Теорія ймовір. та матем. статист.
Вип. 95, 2016, стор. 101–118

Teor. Ĭmovir. ta Matem. Statyst.
No. 95, 2016, pp. 101–118

MULTI-SCALING LIMITS FOR TIME-FRACTIONAL RELATIVISTIC DIFFUSION EQUATIONS WITH RANDOM INITIAL DATA

UDC 519.21

G.-R. LIU AND N.-R. SHIEH

This paper is dedicated to the 65th birthday of Professor Nikolai Leonenko

Abstract. Let $u(t, \mathbf{x})$, $t > 0$, $\mathbf{x} \in \mathbb{R}^n$, be the spatial-temporal random field arising from the solution of a time-fractional relativistic diffusion equation with the time-fractional parameter $\beta \in (0, 1)$, the spatial-fractional parameter $\alpha \in (0, 2)$ and the mass parameter $\mathfrak{m} > 0$, subject to random initial data $u(0, \cdot)$ which is characterized as a subordinated Gaussian field. Compared with [5] written by Anh and Leoeneko in 2002, we not only study the large-scale limits of the solution field u, but also propose a small-scale scaling scheme, which also leads to the Gaussian and the non-Gaussian limits depending on the covariance structure of the initial data. The new scaling scheme involves not only to scale u but also to re-scale the initial data u_0. In the two scalings, the parameters α and $\mathfrak{m}$ play distinct roles in the process of limiting, and the spatial dimensions of the limiting fields are restricted due to the slow decay of the time- fractional heat kernel.

1. Introduction

In this paper, we study the scaling limits of the spatial-temporal random field arising from the solution u of the following random initial value problem

$$\frac{\partial^\beta}{\partial t^\beta} u(t, \mathbf{x}) = (\mathfrak{m} - (\mathfrak{m}^{\frac{2}{\alpha}} - \Delta)^{\frac{\alpha}{2}}) u(t, \mathbf{x}), \quad u(0, \mathbf{x}) = u_0(\mathbf{x}),\ t \geq 0,\ \mathbf{x} \in \mathbb{R}^n, \tag{1}$$

of the time-fractional relativistic diffusion equation (TFRDE), where the time-fractional parameter $\beta \in (0, 1)$, the spatial-fractional parameter $\alpha \in (0, 2)$ and the (normalized) mass parameter $\mathfrak{m} > 0$. This equation is obtained from the classical diffusion equation by replacing the spatial and temporal derivatives by fractional ones, which were introduced to describe physical phenomena such as diffusion in porous media with fractal geometry, kinematics in viscoelastic media, relaxation processes in complex systems [4].

For the operator $(\mathfrak{m} - (\mathfrak{m}^{\frac{2}{\alpha}} - \Delta)^{\frac{\alpha}{2}})$, the prominent case is $\alpha = 1$, for which $-(\mathfrak{m} - \sqrt{\mathfrak{m}^2 - \Delta})$ is regarded as the relativistic Schrödinger operator; see the seminal paper of Carmona *et al.* [11] and Shieh [33] for its relation to Lévy processes. For general $\alpha \in (0, 2)$, one may refer to Ryznar [31], Baeumer *et al.* [7], Kumara *et al.* [21], and the references therein. TFRDEs have also played an essential role in the theory of computer vision; see a special volume edited by Kimmel *et al.* [20], in which P.D.E. and scale-space methods are focused and TFRDEs with $\beta = 1$ are particularly employed.

In this article, the initial data u_0 are modeled by a class of nonlinear functions of homogeneous Gaussian random fields. We study the large-scale and the small-scale

---

2010 *Mathematics Subject Classification.* Primary 60G60, 60H05, 62M15; Secondary 35K15.

Key words and phrases. Large-scale limits; Small-scale limits; Relativistic diffusion equations; Random initial data; Multiple Itô–Wiener integrals; Subordinated Gaussian fields; Hermite ranks.

The first author was supported in part by NCTS/TPE and the Taiwan Ministry of Science and Technology under Grant MOST 104-2115-M-006-016-MY2. This research also received funding from the Headquarter of University Advancement at National Cheng Kung University, which is sponsored by the Ministry of Education, Taiwan, ROC..

limits of the re-scaled solution field. For the large-scale limit (Theorem 1 and Theorem 3), the mass $\mathfrak{m} > 0$ dominates the space-time scaling and also the limiting field. For the small-scale limit (Theorem 2 and Theorem 4), the spatial-fractional parameter α dominates both the scaling factor and the limiting field, and it appears to be irrelevant for $\mathfrak{m}$ being positive or zero.

In our discussions, the large-scale limits in Theorem 1 and Theorem 3 are respectively comparable to the Central Limit Theorem for local functionals of random fields with weak dependence in [10], and to a certain non-Gaussian Central Limit Theorem for which the papers [34, 14] are pioneering. For the small-scale limits in Theorem 2 and Theorem 4, they involve not only the space-time scaling on u but also need to re-scale the initial data u_0; to our knowledge, these are new type results for the literature; see [27] for the authors' very recent study. We use the moment method and the Feymann-type diagrams, which are used notably in [10], to find out the Gaussian limits. On the other hand, we exploit the truncation of Hermite expansions and the multiple Wiener-Itô integrals to find out the non-Gaussian limits.

We remark that, in the non-relativistic case, i.e. $\mathfrak{m} = 0$, the large-scale limits for the random initial value problem with multiple Itô-Wiener integrals as input have been discussed in Anh and Leonenko [2, 5]; subsequent works, together with Burgers' equation, in this direction by the authors and collaborators can be seen in [6, 8, 18, 23, 24, 25, 26, 30] and the references therein. However, the multi-scaling limits due to the different roles of the mass and the fractional-index, the target of this article, are not mentioned in the cited papers. Compare to [3, 5] and our previous work [26] related to the random initial value problem for the fractional diffusion-wave equations, the Laplace operator Δ is extended to the α-fractional relativistic diffusion operator $(\mathfrak{m} - (\mathfrak{m}^{\frac{2}{\alpha}} - \Delta)^{\frac{\alpha}{2}})$. In [3, 5, 26], the large-scale limit of u is discussed but the existence of small-scale limits of u is neglected. In this paper, we show that the random solution u under the small scaling and the large scaling has different limits no matter whether the initial data is long-range dependent or not. Finally, we mention that the study on the PDEs with random initial conditions can be traced back to [19] and [29]. Besides the above mentioned literature, there also has very significant progress on Burgers equation with different types of random input; see the monograph of Woyczyński [37] and the Chapter 6 of Bertoin [9].

The rest of the paper is organized as follows. In Section 2, we present some preliminaries; we state our main results in Section 3, and all the proofs of our results are given in Section 4.

2. Preliminaries

2.1. **Heat kernel for TFRDEs.** In the TFRDE, the fractional temporal derivative $\frac{\partial^\beta}{\partial t^\beta}$ is in the Caputo-Djrbashian sense [12]

$$\frac{d^\beta f}{dt^\beta}(t) = \begin{cases} f^{(m)}(t) & \text{if } \beta = m \in \mathbb{N} \\ \frac{1}{\Gamma(m-\beta)} \int_0^t \frac{f^{(m)}(\tau)}{(t-\tau)^{\beta+1-m}} d\tau & \text{if } \beta \in (m-1, m), \end{cases} \tag{2}$$

where $f^{(m)}$ denotes the ordinary derivative of order m of a causal function f (i.e., f is vanishing for $t < 0$). The spatial operator $(\mathfrak{m} - (\mathfrak{m}^{\frac{2}{\alpha}} - \Delta)^{\frac{\alpha}{2}})$ in (1) is regarded as a psudo-differential operator, see for example the book and the paper by Wong [35, 36].

In this paper, we mainly focus on the case $0 < \beta \leq 1$. In this case, (1) can be derived from the master equation of a continuous-time random walk with the Mittag-Leffler distributed waiting times between jumps (see, for example, Angulo *et al.* [1]). The Mittage-Leffer distribution has the density function $\psi(t)$, $t \geq 0$, as follows

$$\psi_\beta(t) = t^{\beta-1} E_{\beta,\beta}(-t^\beta), \tag{3}$$

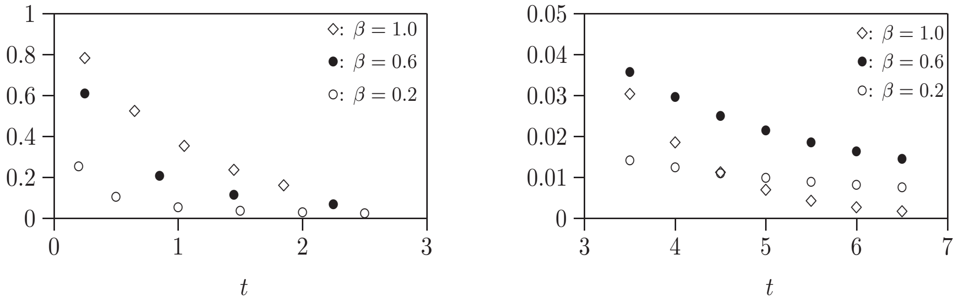


FIGURE 1. The Mittag-Leffler probability density function

where $E_{\beta,\beta}(\cdot)$ is the two-parameter Mittag-Leffler function, which is defined by the series expansion

$$E_{a,b}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(ak+b)}, \quad a,\ b > 0,\ z \in \mathbb{C}. \tag{4}$$

The Mittag-Leffler functions are entire functions on the complex plane and their asymptotic behaviors, when $\beta < 1$, have the inverse power law as follows:

$$|E_{a,b}(z)| \ \sim \ O(\frac{1}{|z|}), \quad |z| \ \to \ \infty \text{ with } |\arg(-z)| < \pi(1-\frac{a}{2}),\ b > 0, \tag{5}$$

where arg: $\mathbb{C} \to (-\pi, \pi)$ and $f(z) \sim O(g(z))$ denotes that $f(z)/g(z)$ remains bounded as z approaches the indicated limit point; see, for example, the classic books by Erdélyi *et.al.* [16] (pp. 206-212, in particular p. 206 (7) and p. 210 (21)) or by Djrbashian [12, Chapter 1]. When $\beta = 1$, the Mittag-Leffler distribution becomes an exponentially distribution since $E_{1,1}(z) = e^z$. The Mittag-Leffler probability density functions $\psi_\beta(t)$ for $\beta = 0.2$, 0.6 and 1 are illustrated in Figure 1. By (5) (see also Figure 1), when $\beta < 1$, the probability density function ψ_β of the waiting time between jumps does not have the exponential decay as ψ_1, so the case $0 < \beta < 1$ is referred as the *sub-diffusive*.

The solution of (1) is given in the convolution form

$$u(t, \mathbf{x}; u_0(\cdot)) = \int_{\mathbb{R}^n} G(t, \mathbf{x} - \mathbf{y}) u_0(\mathbf{y}) d\mathbf{y}, \tag{6}$$

where the heat kernel $G(\cdot,\cdot)$ is defined by its spatial Fourier transform $\widehat{G}(t,\cdot)$ as follows:

$$\widehat{G}(t,\lambda) = \int_{\mathbb{R}^n} e^{i<\lambda,\mathbf{x}>} G(t,\mathbf{x}) d\mathbf{x} = E_{\beta,1}\left(-t^\beta \theta(\lambda)\right) \tag{7}$$

with $\theta(\lambda) = (\mathfrak{m}^{\frac{2}{\alpha}} + |\lambda|^2)^{\frac{\alpha}{2}} - \mathfrak{m}$, where $\lambda \in \mathbb{R}^n$. The derivation of (7) can be found in [32].

In this work, the initial data u_0 in (6) is a second-order homogeneous random field on $\mathbb{R}^n$, and (6) should be understood as a mean-square solution of (1); resulting in a spatial-temporal random solution field u; see [30, Proposition 1] for some discussion on the mean-square solutions of parabolic PDEs with mean-square continuous random initial data.

2.2. **Subordinated Gaussian fields as initial data.** Let $(\Omega, \mathcal{F}, \mathcal{P})$ be an underlying probability space such that all random elements appeared in this article are measurable with respect to it.

Condition A. The initial data of (1) is assumed to be a random field on $\mathbb{R}^n$ given by

$$u_0(\mathbf{x}) = h(\zeta(\mathbf{x})),\ \mathbf{x} \in \mathbb{R}^n, \tag{8}$$

where ζ is a mean-square continuous and homogeneous Gaussian random field with mean zero and variance 1. We suppose that the Gaussian random field ζ has positive covariance function $R(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^n$, and its spectral measure $F(d\lambda)$ has the (spectral) density $f(\lambda)$, $\lambda \in \mathbb{R}^n$; moreover, $h : \mathbb{R} \to \mathbb{R}$ is a (non-random) function such that

$$\mathbb{E}h^2(\zeta(\mathbf{0})) = \int_{\mathbb{R}} h^2(r)p(r)dr < \infty; \quad p(r) = \frac{1}{\sqrt{2\pi}} e^{-\frac{r^2}{2}},\ r \in \mathbb{R}. \tag{9}$$

Condition A implies that the initial data u_0 is a subordinated Gaussian field, which is introduced by Dobrushin [13]; see also [2, 5] for more recent discussions. Under Condition A, by the Bochner-Khintchine theorem, we have the following spectral representation for the covariance function of the Gaussian field ζ:

$$R(\mathbf{x}) = \mathrm{Cov}(\zeta(\mathbf{0}), \zeta(\mathbf{x})) = \int_{\mathbb{R}^n} e^{i<\lambda,\mathbf{x}>} f(\lambda) d\lambda. \tag{10}$$

Moreover, by the Karhunen Theorem, ζ has the representation

$$\zeta(\mathbf{x}) = \int_{\mathbb{R}^n} e^{i<\lambda,\mathbf{x}>} \sqrt{f(\lambda)} W(d\lambda),\ \mathbf{x} \in \mathbb{R}^n, \tag{11}$$

where $W(d\lambda)$ is the standard complex-valued Gaussian white noise on the Fourier domain $\mathbb{R}^n$ such that $W(\Delta_1) = \overline{W(-\Delta_1)}$ and $\mathbb{E}W(\Delta_1)\overline{W(\Delta_2)} = \mathrm{Leb}(\Delta_1 \cap \Delta_2)$ for any $\Delta_1, \Delta_2 \in \mathcal{B}(\mathbb{R}^n)$. See, for example, the book of Leonenko [22, Theorem 1.1.3] for the above facts. The function h has the following expansion:

$$h(r) = C_0 + \sum_{l=1}^{\infty} C_l \frac{H_l(r)}{\sqrt{l!}} \tag{12}$$

in the Hilbert space $L^2(\mathbb{R}, p(r)dr)$, where

$$C_l = \int_{\mathbb{R}} h(r) \frac{H_l(r)}{\sqrt{l!}} p(r) dr, \tag{13}$$

and $\{H_l(r), l = 0, 1, 2, \dots\}$ are the Hermite polynomials, that is,

$$H_l(r) = (-1)^l e^{\frac{r^2}{2}} \frac{d^l}{dr^l} e^{-\frac{r^2}{2}} \quad \text{for} \quad l \in \{0, 1, 2, \dots\}.$$

Accordingly, the *Hermite rank* of the function h is defined by

$$m = \inf\{l \geq 1 : \ C_l \neq 0\}.$$

It is well-known that (see, for example, Major [28, Corollary 5.5 and p. 30]):

$$\mathbb{E}[H_{l_1}(\zeta(\mathbf{y}))H_{l_2}(\zeta(\mathbf{z}))] = \delta_{l_2}^{l_1} l_1! R^{l_1}(\mathbf{y} - \mathbf{z}), \quad \mathbf{y},\ \mathbf{z} \in \mathbb{R}^n, \tag{14}$$

($\delta_{\sigma_2}^{\sigma_1}$ is the Kronecker symbol) and

$$H_l(\zeta(\mathbf{x})) = \int'_{\mathbb{R}^{n \times l}} e^{i<\mathbf{x}, \lambda_1 + \dots + \lambda_l>} \left[\prod_{k=1}^{l} \sqrt{f(\lambda_k)}\right] W(d\lambda_1) \dots W(d\lambda_l), \tag{15}$$

where $\int'$ means that the integral excludes the diagonal hyperplanes $\mathbf{z}_i = \mp \mathbf{z}_j,\ i, j = 1, \dots, l,\ i \neq j$.

We impose two different conditions on the singularity of the spectral density $f(\lambda)$ at $\mathbf{0}$, which yield, respectively, the Gaussian and the non-Gaussian scaling-limits.

Condition B. The spectral density function f of the Gaussian random field ζ in Condition A can be expressed as

$$f(\lambda) = \frac{B(\lambda)}{|\lambda|^{n-\kappa}} \text{ for some } \kappa > \frac{n}{m}, \tag{16}$$

where m is the Hermite rank of the function h, and $B(\cdot) \in \mathrm{C}(\mathbb{R}^n)$ is of suitable decay at infinity to ensure $f \in L^1(\mathbb{R}^n)$.

Condition C. The spectral density function f of the Gaussian random field ζ in Condition A can be expressed as

$$f(\lambda) = \frac{B(\lambda)}{|\lambda|^{n-\kappa}} \quad \text{for some} \quad 0 < \kappa < \frac{n}{m}, \tag{17}$$

where m is the Hermite rank of the function h, $B(\cdot) \in \mathrm{C}(\mathbb{R}^n)$ is of suitable decay at infinity to ensure $f \in L^1(\mathbb{R}^n)$, and $B(\mathbf{0}) > 0$.

Note that, in Condition B and C, we do not assume that $B(\cdot)$ is a radial function, so the field u_0 is not necessary to be isotropic. Condition B means that the density f either is regular at $\mathbf{0}$, or has a singularity for which the order is less than $n(1-1/m)$; while Condition C means that f has a singularity at $\mathbf{0}$ for which the order is higher than $n(1-1/m)$.

By (10) and the convolution theorem, for each $l \in \mathbb{N}$,

$$R^l(\mathbf{x}) = \int_{\mathbb{R}^n} e^{i<\lambda,\mathbf{x}>} f^{*l}(\lambda) d\lambda, \tag{18}$$

where $f^{*l}(\lambda)$ is the l-fold convolution of f. Given that f can be expressed as (16) or (17), the behavior of f^{*l}, $l \in \mathbb{N}$, near the original can be described as follows.

Lemma 1. *Suppose that the spectral density function f has the form,*

$$f(\lambda) = \frac{B(\lambda)}{|\lambda|^{n-\kappa}}, \ \kappa > 0,$$

for some non-negative bounded and continuous function $B(\lambda)$ so that $f \in L^1(\mathbb{R}^n)$. Then, for any $k \geq 2$, there exists a bounded function $B_k \in \mathrm{C}(\mathbb{R}^n \backslash \{\mathbf{0}\})$ such that the k-fold convolution f^{*k} of f can be re-written as

$$f^{*k}(\lambda) = \begin{cases} B_k(\lambda)|\lambda|^{k\kappa - n}, & \text{for } k\kappa < n, \\ B_k(\lambda)\ln(2 + \frac{1}{|\lambda|}), & \text{for } k\kappa = n, \\ B_k(\lambda) \in \mathrm{C}(\mathbb{R}^n), & \text{for } k\kappa > n. \end{cases} \tag{19}$$

Moreover, for any $k_1 > k_2 > n/\kappa$, the inequality $\sup_{\lambda \in \mathbb{R}^n} B_{k_1}(\lambda) \leq \sup_{\lambda \in \mathbb{R}^n} B_{k_2}(\lambda)$ holds.

We refer the reader to the proof of Lemma 1 in [27].

To understand the difference between Conditions B and C, in view of Lemma 1, Condition B implies that the k-fold convolution f^{*k}, $k \geq m$, has no singularity at the origin $\lambda = \mathbf{0}$, which in turn asserts that the spectral density of the random initial data u_0 has no singularity at $\lambda = \mathbf{0}$; while Condition C asserts that the initial data u_0 has a spectral density which is singular at $\lambda = \mathbf{0}$. The situation can be described as, respectively, the short-range and the long-range dependence of the initial field u_0; a central notion in vast applications, as one may refer to the special volume by Doukhan, Oppenheim, and Taqqu [15].

3. Main results

The significant difference between Condition B and Condition C, as remarked at the end of the last section, is employed to obtain the Gaussian and the non-Gaussian scaling-limits. We will present them in the following two subsections.

In the context henceforth, the notation $\Rightarrow$ denotes the convergence of random variables (respectively, random families) in the sense of distribution (respectively, finite-dimensional distributions).

3.1. Gaussian limits with initial data in (A,B).

As mentioned in Section 1, we will present the large-scale and the small-scale limit theorems, which are comparable to the central limit theorem for local functionals of random fields with weak dependence in Breuer and Major [10]. The novel feature is that the mass $\mathfrak{m} > 0$ and the fractional-index α play different roles in the two scales.

Theorem 1. *Let n=1,2 or 3. Consider the mean-square solution $u(t,\mathbf{x};u_0(\cdot))$, $t>0$, $\mathbf{x}\in\mathbb{R}^n$, of (1) with $\mathfrak{m}>0$. The initial data $u_0(\mathbf{x})=h(\zeta(\mathbf{x}))$ are supposed to satisfy Conditions A and B with the Hermite rank $m\geq 1$. When $T\to\infty$,*

$$T^{\frac{n\beta}{4}}\Big\{u(Tt,T^{\frac{\beta}{2}}\mathbf{x};u_0(\cdot))-C_0\Big\}\Rightarrow U(t,\mathbf{x}),$$

where $U(t,\mathbf{x})$, $t>0$, $\mathbf{x}\in\mathbb{R}^n$, is a Gaussian field with the spectral representation

$$U(t,\mathbf{x})=\int_{\mathbb{R}^n}e^{i<\lambda,\mathbf{x}>}\sigma_m E_{\beta,1}(-t^{\beta}\frac{\alpha}{2}\mathfrak{m}^{1-\frac{2}{\alpha}}|\lambda|^2)W(d\lambda),\ \sigma_m=\Big(\sum_{r=m}^{\infty}f^{*r}(\mathbf{0})C_r^2\Big)^{\frac{1}{2}},\quad (20)$$

where $W(d\lambda)$ is a complex-valued standard Gaussian noise measure on $\mathbb{R}^n$ (c.f. (11)).

For the small-scale limit, we need to re-scale the initial data too; thus the notation $u_0(\varepsilon^{-\frac{1}{\alpha}-\chi}\cdot)$ imposed on u_0 emphasizes that the variable of u_0 is under the indicated dilation factor $\varepsilon^{-\frac{1}{\alpha}-\chi}$.

Theorem 2. *Let $u(t,\mathbf{x};u_0(\cdot))$, $t>0$, $\mathbf{x}\in\mathbb{R}^n$, be the mean-square solution of (1) with $\mathfrak{m}>0$ and $2\alpha<(n\wedge 4)$. The initial data $u_0(\mathbf{x})=h(\zeta(\mathbf{x}))$ are supposed to satisfy Conditions A and B with the Hermite rank $m\geq 1$. For any $\chi>0$, when $\varepsilon\to 0$,*

$$\varepsilon^{-\frac{n\chi}{2}}\Big\{u(\varepsilon^{\frac{1}{\beta}}t,\varepsilon^{\frac{1}{\alpha}}\mathbf{x};u_0(\varepsilon^{-\frac{1}{\alpha}-\chi}\cdot))-C_0\Big\}\Rightarrow V(t,\mathbf{x}),\quad (21)$$

where $V(t,\mathbf{x})$, $t>0$, $\mathbf{x}\in\mathbb{R}^n$, is a Gaussian field with the following spectral representation:

$$V(t,\mathbf{x})=\int_{\mathbb{R}^n}e^{i<\lambda,\mathbf{x}>}\sigma_m E_{\beta,1}(-t^{\beta}|\lambda|^{\alpha})W(d\lambda),\ \sigma_m=\Big(\sum_{r=m}^{\infty}f^{*r}(\mathbf{0})C_r^2\Big)^{\frac{1}{2}},\quad (22)$$

where $W(d\lambda)$ is a complex-valued standard Gaussian noise measure on $\mathbb{R}^n$.

3.2. **Non-Gaussian limits with initial data in (A,C).** As in the above subsection, we have the large-scale and the small-scale limits; however, the high singularity order in Condition C assets that our limiting fields are now non-Gaussian. The non-Gaussian limits of the convolution type can be seen in the pioneering papers of Taqqu [34] and Dobrushin and Major [14], and Anh and Leonenko [2, 5].

Theorem 3. *Let $n=1,2$ or 3. Consider the mean-square solution $u(t,\mathbf{x};u_0(\cdot))$, $t>0$, $\mathbf{x}\in\mathbb{R}^n$, of (1) with $\mathfrak{m}>0$. The initial data $\{u_0(\mathbf{x})=h(\zeta(\mathbf{x})),\mathbf{x}\in\mathbb{R}^n\}$ are supposed to satisfy Conditions A and C with $m\geq 1$.*

When $T\to\infty$, we have

$$T^{\frac{\beta m\kappa}{4}}\Big\{u(Tt,T^{\frac{\beta}{2}}\mathbf{x};h(\zeta(\cdot)))-C_0\Big\}\Rightarrow U_m(t,\mathbf{x}),\quad (23)$$

where $U_m(t,\mathbf{x})$ is represented by the following multiple Wiener integrals

$$U_m(t,\mathbf{x})=B^{\frac{m}{2}}(\mathbf{0})\frac{C_m}{\sqrt{m!}}\int_{\mathbb{R}^{n\times m}}^{'}e^{i<\mathbf{x},\lambda_1+\cdots+\lambda_m>}\frac{E_{\beta,1}(-t^{\beta}\frac{\alpha}{2}\mathfrak{m}^{1-\frac{2}{\alpha}}|\lambda_1+\cdots+\lambda_m|^2)}{(|\lambda_1|\ldots|\lambda_m|)^{\frac{n-\kappa}{2}}}\prod_{l=1}^{m}W(d\lambda_l).\quad (24)$$

Theorem 4. *Let $u(t,\mathbf{x};u_0(\cdot))$ be the mean-square solution to (1) with $2\alpha<(n\wedge 4)$. The initial data $\{u_0(\mathbf{x})=h(\zeta(\mathbf{x})),\ \mathbf{x}\in\mathbb{R}^n\}$ are supposed to satisfy Conditions A and C with $m\geq 1$. For any fixed parameter $\chi>0$, when $\varepsilon\to 0$, we have*

$$\varepsilon^{-\frac{m\kappa\chi}{2}}\Big\{u(\varepsilon^{\frac{1}{\beta}}t,\varepsilon^{\frac{1}{\alpha}}\mathbf{x};h(\zeta((\varepsilon^{-\frac{1}{\alpha}-\chi})\cdot)))-C_0\Big\}\Rightarrow V_m(t,\mathbf{x}),\quad (25)$$

where $V_m(t,\mathbf{x})$ is represented by the multiple Wiener integrals

$$V_m(t,\mathbf{x})=B^{\frac{m}{2}}(\mathbf{0})\frac{C_m}{\sqrt{m!}}\int_{\mathbb{R}^{n\times m}}^{'}e^{i<\mathbf{x},\lambda_1+\cdots+\lambda_m>}\frac{E_{\beta,1}(-t^{\beta}|\lambda_1+\cdots+\lambda_m|^{\alpha})}{(|\lambda_1|\ldots|\lambda_m|)^{\frac{n-\kappa}{2}}}\prod_{l=1}^{m}W(d\lambda_l).\quad (26)$$

4. Proofs of Theorems

The following two-scale property of the heat kernel G is the key to our results. In comparison with G, the heat kernel corresponding to the fractional-Laplace operator $-(-\Delta)^{\frac{\alpha}{2}}$, where $\alpha \in (0,2]$, only has one type of scaling limit. We describe the two-scale property of G in terms of its Fourier transform $\widehat{G}$ as follows.

$$\widehat{G}(Tt, T^{-\frac{\beta}{2}}\lambda) = E_{\beta,1}\left(T^{\beta}t^{\beta}(\mathfrak{m} - (\mathfrak{m}^{\frac{2}{\alpha}} + T^{-\beta}|\lambda|^2)^{\frac{\alpha}{2}})\right) \to E_{\beta,1}\left(-t^{\beta}\frac{\alpha}{2}\mathfrak{m}^{1-\frac{2}{\alpha}}|\lambda|^2\right) \quad (27)$$

when $T \to \infty$; (27) is a consequence of the Taylor's expansion,

$$\begin{aligned}\mathfrak{m} - (\mathfrak{m}^{\frac{2}{\alpha}} + T^{-\beta}|\lambda|^2)^{\frac{\alpha}{2}} &= \mathfrak{m} - \left(\mathfrak{m} + \frac{\alpha}{2}(\mathfrak{m}^{\frac{2}{\alpha}})^{\frac{\alpha}{2}-1}T^{-\beta}|\lambda|^2 + \frac{\alpha}{4}(\frac{\alpha}{2}-1)c_T^{\frac{\alpha}{2}-2}T^{-2\beta}|\lambda|^4\right)\\ &= -\frac{\alpha}{2}(\mathfrak{m}^{\frac{2}{\alpha}})^{\frac{\alpha}{2}-1}T^{-\beta}|\lambda|^2 + \frac{\alpha}{4}(1-\frac{\alpha}{2})c_T^{\frac{\alpha}{2}-2}T^{-2\beta}|\lambda|^4\end{aligned}$$

for some $c_T \in (\mathfrak{m}^{\frac{2}{\alpha}}, \mathfrak{m}^{\frac{2}{\alpha}} + T^{-\beta}|\lambda|^2)$. In contrast to the large-scale property (27), when $\varepsilon \to 0$, we have

$$\widehat{G}(\varepsilon^{\frac{1}{\beta}}t, \varepsilon^{-\frac{1}{\alpha}}\lambda) = E_{\beta,1}\left(\varepsilon t^{\beta}\mathfrak{m} - \varepsilon t^{\beta}(\mathfrak{m}^{\frac{2}{\alpha}} + \varepsilon^{-\frac{2}{\alpha}}|\lambda|^2)^{\frac{\alpha}{2}}\right) \to E_{\beta,1}\left(-t^{\beta}|\lambda|^{\alpha}\right). \quad (28)$$

We observe that (28) indeed holds no matter whether $\mathfrak{m}$ is positive or not.

Proofs of Theorems 1 and 2. In the below, we only provide the proof of Theorem 2, and show why the rescaling of the initial data is needed to obtain the desired limit. The proof of Theorem 1 is parallel and does not require the rescaling of the initial data. The method of the proof can be traced back to [10].

Denote

$$Y_\varepsilon(t,\mathbf{x}) = \varepsilon^{-\frac{n\chi}{2}}u(\varepsilon^{\frac{1}{\beta}}t, \varepsilon^{\frac{1}{\alpha}}\mathbf{x}; u_0(\varepsilon^{-\frac{1}{\alpha}-\chi}\cdot)) - C_0.$$

We first apply the Hermite expansion (12) and the property $\int_{\mathbb{R}^n} G(t,\mathbf{x})d\mathbf{x} = 1$, which is obtained by substituting $\lambda = 0$ into (7), to get

$$Y_\varepsilon(t,\mathbf{x}) = \varepsilon^{-\frac{n\chi}{2}}\sum_{l=m}^{\infty}\frac{C_l}{\sqrt{l!}}\int_{\mathbb{R}^n} G(\varepsilon^{\frac{1}{\beta}}t, \varepsilon^{\frac{1}{\alpha}}\mathbf{x} - \mathbf{y})H_l(\zeta(\varepsilon^{-\frac{1}{\alpha}-\chi}\mathbf{y}))d\mathbf{y}.$$

For any $M \in \mathbb{N}$ and any set of real numbers $\{a_1, a_2, \ldots, a_M\}$, denote

$$\xi_\varepsilon = \sum_{j=1}^{M} a_j Y_\varepsilon(t_j, \mathbf{x}_j), \quad (29)$$

where $\{t_1, \ldots, t_M\} \subset \mathbb{R}_+$ and $\{\mathbf{x}_1, \ldots, \mathbf{x}_M\} \subset \mathbb{R}^n$ are arbitrary. In order to apply the Method of Moments to prove the statement of Theorem 2, we need to verify

$$\lim_{\varepsilon\to 0}\mathbb{E}\xi_\varepsilon^p = \begin{cases} 0 & \text{if } p = 2\nu+1,\\ (p-1)!!\left\{\mathbb{E}\Big[\big(\sum_{j=1}^{M}a_jV(t_j,\mathbf{x}_j)\big)^2\Big]\right\}^{\nu} & \text{if } p = 2\nu,\end{cases} \quad (30)$$

where $V(t,\mathbf{x})$ is defined in (22). We remark that calculating the higher (i.e. $p > 2$) moments is needed since ξ_ε is not Gaussian, though the wanted limit is Gaussian. We split ξ_ε into two parts:

$$\xi_\varepsilon = \xi_{\varepsilon,\le N} + \xi_{\varepsilon,>N}, \quad (31)$$

where

$$\xi_{\varepsilon,>N} = \sum_{j=1}^{M} a_j\varepsilon^{-\frac{n\chi}{2}}\sum_{l=N+1}^{\infty}\frac{C_l}{\sqrt{l!}}\int_{\mathbb{R}^n} G(\varepsilon^{\frac{1}{\beta}}t_j, \varepsilon^{\frac{1}{\alpha}}\mathbf{x}_j - \mathbf{y})H_l(\zeta(\varepsilon^{-\frac{1}{\alpha}-\chi}\mathbf{y}))d\mathbf{y}. \quad (32)$$

We first prove that $E[\xi^2_{\varepsilon,>N}] \to 0$ whenever N is chosen large enough. Observe that for any $N \geq m-1$, by (14),

$$\begin{aligned}\mathbb{E}(\xi_{\varepsilon,>N})^2 =&\mathbb{E}\Big[\big(\sum_{j=1}^{M}a_j\varepsilon^{-\frac{n\chi}{2}}\sum_{l=N+1}^{\infty}\frac{C_l}{\sqrt{l!}}\int_{\mathbb{R}^n}G(\varepsilon^{\frac{1}{\beta}}t_j,\varepsilon^{\frac{1}{\alpha}}\mathbf{x}_j-\mathbf{y})H_l(\zeta(\varepsilon^{-\frac{1}{\alpha}-\chi}\mathbf{y}))d\mathbf{y}\big)^2\Big]\\ =&\sum_{j_1,j_2=1}^{M}a_{j_1}a_{j_2}\varepsilon^{-n\chi}\sum_{l=N+1}^{\infty}C_l^2\int_{\mathbb{R}^{2n}}G(\varepsilon^{\frac{1}{\beta}}t_{j_1},\varepsilon^{\frac{1}{\alpha}}\mathbf{x}_{j_1}-\mathbf{y}_1)G(\varepsilon^{\frac{1}{\beta}}t_{j_2},\varepsilon^{\frac{1}{\alpha}}\mathbf{x}_{j_2}-\mathbf{y}_2)\\ &\times R^l(\varepsilon^{-\frac{1}{\alpha}-\chi}(\mathbf{y}_1-\mathbf{y}_2)))d\mathbf{y_1}d\mathbf{y_2}. \end{aligned} \tag{33}$$

By the spectral representation (18) for the k-th power of the covariance function $R(\cdot)$, (33) can be rewritten as

$$\begin{aligned}\mathbb{E}(\xi_{\varepsilon,>N})^2 =&\sum_{j_1,j_2=1}^{M}a_{j_1}a_{j_2}\varepsilon^{-n\chi}\sum_{l=N+1}^{\infty}C_l^2\int_{\mathbb{R}^{2n}}G(\varepsilon^{\frac{1}{\beta}}t_{j_1},\varepsilon^{\frac{1}{\alpha}}\mathbf{x}_{j_1}-\mathbf{y}_1)G(\varepsilon^{\frac{1}{\beta}}t_{j_2},\varepsilon^{\frac{1}{\alpha}}\mathbf{x}_{j_2}-\mathbf{y_2})\\ &\times\int_{\mathbb{R}^n}e^{i<\varepsilon^{-\frac{1}{\alpha}-\chi}(\mathbf{y_1}-\mathbf{y_2}),\lambda>}f^{*l}(\lambda)d\lambda d\mathbf{y_1}d\mathbf{y_2}.\end{aligned}$$

Since

$$\int_{\mathbb{R}^n}e^{i<\varepsilon^{-\frac{1}{\alpha}-\chi}\mathbf{y},\lambda>}G(\varepsilon^{\frac{1}{\beta}}t,\varepsilon^{\frac{1}{\alpha}}\mathbf{x}-\mathbf{y})d\mathbf{y}=e^{i<\varepsilon^{-\chi}\lambda,\mathbf{x}>}E_{\beta,1}\left(-\varepsilon t^{\beta}\theta(\varepsilon^{\frac{1}{\alpha}-\chi}\lambda)\right), \tag{34}$$

$$\begin{aligned}\mathbb{E}(\xi_{\varepsilon,>N})^2 =&\sum_{j_1,j_2=1}^{M}a_{j_1}a_{j_2}\varepsilon^{-n\chi}\sum_{l=N+1}^{\infty}C_l^2\int_{\mathbb{R}^n}e^{i\langle\varepsilon^{-\chi}(\mathbf{x}_{j_1}-\mathbf{x}_{j_2}),\lambda\rangle}f^{*l}(\lambda)\\ &\qquad\times E_{\beta,1}(-\varepsilon t_{j_1}^{\beta}\theta(\varepsilon^{-\frac{1}{\alpha}-\chi}\lambda))E_{\beta,1}(-\varepsilon t_{j_2}^{\beta}\theta(\varepsilon^{-\frac{1}{\alpha}-\chi}\lambda))d\lambda\\ =&\sum_{j_1,j_2=1}^{M}a_{j_1}a_{j_2}\sum_{l=N+1}^{\infty}C_l^2\int_{\mathbb{R}^n}e^{i\langle\mathbf{x}_{j_1}-\mathbf{x}_{j_2},\lambda\rangle}f^{*l}(\varepsilon^{\chi}\lambda)E_{\beta,1}(-\varepsilon t_{j_1}^{\beta}\theta(\varepsilon^{-\frac{1}{\alpha}}\lambda))\\ &\qquad\times E_{\beta,1}(-\varepsilon t_{j_2}^{\beta}\theta(\varepsilon^{-\frac{1}{\alpha}}\lambda))d\lambda.\end{aligned}$$

Because $E_{\beta,1}(-t_{j_1}^{\beta}|\cdot|^{\alpha})E_{\beta,1}(-t_{j_2}^{\beta}|\cdot|^{\alpha})\in L^1(\mathbb{R}^n)$ when $2\alpha>n$ and $f^{*l}(\cdot)$, $l\geq m$, are continuous and uniformly bounded on $\mathbb{R}^n$ (Condition B and Lemma 1 imply that

$$f^{*l}(\lambda)=\int_{\mathbb{R}^n}f^{*m}(\lambda-\eta)f^{*(l-m)}(\eta)d\eta\leq\|B_m\|_\infty\int_{\mathbb{R}^n}f^{*(l-m)}(\eta)d\eta=\|B_m\|_\infty\quad\forall l>m),$$

we have

$$\mathbb{E}(\xi_{\varepsilon,>N})^2\to\sum_{j_1,j_2=1}^{M}a_{j_1}a_{j_2}\sum_{l=N+1}^{\infty}C_l^2f^{*l}(\mathbf{0})\int_{\mathbb{R}^n}e^{i<\lambda,\mathbf{x}_{j_1}-\mathbf{x}_{j_2}>}E_{\beta,1}(-t_{j_1}^{\beta}|\lambda|^{\alpha})E_{\beta,1}(-t_{j_2}^{\beta}|\lambda|^{\alpha})d\lambda \tag{35}$$

when $\varepsilon\to 0$. From (35), for any $\delta>0$, there exists $N_0\in\mathbb{N}$ and $\varepsilon_0>0$ such that

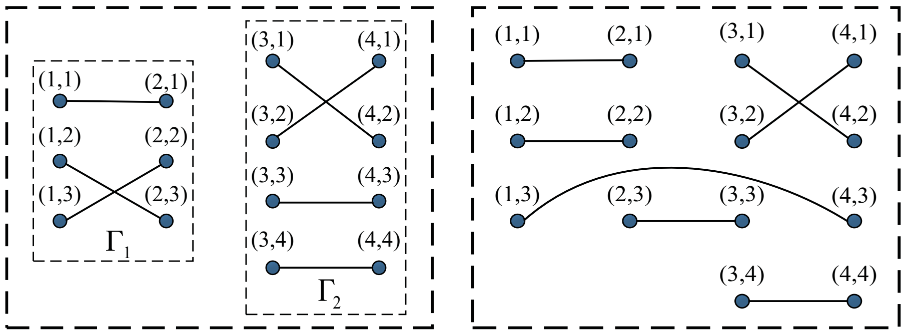
$$\mathbb{E}(\xi_{\varepsilon,>N})^2<\delta,\ \text{for any } N\geq N_0,\ \varepsilon<\varepsilon_0, \tag{36}$$

which implies that it suffices to prove a truncated version of (30) as follows:

$$\lim_{\varepsilon\to 0}\mathbb{E}\xi^p_{\varepsilon,\leq N_0}=\begin{cases}0 & \text{if } p=2\nu+1,\\ (p-1)!!\left\{\mathbb{E}\Big[\big(\sum_{j=1}^{M}a_jV_{m,N_0}(t_j,\mathbf{x}_j)\big)^2\Big]\right\}^{\nu} & \text{if } p=2\nu,\end{cases} \tag{37}$$

where

$$V_{m,N_0}(t,\mathbf{x})=\int_{\mathbb{R}^n}e^{i<\lambda,\mathbf{x}>}\sigma_{m,N_0}E_{\beta,1}\left(-t^{\beta}|\lambda|^{\alpha}\right)W(d\lambda),\ \sigma_{m,N_0}=\left[\sum_{r=m}^{N_0}f^{*r}(\mathbf{0})C_r^2\right]^{\frac{1}{2}}. \tag{38}$$



(a) Regular diagram Γ ($\sharp B(1) = 3$, $\sharp B(2) = 0$, $\sharp B(3) = 4$, $\sharp B(4) = 0$)

(b) Non-regular diagram Γ ($\sharp B(1) = 3$, $\sharp B(2) = 1$, $\sharp B(3) = 3$, $\sharp B(4) = 0$)

FIGURE 2. Illustration of complete diagrams of order $(3, 3, 4, 4)$

By (31) for the definition of $\xi_{\varepsilon,\leq N_0}(= \xi_\varepsilon - \xi_{\varepsilon,>N_0})$ and our rescaling of the initial data,

$$\begin{aligned}
\mathbb{E}(\xi_{\varepsilon,\leq N_0})^p =& \varepsilon^{-\frac{pn\chi}{2}} \sum_{j_1,\ldots,j_p=1}^{M} \sum_{l_1,\ldots,l_p=m}^{N_0} \Big[\prod_{i=1}^{p} a_{j_i} \frac{C_{l_i}}{\sqrt{l_i!}}\Big] \\
&\times \int_{\mathbb{R}^{np}} \Big[\prod_{i=1}^{p} G(\varepsilon^{\frac{1}{\beta}} t_{j_i}, \varepsilon^{\frac{1}{\alpha}} \mathbf{x}_{j_i} - \mathbf{y}_i)\Big] \Big[\mathbb{E}\prod_{i=1}^{p} H_{l_i}(\zeta(\varepsilon^{-\frac{1}{\alpha}-\chi}\mathbf{y}_i))\Big] d\mathbf{y}_1 \ldots d\mathbf{y}_p \\
=& \varepsilon^{-\frac{pn\chi}{2}} \sum_{j_1,\ldots,j_p=1}^{M} \sum_{l_1,\ldots,l_p=m}^{N_0} \Big[\prod_{i=1}^{p} a_{j_i} \frac{C_{l_i}}{\sqrt{l_i!}}\Big] \\
&\times \int_{\mathbb{R}^{np}} \Big[\prod_{i=1}^{p} \varepsilon^{\frac{n}{\alpha}} G(\varepsilon^{\frac{1}{\beta}} t_{j_i}, \varepsilon^{\frac{1}{\alpha}} \mathbf{x}_{j_i} - \varepsilon^{\frac{1}{\alpha}} \mathbf{y}_i)\Big] \Big[\mathbb{E}\prod_{i=1}^{p} H_{l_i}(\zeta(\varepsilon^{-\chi}\mathbf{y}_i))\Big] d\mathbf{y}_1 \ldots d\mathbf{y}_p. \quad (39)
\end{aligned}$$

To analyze $\mathbb{E}(\xi_{\varepsilon,\leq N_0})^p$, we employ the diagram method (see, [10] or [17, p.72]). A graph Γ with $l_1 + \cdots + l_p$ vertices is called a (complete) diagram of order $(l_1, \ldots, l_p)$ if:

(a) the set of vertices V of the graph Γ is of the form $V = \bigcup_{j=1}^{p} W_j$, where $W_j = \{(j, l) : 1 \leq l \leq l_j\}$ is the j-th level of the graph Γ;
(b) each vertex is of degree 1, that is, each vertex is just an endpoint of an edge;
(c) if $((j_1, l_1), (j_2, l_2)) \in \Gamma$ then $j_1 \neq j_2$, that is, the edges of the graph Γ connect only different levels.

Let $\mathrm{T} = \mathrm{T}(l_1, \ldots, l_p)$ be a set of (complete) diagrams of order $(l_1, \ldots, l_p)$. Denote by $E(\Gamma)$ the set of edges of the graph $\Gamma \in \mathrm{T}$. For the edge $e = ((j_1, l_1^{'}), (j_2, l_2^{'})) \in E(\Gamma)$ with $j_1 < j_2$, $1 \leq l_1^{'} \leq l_1$ and $1 \leq l_2^{'} \leq l_2$, we set $d_1(e) = j_1$ and $d_2(e) = j_2$. We call a diagram Γ to be *regular* if its levels can be split into pairs in such a manner that no edge connects the levels belonging to different pairs (see Figure 2(a)). Denote by $\mathrm{T}^* = \mathrm{T}^*(l_1, \ldots, l_p)$ the set of all regular diagrams in T. If $\Gamma \in \mathrm{T}^*$ is a regular diagram, then it implies that p is even and Γ can be divided into $p/2$ sub-diagrams (denoted by $\Gamma_1, \ldots, \Gamma_{p/2}$), which can not be separated again; in this case, we naturally define $d_1(\Gamma_i) \equiv d_1(e)$ and $d_2(\Gamma_i) \equiv d_2(e)$ for any $e \in E(\Gamma_i)$, $i = 1, \ldots, \nu = p/2$. We denote $\sharp E(\Gamma)$ (resp. $\sharp E(\Gamma_j)$) the number of edges belonging to the specific diagram Γ (resp. the sub-diagram Γ_j).

Based on the notations above and let

$$D_p = \{(J, L) : J = (j_1, \ldots, j_p), 1 \leq j_i \leq M,\ L = (l_1, \ldots, l_p), m \leq l_i \leq N_0, i = 1, \ldots, p\},$$

(39) can be rewritten as

$$\mathbb{E}(\xi_{\varepsilon,\leq N_0})^p = \sum_{(J,L)\in D_p} K(J,L) \sum_{\Gamma\in \mathrm{T}^*} F_\Gamma(J,L,\varepsilon) + \sum_{(J,L)\in D_p} K(J,L) \sum_{\Gamma\in \mathrm{T}\setminus\mathrm{T}^*} F_\Gamma(J,L,\varepsilon), \quad (40)$$

where

$$K(J,L) = \prod_{i=1}^{p} a_{j_i} \frac{C_{l_i}}{\sqrt{l_i!}}, \quad (41)$$

$$F_\Gamma(J,L,\varepsilon) = \varepsilon^{-\frac{pn\chi}{2}} \int_{\mathbb{R}^{np}} \Big[\prod_{i=1}^{p} \varepsilon^{\frac{n}{\alpha}} G(\varepsilon^{\frac{1}{\beta}} t_{j_i}, \varepsilon^{\frac{1}{\alpha}}(\mathbf{x}_{j_i} - \mathbf{y}_i))\Big]\Big[\prod_{e\in E(\Gamma)} R(\varepsilon^{-\chi}(\mathbf{y}_{d_1(e)} - \mathbf{y}_{d_2(e)}))\Big] d\mathbf{y}_1 \ldots d\mathbf{y}_p.$$

(37) follows by (40) if we can verify the following two things:

$$\begin{cases} (1) \lim_{\varepsilon\to 0} \sum_{(J,L)\in D_p} K(J,L) \sum_{\Gamma\in\mathrm{T}^*} F_\Gamma(J,L,\varepsilon) = (p-1)!! \Big\{\mathbb{E}\Big[\big(\sum_{j=1}^{M} a_j V_{m,N_0}(t_j,\mathbf{x}_j)\big)^2\Big]\Big\}^{p/2}, \\ (2) \lim_{\varepsilon\to 0} \sum_{(J,L)\in D_p} K(J,L) \sum_{\Gamma\in\mathrm{T}\setminus\mathrm{T}^*} F_\Gamma(J,L,\varepsilon) = 0. \end{cases}$$

Proof of (1): If Γ is a regular diagram in $\mathrm{T}^*(l_1,\ldots,l_p)$, then Γ has an unique decomposition $\Gamma = (\Gamma_1,\ldots,\Gamma_\nu)$, where $\nu = p/2 \in \mathbb{N}$ and $\Gamma_1,\ldots,\Gamma_\nu$ cannot be further decomposed. Accordingly, $F_\Gamma(J,L,\varepsilon)$ can be rewritten as the following $\nu = p/2$ products

$$\begin{aligned} &F_\Gamma(J,L,\varepsilon) \\ =&\varepsilon^{-\frac{pn\chi}{2}} \prod_{i=1}^{\nu} \int_{\mathbb{R}^{2n}} \varepsilon^{\frac{n}{\alpha}} G(\varepsilon^{\frac{1}{\beta}} t_{d_1(\Gamma_i)}, \varepsilon^{\frac{1}{\alpha}}(\mathbf{x}_{d_1(\Gamma_i)} - \mathbf{y}))\varepsilon^{\frac{n}{\alpha}} G(\varepsilon^{\frac{1}{\beta}} t_{d_2(\Gamma_i)}, \varepsilon^{\frac{1}{\alpha}}(\mathbf{x}_{d_2(\Gamma_i)} - \mathbf{y}')) \quad (42) \\ &\times R^{\sharp E(\Gamma_i)}(\varepsilon^{-\chi}(\mathbf{y} - \mathbf{y}'))d\mathbf{y}d\mathbf{y}'. \end{aligned}$$

By

$$R^{\sharp E(\Gamma_i)}(\varepsilon^{-\chi}(\mathbf{y} - \mathbf{y}')) = \varepsilon^{n\chi} \int_{\mathbb{R}^n} e^{i<\mathbf{y}-\mathbf{y}',\lambda>} f^{*\sharp E(\Gamma_i)}(\varepsilon^{\chi}\lambda) d\lambda, \ i = 1,\ldots,\nu,$$

and

$$\int_{\mathbb{R}^n} e^{i<\mathbf{y},\lambda>} \varepsilon^{\frac{n}{\alpha}} G(\varepsilon^{\frac{1}{\beta}}(t_{d_1(\Gamma_i)}, \varepsilon^{\frac{1}{\alpha}}(\mathbf{x}_{d_1(\Gamma_i)} - \mathbf{y}))d\mathbf{y} = e^{i<\lambda,\mathbf{x}_{d_1(\Gamma_i)}>} E_{\beta,1}\left(-\varepsilon t^{\beta}_{d_1(\Gamma_i)} \theta(\varepsilon^{\frac{1}{\alpha}}\lambda)\right),$$

(42) can be rewritten as

$$\begin{aligned} F_\Gamma(J,L,\varepsilon) = \prod_{i=1}^{\nu}\Big[\int_{\mathbb{R}^{2n}} &e^{i\langle\lambda,\mathbf{x}_{d_1(\Gamma_i)} - \mathbf{x}_{d_2(\Gamma_i)}\rangle} E_{\beta,1}\left(-\varepsilon t^{\beta}_{d_1(\Gamma_i)}\theta(\varepsilon^{\frac{1}{\alpha}}\lambda)\right) \\ &\times E_{\beta,1}\left(-\varepsilon t^{\beta}_{d_2(\Gamma_i)}\theta(\varepsilon^{\frac{1}{\alpha}}\lambda)\right) f^{*\sharp E(\Gamma_i)}(\varepsilon^{\chi}\lambda)\, d\lambda\Big]. \end{aligned} \quad (43)$$

Applying the small-scale property illustrated in (28), (43) has the following limit

$$\begin{aligned} &\lim_{\varepsilon\to 0} F_\Gamma(J,L,\varepsilon) \\ =&\prod_{i=1}^{\nu} f^{*\sharp E(\Gamma_i)}(\mathbf{0}) \int_{\mathbb{R}^n} e^{i<\lambda,\mathbf{x}_{d_1(\Gamma_i)} - \mathbf{x}_{d_2(\Gamma_i)}>} E_{\beta,1}(-t^{\beta}_{d_1(\Gamma_i)}|\lambda|^{\alpha}) E_{\beta,1}(-t^{\beta}_{d_2(\Gamma_i)}|\lambda|^{\alpha}) d\lambda, \quad (44) \end{aligned}$$

where $f^{*\sharp E(\Gamma_i)}(\mathbf{0}) < \infty$ follows from Lemma 1 and $\sharp E(\Gamma_i) > n/\kappa$ under Condition B. Meanwhile, because Γ is a regular diagram in $\mathrm{T}(L)$, $K(J,L)$ can be rewritten as follows:

$$K(J,L) = \prod_{i=1}^{\nu} a_{d_1(\Gamma_i)} a_{d_2(\Gamma_i)} \frac{C^2_{\sharp E(\Gamma_i)}}{\sharp E(\Gamma_i)!}. \quad (45)$$

Therefore, by (44) and (45),

$$\begin{aligned}&\lim_{\varepsilon\to 0}\sum_{(J,L)\in D_{2\nu}} K(J,L)\sum_{\Gamma\in \mathrm{T}^*} F_\Gamma(J,L,\varepsilon)\\ &=\sum_{(J,L)\in D_{2\nu}}\sum_{\Gamma\in\mathrm{T}^*}\Big[\prod_{i=1}^{\nu} a_{d_1(\Gamma_i)}a_{d_2(\Gamma_i)}\int_{\mathbb{R}^n} e^{i<\lambda,\mathbf{x}_{d_1(\Gamma_i)}-\mathbf{x}_{d_2(\Gamma_i)}>}\\ &\quad\times E_{\beta,1}(-t^{\beta}_{d_1(\Gamma_i)}|\lambda|^\alpha)E_{\beta,1}(-t^{\beta}_{d_2(\Gamma_i)}|\lambda|^\alpha)d\lambda\Big]\Big[\prod_{i=1}^{\nu} f^{*\sharp E(\Gamma_i)}(\mathbf{0})\frac{C^2_{\sharp E(\Gamma_i)}}{\sharp E(\Gamma_i)!}\Big]. \qquad (46)\end{aligned}$$

Because all components in the first bracket in (46) are independent to the index set L and the summation $\sum_{\Gamma\in\mathrm{T}^*}$ depends only on $\sum_L$, by changing the order of summation, (46) can be rewritten as follows:

$$\begin{aligned}&\lim_{\varepsilon\to 0}\sum_{(J,L)\in D_{2\nu}} K(J,L)\sum_{\Gamma\in \mathrm{T}^*} F_\Gamma(J,L,\varepsilon)\\ &=\sum_{L}\sum_{\Gamma\in\mathrm{T}^*}\sum_{J}\Big[\prod_{i=1}^{\nu} a_{d_1(\Gamma_i)}a_{d_2(\Gamma_i)}\int_{\mathbb{R}^n} e^{i\langle\lambda,\mathbf{x}_{d_1(\Gamma_i)}-\mathbf{x}_{d_2(\Gamma_i)}\rangle}E_{\beta,1}(-t^{\beta}_{d_1(\Gamma_i)}|\lambda|^\alpha)\\ &\qquad\qquad\times E_{\beta,1}(-t^{\beta}_{d_2(\Gamma_i)}|\lambda|^\alpha)\,d\lambda\Big]\\ &\qquad\times\Big[\prod_{i=1}^{\nu} f^{*\sharp E(\Gamma_i)}(\mathbf{0})\frac{C^2_{\sharp E(\Gamma_i)}}{\sharp E(\Gamma_i)!}\Big] \qquad (47)\\ &=\Big[\sum_{j,j'=1}^{M} a_j a_{j'}\int_{\mathbb{R}^n} e^{i\langle\lambda,\mathbf{x}_j-\mathbf{x}_{j'}\rangle}E_{\beta,1}(-t_j^\beta|\lambda|^\alpha)E_{\beta,1}(-t_{j'}^\beta|\lambda|^\alpha)\,d\lambda\Big]^\nu\\ &\quad\times\sum_L\sum_{\Gamma\in\mathrm{T}^*}\Big[\prod_{i=1}^{\nu} f^{*\sharp E(\Gamma_i)}(\mathbf{0})\frac{C^2_{\sharp E(\Gamma_i)}}{\sharp E(\Gamma_i)!}\Big].\end{aligned}$$

To handle the summation $\sum_L\sum_{\Gamma\in\mathrm{T}^*}[\dots]$ in (47), we note that $\prod_{i=1}^{\nu} f^{*\sharp E(\Gamma_i)}(\mathbf{0})\frac{C^2_{\sharp E(\Gamma_i)}}{\sharp E(\Gamma_i)!}$ only depends on $\{\sharp E(\Gamma_i), i=1,\dots,\nu\}$, not on the internal structures of sub-diagrams Γ_i, $i=1,\dots,\nu$. Let s be the number of different integers $r_1,\dots,r_s$ in $\{l_1,\dots,l_{2\nu}\}$ with $m\le r_1<\cdots<r_s\le N_0$, where $1\le s\le\nu$. It implies that the set $\{l_1,\dots,l_{2\nu}\}$ can be split into s subsets $Q_1,\dots,Q_s$ and all elements within Q_i have the common value r_i, $i=1,\dots,s$. For the number of *pairs* within each subset Q_i, we denote it by q_i, which satisfies $q_i\ge 1$, $i=1,\dots,s$, and $q_1+\cdots+q_s=\nu$. Using the notation introduced above, $\sum\limits_L\sum\limits_{\Gamma\in\mathrm{T}^*}[\dots]$ can be rewritten as follows:

$$\begin{aligned}&\sum_L\sum_{\Gamma\in\mathrm{T}^*}\Big[\prod_{i=1}^{\nu} f^{*\sharp E(\Gamma_i)}(\mathbf{0})\frac{C^2_{\sharp E(\Gamma_i)}}{\sharp E(\Gamma_i)!}\Big]\\ &=\sum_{1\le s\le\nu} s!\sum_{m\le r_1<\cdots<r_s=N_0}\sum_{q_1+\cdots+q_s=\nu}\frac{(2\nu)!}{2^\nu q_1!\dots q_s!}(r_1!)^{q_1}\dots(r_s!)^{q_s}\Big[\prod_{i=1}^{s}\big(f^{*r_i}(\mathbf{0})\frac{C^2_{r_i}}{r_i!}\big)^{q_i}\Big]\\ &=(2\nu-1)!!\sum_{1\le s\le\nu} s!\sum_{m\le r_1<\cdots<r_s=N_0}\sum_{q_1+\cdots+q_s=\nu}\frac{\nu!}{q_1!\dots q_s!}\Big[\prod_{i=1}^{s}\big(f^{*r_i}(\mathbf{0})C^2_{r_i}\big)^{q_i}\Big]\\ &=(2\nu-1)!!\Big[\sum_{r=m}^{N_0} f^{*r}(\mathbf{0})C_r^2\Big]^\nu. \qquad (48)\end{aligned}$$

Substituting (48) into (47) yields

$$\lim_{\varepsilon\to 0}\sum_{(J,L)\in D_{2\nu}} K(J,L)\sum_{\Gamma\in \mathrm{T}^*} F_\Gamma(J,L,\varepsilon)$$
$$= (2\nu-1)!!\Big[\sum_{j,j'=1}^{M} a_j a_{j'}\int_{\mathbb{R}^n} e^{i\langle\lambda,\mathbf{x}_j-\mathbf{x}_{j'}\rangle}E_{\beta,1}(-t_j^\beta|\lambda|^\alpha)E_{\beta,1}(-t_{j'}^\beta|\lambda|^\alpha)\,d\lambda\Big]^\nu \quad (49)$$
$$\times\Big[\sum_{r=m}^{N_0} f^{*r}(\mathbf{0})C_r^2\Big]^\nu.$$

By the orthogonal property of the Gaussian white noise measure W (see (11)), the right hand side of (49) is equal to

$$(2\nu-1)!!\Big[\mathbb{E}\Big(\sum_{j=1}^{M} a_j\int_{\mathbb{R}^n} e^{i<\lambda,\mathbf{x}_j>}\sigma_{m,N_0}E_{\beta,1}(-t_j^\beta|\lambda|^\alpha)W(d\lambda)\Big)^2\Big]^\nu \quad (50)$$

with $\sigma_{m,N_0}=(\sum_{r=m}^{N_0} f^{*r}(\mathbf{0})C_r^2)^{\frac{1}{2}}$. The proof of (1) is complete.

Proof of (2): $\lim_{\varepsilon\to 0}\sum_{(J,L)\in D_p} K(J,L)\sum_{\Gamma\in \mathrm{T}\setminus \mathrm{T}^*} F_\Gamma(J,L,\varepsilon)=0.$

By (37), the number of elements in the summation of $\sum_{(J,L)\in D_p}$ is finite, thus it suffices to show that $\lim_{\varepsilon\to 0}F_\Gamma(J,L,\varepsilon)=0$ for arbitrary p, i.e., for each $\Gamma\in \mathrm{T}(l_1,\ldots,l_p)\setminus \mathrm{T}^*$,

$$\varepsilon^{-\frac{pn\chi}{2}}\int_{\mathbb{R}^{np}}\Big[\prod_{i=1}^{p}\varepsilon^{\frac{n}{\alpha}}G(\varepsilon^{\frac{1}{\beta}}t_{j_i},\varepsilon^{\frac{1}{\alpha}}(\mathbf{x}_{j_i}-\mathbf{y}_i))\Big]\Big[\prod_{e\in E(\Gamma)} R(\varepsilon^{-\chi}(\mathbf{y}_{d_1(e)}-\mathbf{y}_{d_2(e)}))\Big]d\mathbf{y}_1\ldots d\mathbf{y}_p \quad (51)$$

$\to 0$ when $\varepsilon\to 0$. Without loss of generality, we prove (51) for $t_{j_i}=1$ and $\mathbf{x}_{j_i}=\mathbf{0}$, $i=1,\ldots,p$, and also just consider the case $l_1\le l_2\le\cdots\le l_p$. Let

$$A_{j,j'}=\big\{e\in E(\Gamma)\mid d_1(e)=j,\ d_2(e)=j'\big\},\ B(i)=\cup_{j'>i}A_{i,j'}, \quad (52)$$

and define $\sharp A_{j,j'}$ and $\sharp B(i)$ to be the numbers of edges in $A_{j,j'}$ and $B(i)$ (see Figure 2), respectively, where $1\le i,\ j<j'\le p$. Based on the notation in (52),

$$\begin{aligned}
F_\Gamma(J,L,\varepsilon)&=\varepsilon^{-\frac{pn\chi}{2}}\int_{\mathbb{R}^{np}}\Big[\prod_{i=1}^{p}\varepsilon^{\frac{n}{\alpha}}G(\varepsilon^{\frac{1}{\beta}},\varepsilon^{\frac{1}{\alpha}}\mathbf{y}_i)\Big]\\
&\qquad\times\Big[\prod_{i;B(i)\neq\phi}\prod_{e\in B(i)}R(\varepsilon^{-\chi}(\mathbf{y}_i-\mathbf{y}_{d_2(e)}))\Big]d\mathbf{y}_1\ldots d\mathbf{y}_p\\
&\le\varepsilon^{-\frac{pn\chi}{2}}\int_{\mathbb{R}^{np}}\Big[\prod_{i=1}^{p}\varepsilon^{\frac{n}{\alpha}}G(\varepsilon^{\frac{1}{\beta}},\varepsilon^{\frac{1}{\alpha}}\mathbf{y}_i)\Big]\\
&\qquad\times\Big[\prod_{i;B(i)\neq\phi}\sum_{e\in B(i)}\frac{1}{\sharp B(i)}R^{\sharp B(i)}(\varepsilon^{-\chi}(\mathbf{y}_i-\mathbf{y}_{d_2(e)}))\Big]d\mathbf{y}_1\ldots d\mathbf{y}_p\\
&\le\varepsilon^{-\frac{pn\chi}{2}}\int_{\mathbb{R}^{np}}\Big[\prod_{i=1}^{p}\varepsilon^{\frac{n}{\alpha}}G(\varepsilon^{\frac{1}{\beta}},\varepsilon^{\frac{1}{\alpha}}\mathbf{y}_i)\Big]\\
&\qquad\times\Big[\prod_{i;B(i)\neq\phi}\sum_{j;A_{i,j}\neq\phi}\frac{1}{\sharp B(i)}R^{\sharp B(i)}(\varepsilon^{-\chi}(\mathbf{y}_i-\mathbf{y}_j))\Big]d\mathbf{y}_1\ldots d\mathbf{y}_p,
\end{aligned} \quad (53)$$

where the first inequality follows from the assumption $R(\cdot)\ge 0$ and the second inequality follows from $\sharp B(i)\le\sharp A_{i,j}$. Henceforth, we denote $G_\varepsilon(\mathbf{y}_i)=\varepsilon^{\frac{n}{\alpha}}G(\varepsilon^{\frac{1}{\beta}},\varepsilon^{\frac{1}{\alpha}}\mathbf{y}_i)$ for all $\mathbf{y}_i$

in (53) and $\widehat{G}_\varepsilon$ to be the Fourier transform of G_ε. To prove (53) $\to 0$, by the spectral representation, it suffices to show that

$$\varepsilon^{-\frac{pn\chi}{2}}\int_{\mathbb{R}^{np}}\Big[\prod_{i=1}^{p}G_\varepsilon(\mathbf{y}_i)\Big]\Big[\prod_{i;B(i)\neq\phi}\int_{\mathbb{R}^n}e^{i<\mathbf{y}_i-\mathbf{y}_{j(i)},\lambda_i>}f^{*\sharp B(i)}(\varepsilon^\chi\lambda_i)\varepsilon^{n\chi}d\lambda_i\Big]d\mathbf{y}_1\dots d\mathbf{y}_p \quad (54)$$

converges to zero when $\varepsilon \to 0$ for each $i \in \{1,\dots,p-1\}$ with $B(i)\neq\phi$ and any $j(i)\in\{j'|A_{i,j'}\neq\phi\}$.

We prove $\lim_{\varepsilon\to 0}(54)=0$ for general non-regular diagrams as follows. By changing the order of integrals,

$$(54) = \varepsilon^{-\frac{pn\chi}{2}}\int_{\mathbb{R}^n}\cdots\int_{\mathbb{R}^n}D(\lambda;\varepsilon)\prod_{i;B(i)\neq\phi}f^{*\sharp B(i)}(\varepsilon^\chi\lambda_i)\varepsilon^{n\chi}d\lambda_i, \quad (55)$$

where $D(\lambda;\varepsilon)=\int_{\mathbb{R}^{np}}\Big[\prod_{i=1}^{p}G_\varepsilon(\mathbf{y}_i)\Big]\Big[\prod_{i;B(i)\neq\phi}e^{i<\mathbf{y}_i-\mathbf{y}_{j(i)},\lambda_i>}\Big]d\mathbf{y}_1\dots d\mathbf{y}_p$. When $\sharp B(i)<l_i$, by Lemma 1, we have

$$f^{*\sharp B(i)}(\lambda)=C_{\sharp B(i)}(\lambda)|\lambda|^{\sharp B(i)\frac{n}{l_i}-n}, \quad (56)$$

where $C_{\sharp B(i)}(\lambda)=B_{\sharp B(i)}(\lambda)|\lambda|^{\sharp B(i)(\kappa-\frac{n}{l_i})}$ and $\lim_{|\lambda|\to 0}C_{\sharp B(i)}(\lambda)=0$ because $\kappa>n/m\geq n/l_i$. When $\sharp B(i)\geq l_i$, $f^{*\sharp B(i)}\in\mathbf{C}(\mathbb{R}^n)$. To summarize, we have

$$f^{*\sharp B(i)}(\lambda)\leq\begin{cases}O(1) & \text{if } \sharp B(i)=l_i,\\ o(|\lambda|^{n(\frac{\sharp B(i)}{l_i}-1)}) & \text{if } 1<\sharp B(i)<l_i\end{cases} \quad (57)$$

when $|\lambda|\to 0$. Thus,

$$(54)\leq\varepsilon^{-\frac{pn\chi}{2}}o(\varepsilon^{\chi n(\sum\frac{\sharp B(i)}{l_i})})Q_\varepsilon, \quad (58)$$

where

$$Q_\varepsilon=\int_{\mathbb{R}^n}\cdots\int_{\mathbb{R}^n}D(\lambda;\varepsilon)\prod_{i;B(i)\neq\phi}|\lambda_i|^{n(\frac{\sharp B(i)}{l_i}-1)}d\lambda_i,$$

which converges to a finite number when $\varepsilon\to 0$ and $\alpha>n/2$. Finally, the convergence of the right hand side of (58) to zero follows by the following inequality ([10, (2.20)])

$$\sum_{i=1}^{p}\frac{\sharp B(i)}{l_i}\geq\frac{p}{2}.$$

The proof of (2) is complete. □

Proof of Theorem 3.
By the solution form (6) and $\int_{\mathbb{R}^n}G(t,\mathbf{x})d\mathbf{x}=1$,

$$\begin{aligned}&T^{\frac{\beta m\kappa}{4}}\Big\{u(Tt,T^{\frac{\beta}{2}}\mathbf{x};h(\zeta(\cdot)))-C_0\Big\}\\ =&T^{\frac{\beta m\kappa}{4}}\Big\{\int_{\mathbb{R}^n}G(Tt,T^{\frac{\beta}{2}}\mathbf{x}-\mathbf{y})\Big[C_0+\sum_{k=m}^{\infty}C_k\frac{H_k(\zeta(\mathbf{y}))}{\sqrt{k!}}\Big]d\mathbf{y}-C_0\Big\}\\ =&\sum_{k=m}^{\infty}T^{\frac{\beta m\kappa}{4}}\frac{C_k}{\sqrt{k!}}\int_{\mathbb{R}^n}G(Tt,T^{\frac{\beta}{2}}\mathbf{x}-\mathbf{y})H_k(\zeta(\mathbf{y}))d\mathbf{y}=:\sum_{k=m}^{\infty}u_{k,T}(t,\mathbf{x}). \end{aligned} \quad (59)$$

By the Slutsky argument [22, p. 6.], Theorem 3 will be proved if we can show that

$$\begin{cases}(1)\ u_{m,T}(t,\mathbf{x})\Rightarrow U_m(t,\mathbf{x}),\\ (2)\ \sum_{k=m+1}^{\infty}u_{k,T}(t,\mathbf{x})\to 0 \text{ in probability}\end{cases} \quad (60)$$

when $T \to \infty$.

Proof of (1): Replacing $H_m(\zeta(\mathbf{y}))$ in the expression of $u_{m,T}(t,\mathbf{x})$ with its Itô-Wiener expansion (15) and using the Fourier transform $\widehat{G}(t,\cdot)$ of $G(t,\cdot)$ in (7), we have

$$\begin{aligned} &u_{m,T}(t,\mathbf{x}) \\ =&T^{\frac{\beta m\kappa}{4}}\frac{C_m}{\sqrt{m!}}\int_{\mathbb{R}^n}G(Tt,T^{\frac{\beta}{2}}\mathbf{x}-\mathbf{y})\left\{\int_{\mathbb{R}^{n\times m}}' e^{i<\mathbf{y},\lambda_1+\cdots+\lambda_m>}\prod_{\sigma=1}^m\sqrt{f(\lambda_\sigma)}W(d\lambda_\sigma)\right\}d\mathbf{y} \\ =&T^{\frac{\beta m\kappa}{4}}\frac{C_m}{\sqrt{m!}}\int_{\mathbb{R}^{n\times m}}' e^{i<T^{\frac{\beta}{2}}\mathbf{x},\lambda_1+\cdots+\lambda_m>}\widehat{G}(Tt,\lambda_1+\cdots+\lambda_m)\prod_{\sigma=1}^m\sqrt{f(\lambda_\sigma)}W(d\lambda_\sigma). \end{aligned} \tag{61}$$

By the definition about $\int_{\mathbb{R}^{n\times m}}'$ in (15) and the self-similarity property $W(T^{-\frac{\beta}{2}}d\lambda)\overset{d}{=}T^{-\frac{n\beta}{4}}W(d\lambda)$, $u_{m,T}$ has the same finite dimensional distributions as $\widetilde{u}_{m,T}$, where

$$\begin{aligned} \widetilde{u}_{m,T}(t,\mathbf{x}) =&\frac{C_m}{\sqrt{m!}}T^{\frac{\beta m(\kappa-n)}{4}}\int_{\mathbb{R}^{n\times m}}' e^{i<\mathbf{x},\lambda_1+\cdots+\lambda_m>}\widehat{G}(Tt,T^{\frac{\beta}{2}}(\lambda_1+\cdots+\lambda_m)) \\ &\times\prod_{\sigma=1}^m\sqrt{f(T^{-\frac{\beta}{2}}\lambda_\sigma)}W(d\lambda_\sigma). \end{aligned} \tag{62}$$

From the isometry property of the multiple Wiener integrals and the integral representation of the limiting field $U_m(t,\mathbf{x})$ in (24),

$$\begin{aligned} &\mathbb{E}|\widetilde{u}_{m,T}(t,\mathbf{x})-U_m(t,\mathbf{x})|^2 \\ =&C_m^2\int_{\mathbb{R}^{nm}}\Big|T^{\frac{\beta m(\kappa-n)}{4}}\widehat{G}(Tt,T^{-\frac{\beta}{2}}(\lambda_1+\cdots+\lambda_m))\prod_{\sigma=1}^m\sqrt{f(T^{-\frac{\beta}{2}}\lambda_\sigma)} \\ &-B(\mathbf{0})^{\frac{m}{2}}\frac{E_{\beta,1}(-t^\beta\frac{\alpha}{2}\mathfrak{m}^{1-\frac{2}{\alpha}}|\lambda_1+\cdots+\lambda_m|^2)}{(|\lambda_1|\dots|\lambda_m|)^{\frac{n-\kappa}{2}}}\Big|^2\prod_{\sigma=1}^m d\lambda_\sigma. \end{aligned} \tag{63}$$

Condition C and the large-scale property (27) allow us to apply the dominated convergence theorem to show that (63) will converge to zero when $T\to\infty$. We note that the convergence in (27) can be shown to be monotone decreasing when $T\uparrow\infty$ for each $t>0$ and $\lambda\in\mathbb{R}^n$. Thus, we get

$$\lim_{T\to\infty}\mathbb{E}|\widetilde{u}_{m,T}(t,\mathbf{x})-U_m(t,\mathbf{x})|^2=0. \tag{64}$$

To summarize, we have proven that $u_{m,T}\overset{d}{=}\widetilde{u}_{m,T}$ and $\widetilde{u}_{m,T}(t,\mathbf{x})\to U_m(t,\mathbf{x})$ in probability when $T\to\infty$. Therefore, the claim (1) follows by the Slutsky argument and the Cramer-Wold theorem.

Proof of (2): By the orthogonal property (14) and (18), we have

$$\begin{aligned} &\mathbb{E}\big[(\sum_{k=m+1}^\infty u_{k,T}(t,\mathbf{x}))^2\big] \\ =&T^{\frac{\beta m\kappa}{2}}\sum_{k=m+1}^\infty C_k^2\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}G(Tt,T^{\frac{\beta}{2}}\mathbf{x}-\mathbf{y})G(Tt,T^{\frac{\beta}{2}}\mathbf{x}-\mathbf{y}')R^k(\mathbf{y}-\mathbf{y}')d\mathbf{y}\,d\mathbf{y}' \\ =&T^{\frac{\beta m\kappa}{2}}\sum_{k=m+1}^\infty C_k^2\int_{\mathbb{R}^n}(\widehat{G}(Tt,\lambda))^2f^{*k}(\lambda)d\lambda \quad \text{(by (18))} \\ =&T^{\frac{\beta(m\kappa-n)}{2}}\Big(\sum_{k=m+1}^{k^*}+\sum_{k=k^*+1}^\infty\Big)C_k^2\int_{\mathbb{R}^n}(\widehat{G}(Tt,T^{-\frac{\beta}{2}}\lambda))^2f^{*k}(T^{-\frac{\beta}{2}}\lambda)d\lambda=:(I)+(II), \end{aligned} \tag{65}$$

where $k^* = \max\{k \in \mathbb{N} \mid k \geq m+1, k\kappa \leq n\}$.
For the case $k^*\kappa < n$, by Lemma 1 and (27),

$$\begin{aligned}\lim_{T\to\infty}(I) =& \lim_{T\to\infty} T^{\frac{\beta(m\kappa-n)}{2}} \sum_{k=m+1}^{k^*} C_k^2 \int_{\mathbb{R}^n} (\widehat{G}(Tt, T^{-\frac{\beta}{2}}\lambda))^2 B_k(T^{-\frac{\beta}{2}}\lambda)|T^{-\frac{\beta}{2}}\lambda|^{k\kappa-n} d\lambda \\ \leq& \lim_{T\to\infty} \sum_{k=m+1}^{k^*} T^{\frac{\beta(m\kappa-k\kappa)}{2}} C_k^2 \, \|B_k\|_\infty \int_{\mathbb{R}^n} \left[E_{\beta,1}(-t^\beta \frac{\alpha}{2}\mathfrak{m}^{1-\frac{2}{\alpha}}|\lambda|^2)\right]^2 |\lambda|^{k\kappa-n} d\lambda \\ \leq& \lim_{T\to\infty} T^{-\frac{\beta\kappa}{2}} \sum_{k=m+1}^{k^*} C_k^2 \, \|B_k\|_\infty \int_{\mathbb{R}^n} \left[E_{\beta,1}(-t^\beta \frac{\alpha}{2}\mathfrak{m}^{1-\frac{2}{\alpha}}|\lambda|^2)\right]^2 |\lambda|^{k\kappa-n} d\lambda = 0.\end{aligned}$$

For the case $k^*\kappa = n$, we still have $\lim_{T\to\infty}(I) = 0$ because

$$\lim_{T\to\infty} T^{\frac{\beta(m\kappa-n)}{2}} C_{k^*}^2 \int_{\mathbb{R}^n} (\widehat{G}(Tt, T^{-\frac{\beta}{2}}\lambda))^2 B_{k^*}(T^{-\frac{\beta}{2}}\lambda) \ln(2 + T^{\frac{\beta}{2}}|\lambda|^{-1}) d\lambda = 0.$$

On the other hand, for any $k > k^*+1$, by Lemma 1, we have $\|f^{*k}\|_\infty \leq \|f^{*(k^*+1)}\|_\infty$, so

$$\lim_{T\to\infty}(II) \leq \lim_{T\to\infty} T^{\frac{\beta(m\kappa-n)}{2}} \sum_{k=k^*+1}^{\infty} C_k^2 \, \|f^{*(k^*+1)}\|_\infty \int_{\mathbb{R}^n} (\widehat{G}(Tt, T^{-\frac{\beta}{2}}\lambda))^2 d\lambda = 0.$$

Therefore, $\lim_{T\to\infty} \mathbb{E}\big[(\sum_{k=m+1}^\infty u_{k,T}(t,\mathbf{x}))^2\big] = 0$ and the claim (2) follows by the Markov inequality. □

Proof of Theorem 4.

The following proof is a hybrid of the proofs of Theorems 2 and 3, we give a full presentation mainly to see how the rescaling of the initial data is proceeded. By the Hermite expansion and the solution form (6), we can rewrite

$$u^\varepsilon(t,\mathbf{x}) = \sum_{k=m}^\infty \varepsilon^{-\frac{\chi m\kappa}{2}} \frac{C_k}{\sqrt{k!}} \int_{\mathbb{R}^n} G(\varepsilon^{\frac{1}{\beta}}t, \mathbf{y}) H_k(\zeta(\varepsilon^{-\frac{1}{\alpha}-\chi}(\varepsilon^{\frac{1}{\alpha}}\mathbf{x} - \mathbf{y}))) d\mathbf{y} =: \sum_{k=m}^\infty I_k^\varepsilon(t,\mathbf{x}). \quad (66)$$

Theorem 4 follows by

$$\begin{cases} (1)\ I_m^\varepsilon(t,\mathbf{x}) \Rightarrow V_m(t,\mathbf{x}), \\ (2)\ \sum\limits_{k=m+1}^{\infty} I_k^\varepsilon(t,\mathbf{x}) \to 0 \text{ in probability} \end{cases} \quad (67)$$

when $\varepsilon \to 0$.

Proof of (1): By substituting the Itô-Wiener expansion (15) for the random field $H_m(\zeta(\cdot))$

into $I_m^\varepsilon(t,\mathbf{x})$ and exchanging the order of integration

$$\begin{aligned}
I_m^\varepsilon(t,\mathbf{x}) &= \frac{C_m}{\sqrt{m!}}\varepsilon^{-\frac{\chi m\kappa}{2}}\int_{\mathbb{R}^n} G(\varepsilon^{\frac{1}{\beta}}t,\mathbf{y})H_m(\zeta(\varepsilon^{-\frac{1}{\alpha}-\chi}(\varepsilon^{\frac{1}{\alpha}}\mathbf{x}-\mathbf{y})))d\mathbf{y}\\
&= \frac{C_m}{\sqrt{m!}}\varepsilon^{-\frac{\chi m\kappa}{2}}\int_{\mathbb{R}^n} G(\varepsilon^{\frac{1}{\beta}}t,\mathbf{y})\int'_{\mathbb{R}^{n\times m}} e^{i\langle\varepsilon^{-\frac{1}{\alpha}-\chi}(\varepsilon^{\frac{1}{\alpha}}\mathbf{x}-\mathbf{y}),\lambda_1+\cdots+\lambda_m\rangle}\\
&\qquad\qquad\qquad\qquad\qquad\qquad\times\prod_{\sigma=1}^m\sqrt{f(\lambda_\sigma)}W(d\lambda_\sigma)d\mathbf{y}\\
&= \frac{C_m}{\sqrt{m!}}\varepsilon^{-\frac{\chi m\kappa}{2}}\int'_{\mathbb{R}^{n\times m}} e^{i\langle\varepsilon^{-\chi}\mathbf{x},\lambda_1+\cdots+\lambda_m\rangle}\widehat{G}(\varepsilon^{\frac{1}{\beta}}t,\varepsilon^{-\frac{1}{\alpha}-\chi}(\lambda_1+\cdots+\lambda_m))\\
&\qquad\qquad\qquad\qquad\qquad\qquad\times\prod_{\sigma=1}^m\sqrt{f(\lambda_\sigma)}W(d\lambda_\sigma)d\mathbf{y}\\
&\stackrel{d}{=} \frac{C_m}{\sqrt{m!}}\varepsilon^{\frac{\chi m(n-\kappa)}{2}}\int'_{\mathbb{R}^{n\times m}} e^{i\langle\mathbf{x},\lambda_1'+\cdots+\lambda_m'\rangle}\widehat{G}(\varepsilon^{\frac{1}{\beta}}t,\varepsilon^{-\frac{1}{\alpha}}(\lambda_1'+\cdots+\lambda_m'))\\
&\qquad\qquad\qquad\qquad\qquad\qquad\times\prod_{\sigma=1}^m\sqrt{f(\varepsilon^\chi\lambda_\sigma')}W(d\lambda_\sigma')\\
&=: \widetilde{I}_m^\varepsilon(t,\mathbf{x}),
\end{aligned}\tag{68}$$

where the last equality follows by the self-similarity property $W(\varepsilon^\chi d\lambda)\stackrel{d}{=}\varepsilon^{\frac{n\chi}{2}}W(d\lambda)$. Now, applying the isometry property of the multiple Wiener integrals to the difference of $\widetilde{I}_m^\varepsilon(t,\mathbf{x})$ and the random field $V_m(t,\mathbf{x})$ in (26), we have

$$\begin{aligned}
&\mathbb{E}|\widetilde{I}_m^\varepsilon(t,\mathbf{x})-V_m(t,\mathbf{x})|^2\\
=&C_m^2\int_{\mathbb{R}^{nm}}\Big|\varepsilon^{\frac{\chi m(n-\kappa)}{2}}\widehat{G}(\varepsilon t^{\frac{1}{\beta}},\varepsilon^{-\frac{1}{\alpha}}(\lambda_1+\cdots+\lambda_m))\prod_{\sigma=1}^m\sqrt{f(\varepsilon^\chi\lambda_\sigma)}\\
&-B(\mathbf{0})^{\frac{m}{2}}E_{\beta,1}(-t^\beta|\lambda_1+\cdots+\lambda_m|^\alpha)(|\lambda_1|\ldots|\lambda_m|)^{\frac{\kappa-n}{2}}\Big|^2\prod_{\sigma=1}^m d\lambda_\sigma\to 0
\end{aligned}\tag{69}$$

when $\varepsilon\to 0$, by Condition C and (28).
By the Markov inequality, (69) implies $\widetilde{I}_m^\varepsilon(t,\mathbf{x})\to V_m(t,\mathbf{x})$ in probability. Because $I_m^\varepsilon(t,\mathbf{x})\stackrel{d}{=}\widetilde{I}_m^\varepsilon(t,\mathbf{x})$, the claim (1) follows by the Cramer-Wold argument.
Proof of (2): From (66), by the orthogonal property (14),

$$\begin{aligned}
&\mathbb{E}\Big(\sum_{k=m+1}^\infty I_k^\varepsilon(t,\mathbf{x})\Big)^2=\sum_{k=m+1}^\infty\mathbb{E}(I_k^\varepsilon(t,\mathbf{x}))^2\\
=&\sum_{k=m+1}^\infty\varepsilon^{-\chi m\kappa}C_k^2\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}G(\varepsilon^{\frac{1}{\beta}}t,\mathbf{y})G(\varepsilon^{\frac{1}{\beta}}t,\mathbf{y}')R^k(\varepsilon^{-\frac{1}{\alpha}-\chi}(\mathbf{y}-\mathbf{y}'))d\mathbf{y}d\mathbf{y}'\\
=&\sum_{k=m+1}^\infty\varepsilon^{-\chi m\kappa}C_k^2\int_{\mathbb{R}^n}(\widehat{G}(\varepsilon^{\frac{1}{\beta}}t,\varepsilon^{-\frac{1}{\alpha}-\chi}\lambda))^2f^{*k}(\lambda)d\lambda\\
=&\Big(\sum_{k=m+1}^{k^*}+\sum_{k=k^*+1}^\infty\Big)\varepsilon^{\chi(n-m\kappa)}C_k^2\int_{\mathbb{R}^n}(\widehat{G}(\varepsilon^{\frac{1}{\beta}}t,\varepsilon^{-\frac{1}{\alpha}}\lambda))^2f^{*k}(\varepsilon^\chi\lambda)d\lambda=:(I)+(II),
\end{aligned}$$

where $k^* = \max\{k \in \mathbb{N} \mid k \geq m+1, k\kappa \leq n\}$.
For the case $k^*\kappa < n$, by Lemma 1,

$$\begin{aligned}\lim_{\varepsilon\to 0}(I) =& \lim_{\varepsilon\to 0}\sum_{k=m+1}^{k^*} \varepsilon^{\chi(n-m\kappa)} C_k^2 \int_{\mathbb{R}^n} (\widehat{G}(\varepsilon^{\frac{1}{\beta}}t, \varepsilon^{-\frac{1}{\alpha}}\lambda))^2 B_k(\varepsilon^{\chi}\lambda)|\varepsilon^{\chi}\lambda|^{k\kappa-n} d\lambda \\ \leq& \lim_{\varepsilon\to 0}\sum_{k=m+1}^{k^*} \varepsilon^{\chi\kappa(k-m)} C_k^2 \,\|B_k\|_\infty \int_{\mathbb{R}^n} \left[E_{\beta,1}(-t^\beta|\lambda|^\alpha)\right]^2 |\lambda|^{k\kappa-n} d\lambda = 0.\end{aligned}$$

For the case $k^*\kappa = n$, we still have $\lim_{\varepsilon\to 0}(I) = 0$ because

$$\lim_{\varepsilon\to 0}\varepsilon^{\chi(n-m\kappa)} C_{k^*}^2 \int_{\mathbb{R}^n} (\widehat{G}(\varepsilon^{\frac{1}{\beta}}t, \varepsilon^{-\frac{1}{\alpha}}\lambda))^2 B_{k^*}(\varepsilon^{\chi}\lambda) \ln(2 + |\varepsilon^{\chi}\lambda|^{-1}) d\lambda = 0.$$

On the other hand, by the assumption $\kappa < n/m$ in Condition C and Lemma 1, for any $k > k^* + 1$, we have $\|f^{*k}\|_\infty \leq \|f^{*(k^*+1)}\|_\infty$, so

$$\lim_{\varepsilon\to 0}\,(II) \leq \lim_{\varepsilon\to 0}\sum_{k=k^*+1}^{\infty} \varepsilon^{\chi(n-m\kappa)} C_k^2 \,\|f^{*(k^*+1)}\|_\infty \int_{\mathbb{R}^n} \left[E_{\beta,1}(-t^\beta|\lambda|^\alpha)\right]^2 d\lambda = 0.$$

Hence, $\lim_{\varepsilon\to 0} \mathbb{E}\big[(\sum_{k=m+1}^{\infty} I_k^{\varepsilon}(t,\mathbf{x}))^2\big] = 0$ and the claim (2) follows by the Markov inequality. □

Acknowledgement

The authors are indebted to the inspiring lectures of Professor N. N. Leonenko at National Taiwan University for the perspective on *Random Fields: modeling, inferences, and applications* in 2006. The authors also thank the anonymous reviewers for their valuable comments and suggestions to make the paper more precise and readable.

References

1. J. M. Angulo, V. V. Anh, R. McVinish, and M. D. Ruiz-Medina, Fractional kinetic equations driven by Gaussian or infinitely divisible noise, *Adv. Appl. Probab.* **37** (2005), 366-392.
2. V. V. Anh and N. N. Leonenko, Non-Gaussian scenarios for the heat equation with singular initial data, *Stoch. Processes Appl.* **84** (1999), 91-114 .
3. V. V. Anh and N. N. Leonenko, Scaling laws for fractional diffusion-wave equations with singular data, *Statis. Probab. Lett.* **48** (2000), 239-252.
4. V. V. Anh and N. N. Leonenko, Spectral analysis of fractional kinetic equations with random data, *J. Statist. Phys.* **104** (2001), 1349-1387.
5. V. V. Anh and N. N. Leonenko, Renormalization and homogenization of fractional diffusion equations with random data, *Probab. Theory Rel. Fields* **124** (2002), 381-408.
6. V. V. Anh, N. N. Leonenko, and L. M. Sakhno, Higher-order spectral densities of fractional random fields, *J. Statist. Phys.* **26** (2003), 789-814.
7. B. Baeumer, M. M. Meerschaert, and M. Naber, Stochastic models for relativistic diffusion, *Phys. Rev. E* **82** (2010), 1132-1136.
8. O. E. Barndorff-Nielsen and N. N. Leonenko, Burgers turbulence problem with linear or quadratic external potential, *J. Appl. Probab.* **42** (2005), 550-565.
9. J. Bertoin, *Subordinators: Examples and Applications.* In Lect. Note in Math. 1717, Springer 1999.
10. P. Breuer and P. Major, Central limit theorems for nonlinear functionals of Gaussian fields. *J. Multiv. Anal.*, **13** (1983), 425-441.
11. R. Carmona, W. C. Masters, and B. Simon, Relativistic Schrodinger operators: Asymptotic behaviour of the eigenfunctions, *J. Funct. Anal.* **91** (1990), 117-142.
12. M. M. Djrbashian; *Harmonic Analysis and Boundary Value Problems in Complex Domain.* Birkhäuser, 1993.
13. R. L. Dobrushin, Gaussian and their subordinated self-similar random generalized fields, *Ann. Probab.* **7** (1979), 1-28.
14. R. L. Dobrushin and P. Major, Non-central limit theorems for nonlinear functionals of Gaussian fields, *Z. Wahrsch. verw. Geb.* **50** (1979), 1-28.

15. P. Doukhan, G. Oppenheim, and M.S. Taqqu, *Theory and Applications of Long-Range Dependence.* Birkhäuser 2003.
16. A. Erdély, W. Magnus, F. Obergettinger, F.G. Tricomi, *Higher Transcendental Functions, Volume 3*, McGraw-Hill 1995.
17. A. V. Ivanov and N. N. Leonenko, *Statistical Analysis of Random Fields*, Kluwer Academic Publishers, Dordrecht, 1989.
18. M. Ya. Kelbert, N. N. Leonenko, and M. D. Ruiz-Medina, Fractional random fields associated with stochastic fractional heat equations, *Adv. Appl. Prob.* **37** (2005), 108-133.
19. Kampe, J. de Feriet, Random solutions of the partial differential equations, in Proc. 3rd Berkeley Symp. Math. Statist. Probab. Volume III (University of California Press, Berkeley, CA, 1955), pp. 199-208.
20. R. Kimmel, N. Sochen, and J. Weickert, Scale-Space and PDE Methods in Computer Vision, Lecture Notes in Computer Science V. 3459. Springer, 2005.
21. A. Kumara, M. M. Meerschaert, P. Vellaisamy, Fractional normal inverse Gaussian diffusion, *Statist. Probab. Lett.* **81** (2011), 146-152.
22. N. N. Leonenko, *Limit Theorems for Random Fields with Singular Spectrum.* Kluwer Academic 1999.
23. N. N. Leonenko and O. O. Mel'nikova, Rescaling and homogenization of solutions of the heat equation with a linear potential and of the corresponding Burgers equation with random data, *Theory Probab. Math. Statist.* **62** (2001), 77–88.
24. N. N. Leonenko and W. A. Woyczyński, Scaling limits of solution of the heat equation with non-Gaussian data, *J. Statist. Phys.* **91** (1998), 423-438.
25. N. N. Leonenko and W. A. Woyczyński, Exact parabolic asymptotics for singular $n-$D Burgers' random fields: Gaussian approximation, *Stochastic Processes Appl.* **76** (1998), 141-165.
26. G.-R. Liu and N.-R. Shieh, Scaling limits for time-fractional diffusion-wave systems with random initial data, *Stoch. Dyn.* **10** (2010), 1-35.
27. G.-R. Liu and N.-R. Shieh, Multi-scaling limits for relativistic diffusion equations with random initial data, *Trans. Amer.Math. Soc.* **367** (2015), no. 5, 3423-3446.
28. P. Major, *Muliple Wiener-Itô Integrals*, Lect. Note in Math. 849, Springer 1981.
29. M. Rosenblatt, Remark on the Burgers equation, *J. Math. Phys.* **9** (1968), 1129-1136.
30. M. D. Ruiz-Medina, J. M. Angulo, and V. V. Anh, Scaling limit solution of the fractional Burgers equation, *Stoch. Process. Appl.* **93** (2001), 285-300.
31. M. Ryznar, Estimates of Green function for relativistic α-stable process, *Potential Anal.* **17** (2002), 1-23.
32. N.-R. Shieh, On time-fractional relativistic diffusion equations, *J. Pseudo-Differ. Oper. Appl.* **3** (2012), 229-237.
33. N.-R. Shieh, Free fields associated with the relativistic operator $(\mathfrak{m} - \sqrt{\mathfrak{m}^2 - \Delta})$, *J. Pseudo-Differ. Oper. Appl.* **3** (2012), 309-319.
34. M. S. Taqqu, Convergence of integrated processes of arbitrary Hermite rank, *Z. Wahrsch. verw. Geb.* **50** (1979), 53-83.
35. M. W. Wong; A contraction semigroup generated by a pseudo-differential operator. *Diff. and Int. Eq.* **5** (1992), 193-200.
36. M. W. Wong; *An Introduction to Pseudo-Differential Operators, 2nd Edition.* World Scientific, 1999.
37. W. Woyczyński, *Burgers-KPZ turbulence*, Göttingen Lectures. Lect. Note in Math. 1700, Springer 1998.

Current address: Department of Mathematics, National Cheng Kung University, Tainan 701, Taiwan
E-mail address: `girenliu@mail.ncku.edu.tw`

Current address: Department of Mathematics, National Taiwan University, Taipei 10617, Taiwan
E-mail address: `shiehnr@ntu.edu.tw`

Received 23/September/2016