) + \sum_{n=1}^{\infty} \int_0^y \mathbb{E}_0\Bigg(\exp\left(-\lambda\alpha - \lambda \sum_{k=2}^{n} \alpha\circ\theta_{\gamma(k-1)}\right); \beta\circ\theta_\alpha \in dx, \\
&\qquad\qquad \sum_{k=2}^{n-1} \beta\circ\theta_\alpha\circ\theta_{\gamma(k-1)} < y - x \le \sum_{k=2}^{n} \beta\circ\theta_\alpha\circ\theta_{\gamma(k-1)}\Bigg) \\
&\quad = P_\varepsilon(\beta \ge y) \\
&\qquad + \sum_{n=1}^{\infty} \int_0^y \mathbb{E}_0\left(e^{-\lambda\alpha}; \beta\circ\theta_\alpha \in dx\right) \\
&\qquad\qquad \times \mathbb{E}_0\Bigg(\exp\left(-\lambda \sum_{k=2}^{n} \alpha\circ\theta_{\gamma(k-2)}\right); \\
&\qquad\qquad \sum_{k=2}^{n-1} \beta\circ\theta_\alpha\circ\theta_{\gamma(k-2)} < y - x \le \sum_{k=2}^{n} \beta\circ\theta_\alpha\circ\theta_{\gamma(k-2)}\Bigg)
\end{aligned}$$

$$= P_\varepsilon(\beta \geq y)$$
$$+ \sum_{n=1}^{\infty} \int_0^y P_\varepsilon(\beta \in dx) \mathbb{E}_0 \left(e^{-\lambda\alpha}\right)$$
$$\times \mathbb{E}_0 \left(\exp\left(-\lambda \sum_{k=1}^{n-1} \alpha \circ \theta_{\gamma(k-1)} \right); \right.$$
$$\left. \sum_{k=1}^{n-2} \beta \circ \theta_\alpha \circ \theta_{\gamma(k-1)} < y - x \leq \sum_{k=1}^{n-1} \beta \circ \theta_\alpha \circ \theta_{\gamma(k-1)} \right)$$
$$= P_\varepsilon(\beta \geq y)$$
$$+ \int_0^y P_\varepsilon(\beta \in dx) \mathbb{E}_0 \left(e^{-\lambda\alpha}\right) \sum_{n=0}^{\infty} \mathbb{E}_0 \left(\exp\left(-\lambda \sum_{k=1}^{n} \alpha \circ \theta_{\gamma(k-1)} \right); N_\varepsilon(y-x) = n \right)$$
$$= P_\varepsilon(\beta \geq y) + \int_0^y P_\varepsilon(\beta \in dx) \mathbb{E}_0 \left(e^{-\lambda\alpha}\right) \mathbb{E}_0 \exp\left(-\lambda \left(V_\varepsilon^{-1}(y-x) - (y-x) \right) \right).$$

Let us denote $Z(y) := \mathbb{E}_0 \exp(-\lambda(V_\varepsilon^{-1}(y) - y))$, $F(x) := P_x(\beta < x)$, $\overline{F}(x) := 1 - F(x)$, $A := \mathbb{E}_0(e^{-\lambda\alpha})$. We obtain an integral equation

$$Z(y) = \overline{F}(x) + A \int_0^y Z(y-x)\, dF(x),$$

with a solution which can be written as follows

$$Z(y) = \sum_{n=0}^{\infty} A^n \left(F^{(n)}(y) - F^{(n+1)}(y) \right),$$

where $F^{(n)}$ is n-times convolution of distribution F. Let us consider a sequence of independent and identically distributed random values $|b(n)|$, $n = 1, 2, \ldots$. Let P_ε^* is the distribution of a renewal process $N_\varepsilon(y)$ with this sequence of lengths of intervals, and $\mathbb{E}_\varepsilon^*$ is the corresponding expectation. Then

$$\mathbb{E}_\varepsilon^* A^{N_\varepsilon(y)} = \sum_{n=0}^{\infty} A^n P_\varepsilon^*(N_\varepsilon(y) = n) = \sum_{n=0}^{\infty} A^n \left(F^{(n)}(y) - F^{(n+1)}(y) \right),$$

Thus

$$\mathbb{E}_0 \exp\left(-\lambda V_\varepsilon^{-1}(y)\right) = e^{-\lambda y} \mathbb{E}_\varepsilon^* \left(\mathbb{E}_0 e^{-\lambda\alpha} \right)^{N_\varepsilon(y)}.$$

On the other hand it is clear that there exists a version of the process $N_\varepsilon(y)$, measurable with respect to the basic sigma-algebra, and adapted to the natural filtration of the original process, and having identical distribution with respect to measure P_0. Preserving denotations we can write

$$\mathbb{E}_\varepsilon^* \left(\mathbb{E}_0 e^{-\lambda\alpha} \right)^{N_\varepsilon(y)} = \mathbb{E}_0 \left(\mathbb{E}_0 e^{-\lambda\alpha} \right)^{N_\varepsilon(y)}.$$

Moreover, measures P_0 and P_0^0 coincide on sigma-algebra F^*, generated by all the random values $\beta^\varepsilon \circ \theta_{\alpha^\varepsilon} \circ \theta_{\gamma(k)^\varepsilon}$, $\varepsilon > 0$, $k = 1, 2, \ldots$. From here

$$\mathbb{E}_0 \left(\mathbb{E}_0 e^{-\lambda\alpha} \right)^{N_\varepsilon(y)} = \mathbb{E}_0^0 \left(\mathbb{E}_0 e^{-\lambda\alpha} \right)^{N_\varepsilon(y)}.$$

Taking into account that α depends on ε and using our former denotations we can write

$$\mathbb{E}_0 e^{-\lambda\alpha} = K(\lambda, \varepsilon) = 1 - C(\lambda)\varepsilon + o(\varepsilon).$$

We will show that the process $W_\varepsilon(y) := \varepsilon N_\varepsilon(y)$ tends weakly to a limit $W(y)$ as $\varepsilon \to 0$, which is an inverse process with independent increments with known parameters, and measurable with respect to sigma-algebra F^*. Actually, the process $W_\varepsilon(y)$ does not decrease and is characterized completely by the process $W_\varepsilon^{-1}(t)$. The latter has independent positive jumps on the lattice with a pitch ε. Hence it is a process with independent increments. Evidently a limit of a sequence of such processes, if it exists, is a process with independent increments too. Its existence follows from evaluation of Laplace transform of its increment. We have

$$\begin{aligned}\mathbb{E}_0^0 e^{-\lambda W_\varepsilon^{-1}(t)} &= \mathbb{E}_0^0 \exp\left(-\lambda \sum_{k=1}^{[t/\varepsilon]} |b(k)|\right) = \left(\mathbb{E}_\varepsilon e^{-\lambda\beta}\right)^{[t/\varepsilon]} \\ &= \left(1 + g'_{(0,\infty)}(\lambda, 0)\varepsilon + o(\varepsilon)\right)^{[t/\varepsilon]} \to e^{g'_{(0,\infty)}(\lambda,0)t}, \qquad \varepsilon \to 0.\end{aligned}$$

Using the sufficient condition of weak convergence of processes in terms of convergence of their points of the first exit from open sets ([4], p. 287), we obtain

$$\mathbb{E}_0 \exp\left(-\lambda V^{-1}(y)\right) = \mathbb{E}_0^0 \exp\left(-\lambda y - C(\lambda) W(y)\right),$$

what can be considered as description of the direct time change in terms of the process with instantaneous reflection and the main characteristic of delaying, function $C(\lambda)$. □

We use this formula for deriving the Laplace transform of a difference between the first exit times from an one-sided neighborhood of the boundary point for processes with delayed and instantaneous reflection.

Denote

$$\begin{gathered}\beta^r := \sigma_{(0,r)}, \qquad \gamma^r(0) = 0, \\ \gamma^r := \alpha \dot{+} \beta^r, \qquad \gamma^r(n) := \gamma^r(n-1) \dot{+} \gamma^r, \qquad n \geq 1, \\ b^r(n) = \left[\gamma^r(n-1) \dot{+} \alpha, \gamma^r(n)\right), \qquad n \geq 1, \\ M_\varepsilon^r := \inf\left\{n \geq 0 \colon X(\gamma^r(n)) \geq r\right\}.\end{gathered}$$

Hence

$$\begin{gathered}P_0(M_\varepsilon^r = n) = P_0(X(\gamma^r(1)) = 0, \qquad \ldots, \qquad X(\gamma^r(n-1)) = 0, \\ X(\gamma^r(n-1)) = r) = (p(\varepsilon, r))^{n-1}(1 - p(\varepsilon, r)),\end{gathered}$$

where $p(\varepsilon, r) := P_0(X(\gamma^r(1)) = 0)$.

Theorem 2. *A difference between the first exit times from a semi-closed interval* $[0, r)$ *for processes with delayed and instantaneous reflection obeys to the relation*

$$\mathbb{E}_0 \exp\left(-\lambda\left(\sigma_{[0,r)} - \sigma^0_{[0,r)}\right)\right) = \frac{-G'_{(0,r)}(0+)}{C(\lambda) - G'_{(0,r)}(0+)}, \tag{3}$$

where $G_{(0,r)}(x) = g_{(0,r)}(0, x)$.

Short proof. Let $X(0) = 0$. Then evidently, $\sigma_{[0,r)} = \gamma^r_{M_\varepsilon^r}$ for any $\varepsilon < r$. On the other hand, it is clear, that $\gamma^r = \gamma$ on the set $\{X(\gamma^r) = 0\}$, and by induction we conclude that

$$\gamma^r(n) = \gamma(n) \quad \text{on the set} \quad \bigcap_{k=1}^{n} \{X(\gamma^r(k)) = 0\}.$$

From here

$$\begin{aligned}\gamma^r(n) I(M_\varepsilon^r = n) &= (\gamma^r(n-1) \dot{+} \gamma^r) I\left(\bigcap_{k=1}^{n-1} \{X(\gamma^r(k)) = 0\}\right) \cap \{X(\gamma^r(n)) = r\} \\ &= (\gamma(n-1) \dot{+} \gamma^r) I(M_\varepsilon^r = n).\end{aligned}$$

Let us denote $\sigma^0_{[0,r)}$ the first exit time from interval $[0,r)$ of the process with instantaneous reflection (formally it means $\sigma^0_{[0,r)} = V(\sigma_{[0,r)})$). Then $V^{-1}(\sigma^0_{[0,r)}) = \sigma_{[0,r)}$, and from formula (1) it follows that

$$\begin{aligned}
&\mathbb{E}_0 \exp\left(-\lambda\left(\sigma_{[0,r)} - \sigma^0_{[0,r)}\right)\right) = \mathbb{E}^0_0 \exp\left(-C(\lambda)W\left(\sigma^0_{[0,r)}\right)\right) \\
&\quad = \mathbb{E}^0_0 \exp\bigl(-C(\lambda)W(\gamma^r(M^r_\varepsilon))\bigr) = \sum_{n=1}^{\infty} \mathbb{E}^0_0 \left(\exp\bigl(-C(\lambda)W(\gamma^r(n))\bigr); M^r_\varepsilon = n\right) \\
&\quad = \sum_{n=1}^{\infty} \mathbb{E}^0_0 \exp\left(-C(\lambda)W\left(\gamma(n-1)\dot{+}\gamma^r\right); M^r_\varepsilon = n\right) \\
&\quad = \sum_{n=1}^{\infty} \mathbb{E}^0_0 \left(\exp(-C(\lambda)W\left(\sum_{k=1}^{n-1}(|a(k)| + |b^r(k)|) + |a(n)| + |b^r(n)|\right); M^r_\varepsilon = n\right).
\end{aligned}$$

Taking into account P^0_0-almost sure convergence $\sum_{k=1}^{M^r_\varepsilon}(|a(k)| \to 0$ as $\varepsilon \to 0$, we have

$$\begin{aligned}
&\lim_{\varepsilon\to 0} \sum_{n=1}^{\infty} \mathbb{E}^0_0 \left(\exp(-C(\lambda)W\left(\sum_{k=1}^{n-1}(|a(k)| + |b^r(k)|) + |a(n)| + |b^r(n)|\right); M^r_\varepsilon = n\right) \\
&\quad = \lim_{\varepsilon\to 0} \mathbb{E}^0_0 \left(\exp(-C(\lambda)W\left(\sum_{k=1}^{M^r_\varepsilon - 1} |b^r(k)| + |b^r(M^r_\varepsilon)|\right)\right) \\
&\quad = \lim_{\varepsilon\to 0} \mathbb{E}^0_0 \left(\exp(-C(\lambda)\varepsilon N_\varepsilon\left(\sum_{k=1}^{M^r_\varepsilon - 1} |b^r(k)| + |b^r(M^r_\varepsilon)|\right)\right).
\end{aligned}$$

From the definition of the process $N_\varepsilon(t)$ it folows that

$$N_\varepsilon\left(\sum_{k=1}^{n-1} |b^r(k)| + |b^r(n)|\right) = n, \qquad n = 1, 2 \ldots .$$

Consecuently

$$\begin{aligned}
\mathbb{E}_0 \exp\left(-\lambda\left(\sigma_{[0,r)} - \sigma^0_{[0,r)}\right)\right) &= \lim_{\varepsilon\to 0} \mathbb{E}^0_0 \exp\left(-C(\lambda)\varepsilon M^r_\varepsilon\right) \\
&= \lim_{\varepsilon\to 0} \sum_{n=1}^{\infty} e^{-C(\lambda)\varepsilon n}(p(\varepsilon,r))^{n-1}(1 - p(\varepsilon,r)) \\
&= \lim_{\varepsilon\to 0} e^{-C(\lambda)\varepsilon} \frac{1 - p(\varepsilon,r)}{1 - e^{-C(\lambda)\varepsilon}p(\varepsilon,r)}.
\end{aligned}$$

and taking into account that

$$\begin{aligned}
p(\varepsilon,r) &= P_0(X(\gamma^r_\varepsilon) = 0) = P_0(X(\alpha_\varepsilon \dot{+} \beta^r_\varepsilon) = 0) = P_0(X(\beta^r_\varepsilon) \circ \theta_{\alpha_\varepsilon} = 0) \\
&= P_\varepsilon(X(\beta^r_\varepsilon) = 0) := G_{(0,r)}(\varepsilon),
\end{aligned}$$

and that the last expression (the partial case $g_{(0,r)}(\lambda,\varepsilon)$ for $\lambda = 0$) has an asymptotic $G_{(0,r)}(\varepsilon) = 1 + G'_{(0,r)}(0+)\varepsilon + o(\varepsilon)$, we obtain at last

$$\mathbb{E}_0 \exp\left(-\lambda\left(\sigma_{[0,r)} - \sigma^0_{[0,r)}\right)\right) = \frac{-G'_{(0,r)}(0+)}{C(\lambda) - G'_{(0,r)}(0+)}. \qquad \square$$

It is interesting to note that for a linear function $C(\lambda) = k\lambda$, when a reflecting locally Markov process is globally Markov [5], the difference between the first exit times from a semi-closed interval $[0,r)$ for processes with delayed and instantaneous reflection has the exponential distribution with parameter $-G'_{(0,r)}(0+)/k$.

4. Example

Let us consider the standard Wiener process truncated in its negative values

$$\overline{w}(t) = \begin{cases} 0, & w(t) \le 0, \\ w(t), & w(t) > 0. \end{cases}$$

In frames of the semi-Markov model of reflection it is characterized by the function

$$K(\lambda, r) = \frac{h'_{(0,r)}(\lambda, 0+)}{C(\lambda) - g'_{(0,r)}(\lambda, 0+)} = \frac{\sqrt{2\lambda}/\sinh r\sqrt{2\lambda}}{C(\lambda) - \sqrt{2\lambda}\cosh r\sqrt{2\lambda}/\sinh r\sqrt{2\lambda}}.$$

Taking into account the origin of this process one can write

$$K(\lambda, r) = \mathbb{E}_0^w \exp\left(-\lambda\sigma_{(-\infty,r)}\right) = \exp\left(-r\sqrt{2\lambda}\right).$$

Comparing derivatives at zero of these two representations of the same function, we obtain $C(\lambda) = \sqrt{2\lambda}$. Now we can obtain the main characteristic of delay of this process under reflection (including lengths of all the intervals of constancy) from the first hitting time of the level 0 up to the first hitting time of the level r:

$$\mathbb{E}_0 \exp\left(-\lambda\left(\sigma_{[0,r)} - \sigma^0_{[0,r)}\right)\right) = \frac{1/r}{\sqrt{2\lambda} + 1/r},$$

what relates to tabulated values of Laplace transforms, and here is not exposed.

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