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NON-CENTRAL LIMIT THEOREMS AND CONVERGENCE RATES

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This paper is dedicated to Professor N.N. Leonenko on the occasion of his 65-th birthday

Abstract. This paper surveys some recent developments in non-central limit theorems for long-range dependent random processes and fields. We describe an increasing domain framework for asymptotic behavior of functionals of random processes and fields. Recent results on the rate of convergence to the Hermite-type distributions in non-central limit theorems are presented. The use of these results is demonstrated through an application to the case of Rosenblatt-type distributions.

1. Introduction

In this paper we review some developments of non-central asymptotic theory. The main purpose of this article is two-fold. First, we give a survey of some key results on non-central limit theorems for functionals of random processes and fields. Second, we discuss some new results on the rate of convergence in non-central limit theorems and present an application that improves the known results for the Rosenblatt-type case. These results highlight some of Professor Leonenko's contributions in this area.

The case when the summands/integrands are functionals of a long-memory Gaussian process or a long-range dependent random field is of great importance in the asymptotic theory. It was shown that, compared with classical Donsker-Prohorov results, long-range dependence can result in non-Gaussian limits. The non-central limit theorems for functionals of long-memory Gaussian processes were investigated by Rosenblatt [23], Dobrushin and Major [6, 7], Taqqu [24, 25], Giraitis and Surgailis [9, 10], Oppenheim, Haye, and Viano [22], Doukhan, Oppenheim, and Taqqu [8]. For multidimensional versions of this type of results see [1, 17, 21, 27]. Detailed accounts of results on long-range dependent random fields, their asymptotics and statistical applications can be found in the monographs [11, 15] by Professor Leonenko.

There is a very rich literature on non-central limit theorems; therefore in this article we cite only some key publications related to the integral functionals. At the same time there has been little fundamental theoretical study on rates of convergence in non-central limit theorems. The rate of convergence to the Gaussian distribution for a local functionals of Gaussian random fields with long-range dependence was first obtained in [14]. The only publications, which are known to the authors, on the rate of convergence to Hermite-type distributions are [1, 2, 5, 16].

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Non-central limit theorems are not only of their own interest but also have numerous applications. In spatial statistics, models with long-range dependence have been used to study data in physics, environmental sciences, geology and image analysis, just to mention a few examples. Recently, asymptotics of functionals of random fields played a crucial role in investigations of cosmic microwave background radiation, see [19].

The rest of the paper is organized as follows. In Section 2 we recall some basic definitions and formulae. Section 3 discusses non-central limit theorems. Results on the rate of convergence in non-central limit theorems are presented in Section 4. In Section 5 we obtain the upper bound on the rate of convergence to the Rosenblatt-type distribution. Finally, some open problems are presented in Section 6.

2. Notations and definitions

In what follows $|\cdot|$ and $\|\cdot\|$ denote the Lebesgue measure and the Euclidean distance in $\mathbb{R}^d$, respectively.

Definition 1. [24] *A measurable function $L:(0,\infty)\to(0,\infty)$ is called slowly varying if for all $t>0$,*

$$\lim_{\lambda\to\infty}\frac{L(\lambda t)}{L(\lambda)}=1.$$

Definition 2. [4] *A measurable function $g:(0,\infty)\to(0,\infty)$ is said to be regularly varying, denoted $g(\cdot)\in R_\tau$, if there exists τ such that, for all $t>0$, it holds that*

$$\lim_{\lambda\to\infty}\frac{g(\lambda t)}{g(\lambda)}=t^\tau.$$

Definition 3. [4] *Let $L(\cdot)$ be a slowly varying function and let $g:(0,\infty)\to(0,\infty)$ be measurable and $g(x)\to 0$ as $x\to 0$. Then $L(\cdot)$ is said to be slowly varying with remainder of type 2, or that it belongs to the class SR2, if*

$$\forall\lambda>1:\quad \frac{L(\lambda x)}{L(x)}-1\sim k(\lambda)g(x),\quad x\to\infty,$$

for some function $k(\cdot)$.

If there exists λ such that $k(\lambda)\neq 0$ and $k(\lambda\mu)\neq k(\mu)$ for all μ, then $g(\cdot)\in R_\tau$ for some $\tau\le 0$ and $k(\lambda)=ch_\tau(\lambda)$, where

$$h_\tau(\lambda)=\begin{cases}\ln(\lambda), & if\ \tau=0,\\ \frac{\lambda^\tau-1}{\tau}, & if\ \tau\neq 0,\end{cases}$$

and c is a constant.

Definition 4. *The polynomials defined by*

$$H_n(x)=(-1)^n e^{\frac{x^2}{2}}\frac{\mathrm{d}^n}{\mathrm{d}x^n}e^{-\frac{x^2}{2}}$$

are called Hermite polynomials of rank $n\in\mathbb{N}$. The first three Hermite polynomials are $H_0(x)=1$, $H_1(x)=x$, and $H_2(x)=x^2-1$.

The Hermite polynomials form a complete orthogonal system in the Hilbert space

$$L_2(\mathbb{R},\phi(w)\,dw)=\left\{G:\int_{\mathbb{R}}G^2(w)\phi(w)\,\mathrm{d}w<\infty\right\},\quad \phi(w):=\frac{1}{\sqrt{2\pi}}e^{-\frac{w^2}{2}}.$$

An arbitrary function $G(\cdot)\in L_2(\mathbb{R},\phi(w)\,dw)$ admits the mean-square convergent expansion

$$G(w)=\sum_{j=0}^{\infty}\frac{C_jH_j(w)}{j!},\qquad C_j:=\int_{\mathbb{R}}G(w)H_j(w)\phi(w)\,\mathrm{d}w, \tag{1}$$

where, by Parseval's identity,

$$\sum_{j=0}^{\infty} \frac{C_j^2}{j!} = \int_{\mathbb{R}} G^2(w)\phi(w)\,\mathrm{d}w.$$

Definition 5. [4] *Let $G(\cdot) \in L_2(\mathbb{R}, \phi(w)\,dw)$ and assume there exists an integer $\kappa \geq 1$ such that $C_j = 0$, for all $0 \leq j \leq \kappa - 1$, but $C_\kappa \neq 0$. Then κ is called the Hermite rank of $G(\cdot)$ and denoted by $H\,\mathrm{rank}\,G$.*

Definition 6. *Fractional Brownian motion is a continuous-time Gaussian process $B_H(t)$ that starts at zero almost surely, with $EB_H(t) = 0$ and covariance function*

$$EB_H(t)B_H(s) = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H}),$$

for any $t, s \in \mathbb{R}$. The parameter $H \in (0,1)$ is called the Hurst index.

Definition 7. *d-dimensional fractional Brownian sheet is a centered Gaussian random field $\{B_{\mathbf{t}}^{\mathbf{H}} : \mathbf{t} = (t_1, \ldots, t_d) \in \mathbb{R}^d\}$ with Hurst multi-index $\mathbf{H} = (H_1, \ldots, H_d) \in (0,1)^d$. It is equal to zero almost surely on the hyperplanes $\{\mathbf{t} : t_i = 0\}$, $1 \leq i \leq d$, and its covariance function is given by*

$$B_{\mathbf{H}}(\mathbf{t}, \mathbf{s}) = EB_{\mathbf{t}}^{\mathbf{H}} B_{\mathbf{s}}^{\mathbf{H}} = \prod_{i=1}^{d} \frac{|t_i|^{2H_i} + |s_i|^{2H_i} - |t_i - s_i|^{2H_i}}{2}.$$

Consider a measurable mean square continuous zero-mean homogeneous isotropic real-valued random field $\eta(x)$, $x \in \mathbb{R}^d$, defined on a probability space $(\Omega, \mathcal{F}, P)$, with covariance function $\mathrm{B}(r) := \mathbf{Cov}\,(\eta(x), \eta(y))$ where $r := \|x - y\|$, $x, y \in \mathbb{R}^d$. In this case there exists the following representation of $\mathrm{B}(\cdot)$:

$$\mathrm{B}(r) = \int_0^\infty Y_d(rz)\,\mathrm{d}\Phi(z),$$

where $\Phi(\cdot)$ is an isotropic spectral measure, and the function $Y_d(\cdot)$ is defined by

$$Y_d(z) := 2^{(d-2)/2}\Gamma\left(\frac{d}{2}\right) J_{(d-2)/2}(z)\; z^{(2-d)/2}, \quad z \geq 0,$$

$J_{(d-2)/2}(\cdot)$ being the Bessel function of the first kind of order $(d-2)/2$.

Definition 8. *The random field $\eta(x)$, $x \in \mathbb{R}^d$, as defined above, is said to possess an absolutely continuous spectrum if there exists a non-negative function $f(\cdot)$ such that*

$$\Phi(z) = 2\pi^{d/2}\Gamma^{-1}(d/2) \int_0^z u^{d-1} f(u)\,\mathrm{d}u, \quad z \geq 0, \quad u^{d-1} f(u) \in L_1(\mathbb{R}_+).$$

The function $f(\cdot)$ is called the isotropic spectral density function of the field $\eta(x)$.

The field $\eta(\cdot)$ with an absolutely continuous spectrum has the isonormal spectral representation

$$\eta(x) = \int_{\mathbb{R}^d} e^{i(\lambda, x)} \sqrt{f(\|\lambda\|)}\, W(\mathrm{d}\lambda),$$

where $W(\cdot)$ is the complex Gaussian white noise measure on $\mathbb{R}^d$.

In the following we will consider random processes or fields with covariance functions satisfying the following assumption:

Assumption L. *Let $X(x)$, $x \in \mathbb{R}^d$, be a homogeneous isotropic Gaussian random field with $EX(x) = 0$ and the covariance function $B(\cdot)$ such that*

$$B(0) = 1, \quad B(x) = EX(0)X(x) = \|x\|^{-\alpha} L(\|x\|),$$

where $L(\|\cdot\|)$ is a slowly varying function, and $\alpha \in (0, d/\kappa)$, where $\kappa = H\,\mathrm{rank}\,G$ in Definition 5.

It follows from the assumption $\alpha \in (0, d/\kappa)$ that the covariance function $B(\cdot)$ is non-integrable. I this case we will call the random field $X(\cdot)$ strongly dependent (or long-range dependent). For each Hermite rank κ of $G(\cdot)$ we will use a corresponding class of covariance functions in Assumption L. These classes are ordered by inclusion when κ increases.

Definition 9. *Let Y_1 and Y_2 be arbitrary random variables. The uniform (Kolmogorov) metric for the distributions of Y_1 and Y_2 is defined by the formula*

$$\rho\left(Y_1, Y_2\right) = \sup_{z\in\mathbb{R}} \left|P\left(Y_1 \le z\right) - P\left(Y_2 \le z\right)\right|.$$

Definition 10. *Let Y_1 and Y_2 be arbitrary random variables. The total variation metric for the distributions of Y_1 and Y_2 is defined by the formula*

$$\rho_{TV}\left(Y_1, Y_2\right) = \sup_{A\in\mathcal{F}} \left|P\left(Y_1 \in A\right) - P\left(Y_2 \in A\right)\right|.$$

3. Non-central limit theorems

Let X_i, $i = 1, 2, \ldots$, be a stationary Gaussian sequence of identically distributed random variables with $EX_i = 0$ and $EX_i^2 = 1$, $G(\cdot)$ be a function such that $EG(X_i) = 0$, the variance of $G(X_i)$ is finite and $H\,\mathrm{rank}\,G = \kappa$.

Motivated by the dependence structure of some geophysical phenomena [18], Taqqu [24] was one of the first authors to consider non-central limits of the normalized sums

$$Z_N(t) = \frac{1}{d_N}\sum_{i=1}^{[Nt]} G(X_i), \quad 0 \le t \le 1, \tag{2}$$

where N is a positive integer number, $[x]$ denotes the integer part of x, and d_N^2 is asymptotically proportional to $\mathrm{Var}\sum_{i=1}^{N} G(X_i)$.

The necessary condition for the sum (2) to be convergent is that $d_N^2 \sim N^{2H}L(N)$ as $N \to \infty$ for some constant H and some slowly varying function $L(\cdot)$, see [13]. The case $H = 1/2$ is well known and leads to the central limit theorem. Taqqu [24] focused his attention on the case $\frac{1}{2} < H < 1$. Under the assumption $EX_iX_{i+k} \sim k^{-\alpha}L(k)$ as $k \to \infty$ for some slowly varying function $L(\cdot)$ and the constant $\alpha < \frac{1}{\kappa}$, he proved that the sum (2) is convergent and the limit is an a.s. continuous process depending on κ. It was shown that in the case $\kappa \ge 2$ this process is not Gaussian. Note that the conditions used by Taqqu are standard and all the results in this present paper assume those conditions, or their multidimensional versions.

One of the important results of [24] is the reduction theorem:

Theorem 1. *Let the Hermite rank of G be equal to κ. Under the assumptions of Theorem* 4.1 [24], *if, for $N \to \infty$ the finite-dimensional distributions of*

$$Z_{N,\kappa}(t) = \frac{1}{d_N}\sum_{i=1}^{[Nt]} H_\kappa(X_i)$$

converge, then the finite-dimensional distributions of $Z_N(t)$ also converge and their limits are the same.

This theorem allows to consider H_κ instead of G to investigate the finite-dimensional distributions of the limit processes. As a result, it simplifies the theory, as now we only have to deal with Hermite polynomials instead of general functions from $L_2(\mathbb{R}, \phi(w)\,dw)$.

The paper [24] also derives the limits for the cases $\kappa = 1$ and $\kappa = 2$.

Theorem 2. [24] *Let HrankG=1. Under the assumptions of Theorem* 1*, the process $Z_N(t)$, defined by* (2)*, converges weakly, as $N \to \infty$, to $E(XG(X))B_H(t)$, where $B_H(t)$*

is a fractional Brownian motion with parameter H and X is a random variable with the same distribution as $X_i, i = 1, 2, \ldots$.

Theorem 3. [24] *Let $HrankG = 2$. Under the assumptions of Theorem 1, the process $Z_N(t)$, defined by (2), converges weakly, as $N \to \infty$, to $E(X^2G(X))\bar{Z}(t)$, where X is a random variable with the same distribution as $X_i, i = 1, 2, \ldots$, $\bar{Z}(\cdot)$ is a separable, a.s. continuous, semi-stable of order $H = 1 - \alpha$, zero-mean stochastic process with strictly stationary increments, such that $\bar{Z}(0) = 0$ a.s., and $E|Z|^\gamma < \infty$, $\gamma \le \frac{1}{H}$, see* [24]*. The characteristic function of the random vector $(\bar{Z}(t_1), \bar{Z}(t_2), \ldots, \bar{Z}(t_p))$ admits the following representation valid for small values of $|u_1|, |u_2|, \ldots, |u_p|$:*

$$\phi(u_1, u_2, \ldots, u_p) = \exp\left\{ \frac{1}{2} \sum_{j=2}^{\infty} \frac{(2i)^j}{j} \sum_{\substack{s_1+\cdots+s_p=j \\ s_1,\ldots,s_p \ge 0}} \frac{p!}{s_1! s_2! \ldots s_p!} u_1^{s_1} u_2^{s_2} \ldots u_p^{s_p} S_\alpha(a^{(j)}) \right\},$$

where $a^{(j)} \leftrightarrow (t^{(p)}, s^{(p)})$, that is, $a^{(j)} = (a_1, \ldots, a_j)$, where the first s_1 parameters a_i are equal to t_1, the next s_2 parameters are equal to t_2, ..., the last s_j parameters a_i are equal to t_j, and

$$S_\alpha(a^{(j)}) = \int_0^{a_1} \mathrm{d}x_1 \int_0^{a_2} \mathrm{d}x_2 \ldots \int_0^{a_j} \mathrm{d}x_j |x_1-x_2|^{-\alpha}|x_2-x_3|^{-\alpha} \ldots |x_{j-1}-x_j|^{-\alpha}|x_j-x_1|^{-\alpha}.$$

Note that in Theorem 3 the distribution of $\bar{Z}$ coincides with the Rosenblatt distribution. Hence, the limit is non-Gaussian.

In [7] the authors considered a similar sum with X_i replaced by a stationary d-dimensional isotropic random field with mean zero, variance 1 and correlation function

$$r(\mathbf{n}) \sim \|\mathbf{n}\|^{-\alpha} L(\|\mathbf{n}\|)\, a\left(\frac{\mathbf{n}}{\|\mathbf{n}\|}\right), \quad \min_{1\le i\le d} n_i \to \infty,$$

where $\mathbf{n} = (n_1, \ldots, n_d)$ denotes the vector between two points in d-dimensional space, $L(\cdot)$ is a slowly varying function, $a(\cdot)$ is a continuous function on the unit sphere in $\mathbb{R}^d$ and $0 < \alpha < d/\kappa$. The main result of [7] is a theorem that provides the limit random field in terms of multiple Wiener-Itô stochastic integrals:

Theorem 4. *Consider the sum*

$$Z_{\mathbf{n}}^N = N^{\frac{\kappa\alpha}{2}-d}(L(N))^{-\kappa/2} \sum_{i \in B_{\mathbf{n}}^N} G(X_i), \quad 0 < \alpha < d/\kappa, \mathbf{n} \in \mathbb{Z}^d, N = 1, 2, \ldots,$$

where $B_{\mathbf{n}}^N = \{j | j \in \mathbb{Z}^d, n^{(i)}N \le j^{(i)} \le (n^{(i)}+1)N, i = 1, \ldots, d\}$, $L(\cdot)$ is a slowly varying function, and $X = \{X_i, i \in B_n^N\}$ is a stationary d-dimensional random field. Then its finite dimensional distributions converge to those of the random field $Z_{\mathbf{n}}$, given by the formula

$$Z_{\mathbf{n}} = \int'_{\mathbb{R}^{d\kappa}} e^{i(\mathbf{n}, x_1+\cdots+x_\kappa)} \prod_{j=1}^d \frac{e^{i(x_1^j+\cdots+x_\kappa^j)-1}}{i(x_1^j+\cdots+x_\kappa^j)} Z_{Q_0}(\mathrm{d}x_1) \ldots Z_{Q_0}(\mathrm{d}x_\kappa),$$

where $Q_0(A) = \lim_{N\to\infty} \frac{N^\alpha}{L(N)} Q(N^{-1}A)$, with $Q(\cdot)$ being the spectral measure of X, and $\int'_{\mathbb{R}^{d\kappa}}$ denotes the multiple Wiener-Itô stochastic integral.

For the one dimensional continuous case, [25] proved an analogous result for the random variables $\int_0^{rt} G(X(s))\mathrm{d}s$.

Leonenko and Ivanov in their monograph [11] generalized the above results. The main object of their studies were functionals

$$X_r(t) = A(r, d) \int_{\Delta(rt^{1/d})} G(\eta(s))\mathrm{d}s, \quad 0 \le t \le 1, \tag{3}$$

where $\Delta(r)$, $r > 0$, is the homothetic image of the bounded set $\Delta \subset \mathbb{R}^d$, with the centre of homothety at the origin and the coefficient $r > 0$, and $\eta(\cdot)$ is a random field defined on $\mathbb{R}^d$.

For these functionals the conditions for convergence to the Gaussian and non-Gaussian processes were provided. Also, a reduction theorem, analogous to the one obtained in [24], was proved. Once again, this allows to consider only Hermite polynomials of different ranks κ instead of an arbitrary function $G(\cdot)$.

In the non-central limit case, similarly to [7], Ivanov and Leonenko [11] obtained the limit process of (3) in terms of the Wiener-Itô stochastic integrals, where $\Delta(r)$ is a multidimensional ball $v(r) = \{x \in \mathbb{R}^d : \|x\| \le r\}$.

Theorem 5. [11] *Let $\eta(x)$, $x \in \mathbb{R}^d$, be a homogeneous isotropic Gaussian random field with $E\eta(x) = 0$. If Assumption* XXI [11] *holds and $\alpha \in (0, \min\{\frac{d}{\kappa}, \frac{d+1}{2}\})$, then, as $r \to \infty$ the finite-dimensional distributions of*

$$X_{r,\kappa}(t) := r^{(\kappa\alpha)/2-d} L^{-\kappa/2}(r) \int_{v(rt^{1/d})} H_\kappa(\eta(x))\,\mathrm{d}x, \quad t \in [0,1],$$

converge weakly to the finite-dimensional distributions of

$$X_\kappa(t, v(1)) := c_2^{\kappa/2}(d,\alpha)\sqrt{t} \int'_{\mathbb{R}^{d\kappa}} \frac{J_{d/2}(\|\lambda_1 + \cdots + \lambda_\kappa\|\, t^{1/d})}{\|\lambda_1 + \cdots + \lambda_\kappa\|^{d/2}} \frac{W(\mathrm{d}\lambda_1)\ldots W(\mathrm{d}\lambda_\kappa)}{\|\lambda_1\|^{(d-\alpha)/2} \cdots \|\lambda_\kappa\|^{(d-\alpha)/2}}.$$

Further generalization was obtained by Leonenko and Olenko in [17], where an arbitrary set Δ was considered. Let $K_\Delta(\cdot)$ be the Fourier transform of the indicator function of the set Δ:

$$K_\Delta(x) := \int_\Delta e^{i(x,u)}\,\mathrm{d}u, \quad x \in \mathbb{R}^d. \tag{4}$$

Theorem 6. *Let $\eta(x)$, $x \in \mathbb{R}^d$, be a homogeneous isotropic Gaussian random field with $E\eta(x) = 0$. Then, under Assumptions* 1 *and* 2 *from* [17], *as $r \to \infty$ the distributions of*

$$X_{r,\kappa} := r^{(\kappa\alpha)/2-d} L^{-\kappa/2}(r) \int_{\Delta(r)} H_\kappa(\eta(x))\,\mathrm{d}x$$

converge weakly to the distributions of

$$X_\kappa(\Delta) := c_2^{\kappa/2}(d,\alpha) \int'_{\mathbb{R}^{d\kappa}} K_\Delta(\lambda_1 + \cdots + \lambda_\kappa) \frac{W(\mathrm{d}\lambda_1)\ldots W(\mathrm{d}\lambda_\kappa)}{\|\lambda_1\|^{(d-\alpha)/2} \cdots \|\lambda_\kappa\|^{(d-\alpha)/2}}.$$

Remark 1. The probability distribution of $X_2(\Delta)$ is called the Rosenblatt-type distribution. This distribution belongs to the class of Hermite-type distributions and is the most studied distribution from this class. For more information about this distribution we refer to [26] and the references therein.

Remark 2. There are some well-known particular cases of the kernel $K_\Delta(\cdot)$ used in the literature.

In the case of Δ being a multidimensional unit ball $v(1) = \{x \in \mathbb{R}^d : \|x\| \le 1\}$,

$$K_{v(1)}(x) = \int_{v(1)} e^{i(x,u)} du = (2\pi)^{d/2} \frac{J_{d/2}(\|x\|)}{\|x\|^{d/2}}, \quad x \in \mathbb{R}^d.$$

If Δ is the multidimensional rectangle $[\mathbf{a}, \mathbf{b}] := \{x \in \mathbb{R}^d : x_j \in [a_j, b_j],\ j = 1, ..., d\}$ and $a_j < 0 < b_j$, $j = 1, ..., d$, then

$$K_{[\mathbf{a},\mathbf{b}]}(x) = \int_{[\mathbf{a},\mathbf{b}]} e^{i(x,u)} du = \prod_{j=1}^{d} \frac{e^{ib_j x_j} - e^{ia_j x_j}}{ix_j}.$$

The representation (4) allows to compute $K_\Delta(\cdot)$ and generalize classical asymptotic results for specific increasing domains. For example, let us consider the two-dimensional rhombus A with centre at zero. In this case,

$$K_A(x_1, x_2) = \int_A e^{i(x_1u_1+x_2u_2)}\mathrm{d}u_1\mathrm{d}u_2 = |u_1 - u_2 = z; u_1 + u_2 = y|$$

$$= \frac{1}{2}\int_{-1}^{1}\int_{-1}^{1} e^{\frac{iy}{2}(x_1+x_2)}e^{\frac{iz}{2}(x_1-x_2)}\mathrm{d}y\mathrm{d}z = 4\frac{\cos(x_2) - \cos(x_1)}{x_1^2 - x_2^2}.$$

As an alternative to the direct probability approach, [20] offered a new method that is based on the combination of Malliavin calculus and Stein's method. At the present time, the Stein-Malliavin approach is well developed for stochastic processes and is still developing fast. However, many problems concerning non-central limit theorems for random fields remain unsolved. The most general results, that are known to the authors, that concern non-central limit theorems for the functionals of random fields based on this approach can be found in [3]. These results coincide with those of Theorem 4, but were obtained under weaker conditions.

In [12] Leonenko and his co-authors also used the Stein-Malliavin method to investigate weighted functionals of random processes. The following central limit theorem was proved.

Theorem 7. *Consider a Gaussian stationary process $X(t)$ and $G(\cdot) \in L_2(\mathbb{R}, \phi(w)\,dw)$, where $\phi(w) = e^{-\frac{w^2}{2}}/\sqrt{2\pi}$. Consider the random vector*

$$Z_T = W_T^{-1}\int_0^T \mathbf{w}(t)G(X(t))\nu(\mathrm{d}t),$$

where $\mathbf{w}(t) = (w_1(t), \dots, w_d(t))'$, $W_T^2 = diag(W_{iT}^2)_{i=1}^d$, $W_{iT}^2 = \int_0^T w_i^2\nu(\mathrm{d}t)$, and $\nu(\mathrm{d}t)$ represents a counting measure in the case of discrete time and the Lebesgue measure in the case of continuous time. Suppose that conditions (A1)-(A4), (B1)-(B3)*, and* (C) *of* [12] *hold, then for $T \to \infty$, Z_T converges in distribution to the Gaussian random vector Z with zero mean and covariance matrix*

$$B = 2\pi\int_{\mathbb{R}} G^2(x)\phi(x)\mathrm{d}x\int_\Lambda f^{*(j)}(\lambda)\mu(\mathrm{d}\lambda),$$

where $\mu(\cdot)$ is the weak-sense limit of the family of matrix-valued measures introduced in (4.1)[12] *and*

$$f^{*(j)}(\lambda) = \int_{\Lambda^{j-1}} f(\lambda - \lambda_2 - \dots - \lambda_j)\prod_{i=2}^{j} f(\lambda_i)\mathrm{d}\lambda_2\dots\mathrm{d}\lambda_j,$$

where $f(\cdot)$ is a spectral density of $X(\cdot)$ and the set $\Lambda = (-\pi, \pi]$ in the discrete case, and $\Lambda = \mathbb{R}$ in the continuous case.

The approaches taken in [11, 21] complemented this line of investigations for the case of random fields, but used direct probability methods. In particular, [21] investigated the limit behaviour of weighted linear functionals of Gaussian random fields. Limit theorems for random fields, which also hold for the case of stochastic processes, were established. It was shown that for a wide class of weighted functionals, that include the Donsker scheme, the limit is not affected by the location of the spectral singularity. On the other hand, for general schemes, in contrast to the Donsker scheme, the singularity may play a significant role even for the linear case.

4. Rate of convergence in non-central limit theorems

For practical applications, it is not enough to know only the limit of a process, but the speed of convergence to this limit is also required. Leonenko and Anh in [16] used a direct probability approach to obtain the first result for the speed of convergence in a non-central limit theorem for random fields with long-range dependence.

There is only one similar result for the multidimensional case in the literature. In [5] the author considered the generalized increments of fractional Brownian sheet (fBs) that form long-range dependent random fields. The following theorem was obtained:

Theorem 8. *Let $\kappa = HrankG \geq 2$ and $B^{\mathbf{H}}$ be a d-dimensional fBs with multi-Hurst index $\mathbf{H} \in (\mathbf{1} - \mathbf{1}/2\kappa, \mathbf{1})$. Consider the process*

$$V_{\mathbf{N}} := [\mathbf{N}]^{\kappa-1-\kappa\mathbf{H}} \sum_{\mathbf{i}\in[\mathbf{0},\mathbf{N}-\mathbf{1}]} H_\kappa\left([\mathbf{N}]^{\mathbf{H}}\Delta_{\mathbf{i},\mathbf{N}}(B^{\mathbf{H}})\right),$$

where $\Delta_{\mathbf{i},\mathbf{N}}(B^{\mathbf{H}})$ denotes the generalized increments of $B^{\mathbf{H}}$ on the block

$$\Delta_{\mathbf{i},\mathbf{N}} = \left[\frac{\mathbf{i}}{\mathbf{N}}, \frac{\mathbf{i}+\mathbf{1}}{\mathbf{N}}\right] = \prod_{j=1}^{d}\left[\frac{i_j}{N_j}, \frac{i_j+1}{N_j}\right]$$

given by

$$\Delta_{\mathbf{i},\mathbf{N}}(B^{\mathbf{H}}) := \sum_{\epsilon\in\{0,1\}^d} (-1)^{d-\sum_{j=1}^{d}\epsilon_k} B^{\mathbf{H}}_{(i_1+\epsilon_1)/N_1,\dots,(i_d+\epsilon_d)/N_d},$$

where $\mathbf{N} = (N_1,\dots,N_d)$ and for $\mathbf{a} = (a_1,\dots,a_d)$, $[\mathbf{a}]^{\mathbf{b}} = \prod_{i=1}^{d} a_i^{b_i}$.

Then,

$$d_{TV}\left(V_{\mathbf{N}}, Z\right) \leq O\left(\sum_{i=1}^{d} N_i^{(2\kappa-1-2\kappa H_i)/2\kappa}\right), \quad \min_{1\leq i\leq d} N_i \to +\infty,$$

where d_{TV} denotes the total variation distance, and Z is a value of a d-dimensional Hermite sheet $Z^\kappa(\mathbf{t})$ at $(1,\dots,1)$. $Z^\kappa(\mathbf{t}) = I_\kappa(h(\mathbf{t}))$, $t\in[0,1]^d$, where I_κ denotes multiple Wiener–Itô stochastic integral and $h(\mathbf{t})$ is the limit of the expression

$$h_{\mathbf{N}}(\mathbf{t}) = \frac{\mathbf{N}^{\kappa-1}}{\kappa!} \sum_{\mathbf{i}\in[\mathbf{0},(\mathbf{N}-\mathbf{1})\mathbf{t}]} \mathbf{1}^{\otimes\kappa}_{\Delta_{\mathbf{i},\mathbf{N}}}.$$

Finally, in [1], Anh, Leonenko and Olenko obtained the rate of convergence for the functionals of the type (3), where the integration is taken over an arbitrary set Δ. Recall that $X_r(1) = r^{(\kappa\alpha)/2-d}L^{-\kappa/2}(r)\int_{\Delta(r)} G(\eta(x))\,\mathrm{d}x$.

Theorem 9. *Let $H\mathrm{rank}\,G = 2$, then, under Assumption* L *and Assumption* 2 *from* [1], *for any $\varkappa < \frac{1}{3}\min\left(\frac{\alpha(d-2\alpha)}{d-\alpha}, \varkappa_1\right)$,*

$$\rho\left(X_r(1), X_2(\Delta)\right) = o(r^{-\varkappa}), \quad r\to\infty,$$

where $\rho(\cdot)$ denotes Kolmogorov's distance, and

$$\varkappa_1 := 2\min\left(q, \left(\frac{2}{d-2\alpha} + \frac{2}{d+1-2\alpha} + \frac{1}{\upsilon}\right)^{-1}\right).$$

Here α is a parameter in Assumption L, *q gives the order for the upper bound of the slowly varying function with remainder $L(\cdot)$, and υ describes the magnitude of deviations of the spectral density from $c_2(d,\alpha)\left\|\lambda\right\|^{\alpha-d} L\left(1/\left\|\lambda\right\|\right)$ at the origin.*

Additionally, it was proved that the Rosenblatt-type distribution posseses a bounded probability density function.

The results similar to [1] were proved for an arbitrary Hermite rank in [2]. The results were obtained under the assumption that the limiting distribution has to possess a bounded probability density function. Also, [1] used more natural and general assumptions on spectral densities. Namely,

Assumption S. *The random field $\eta(x)$, $x \in \mathbb{R}^d$, has the spectral density*

$$f(\|\lambda\|) = c_2(d,\alpha)\,\|\lambda\|^{\alpha-d}\,L\left(\frac{1}{\|\lambda\|}\right),$$

where

$$c_2(d,\alpha) := \frac{\Gamma\left(\frac{d-\alpha}{2}\right)}{2^\alpha \pi^{d/2}\Gamma\left(\frac{\alpha}{2}\right)},$$

and $L(\|\cdot\|)$ is a locally bounded function which is slowly varying and satisfies for sufficiently large r the condition

$$\left|1-\frac{L(tr)}{L(r)}\right| \le C\,g(r)h_\tau(t),\ \ t\ge 1, \tag{5}$$

where $g(\cdot) \in R_\tau$, $\tau \le 0$, such that $g(x) \to 0$, $x \to \infty$, and $h_\tau(t)$ is defined by (3).

The following rate of convergence was obtained:

Theorem 10. [2] *Under Assumption* S *and Assumption* 1 *from* [2], *if $\tau \in \left(-\frac{d-\kappa\alpha}{2}, 0\right)$, then for any $\varkappa < \frac{1}{3}\min\left(\frac{\alpha(d-\kappa\alpha)}{d-(\kappa-1)\alpha}, \varkappa_1\right)$*

$$\rho\left(X_r(1), X_\kappa(\Delta)\right) = o(r^{-\varkappa}), \quad r\to\infty.$$

If $\tau = 0$ then

$$\rho\left(X_r(1), X_\kappa(\Delta)\right) = g^{\frac{2}{3}}(r), \quad r\to\infty,$$

where $\rho(\cdot)$ denotes Kolmogorov's distance, τ and $g(\cdot)$ are given in Assumption S, *and $\varkappa_1 := \min\left(-2\tau, \frac{1}{\frac{1}{d-2\alpha}+\cdots+\frac{1}{d-\kappa\alpha}+\frac{1}{d+1-\kappa\alpha}}\right)$.*

5. Rate of convergence to Rosenblatt-type distributions

In this section we specify a novel approach suggested in [2] to the case of Rosenblatt-type distributions (compare with the result in [1]).

In what follows we use the symbol C with subscripts to denote various constants which are not important for our discussion.

Theorem 11. *Let Assumptions S and* 1 [2] *hold and $H\mathrm{rank}\, G = 2$.*

If $\tau \in \left(-\frac{d-2\alpha}{2}, 0\right)$ then for any $\varkappa < \frac{1}{3}\min\left(\frac{\alpha(d-2\alpha)}{d-\alpha}, -2\tau, \frac{1}{\frac{1}{d-2\alpha}+\frac{1}{d+1-2\alpha}}\right)$

$$\rho\left(\frac{2}{C_2}X_r(1), X_2(\Delta)\right) = o(r^{-\varkappa}), \quad r\to\infty,$$

where C_2 is a first non-zero coefficient in the Hermite expansion of G.

Proof. Since $H\mathrm{rank}\, G = 2$, it follows that $K_r = \int_{\Delta(r)} G(\eta(x))\,\mathrm{d}x$ can be represented in the space of squared-integrable random variables $L_2(\Omega)$ as

$$K_r = K_{r,2} + S_r := \frac{C_2}{2}\int_{\Delta(r)} H_2(\eta(x))\,\mathrm{d}x + \sum_{j\ge 3}\frac{C_j}{j!}\int_{\Delta(r)} H_j(\eta(x))\,\mathrm{d}x,$$

where C_j are coefficients of the Hermite series (1) of the function $G(\cdot)$.

Notice that $EK_{r,2} = ES_r = EX_2(\Delta) = 0$, and

$$X_{r,2} = \frac{2\,K_{r,2}}{C_2\, r^{d-\alpha} L(r)}.$$

It follows from Assumption 1 [2] that $|L(u)/u^\alpha| = |B(u)| \le B(0) = 1$. Thus, by estimator (17) in [17],

$$\mathbf{Var}\, S_r \le |\Delta|^2 r^{2d-2\alpha} L^2(r) \sum_{j\ge 3} \frac{C_j^2}{j!} \int_0^{diam\{\Delta\}} z^{-2\alpha} \frac{L^2\,(rz)}{L^2(r)} \frac{L\,(rz)}{(rz)^\alpha} \psi_\Delta(z)\,\mathrm{d}z.$$

We represent the last integral as the sum of two integrals I_1 and I_2 with the ranges of integration $[0, r^{-\beta_1}]$ and $(r^{-\beta_1}, diam\,\{\Delta\}]$ respectively, where $\beta_1 \in (0,1)$.

Using the same arguments as in [2], one can estimate these integrals as

$$I_1 \le \frac{C\, r^{-\beta_1(d-2\alpha-\delta)}}{(d-2\alpha-\delta)\,|\Delta|},$$

and

$$I_2 \le C \cdot o(r^{-(\alpha-\delta)(1-\beta_1)}),$$

when r is sufficiently large and δ is an arbitrary number in $(0, \min(\alpha, d-2\alpha))$.

Therefore, we obtain that for sufficiently large r

$$\mathbf{Var}\, S_r \le C r^{2d-2\alpha} L^2(r) r^{-\frac{\alpha(d-2\alpha)}{d-\alpha}+\delta}.$$

It follows from Lemma 3 [1] that

$$\rho\,(X_2(\Delta)+\varepsilon, X_2(\Delta)) \le \varepsilon \sup_{z\in\mathbb{R}} p_{X_2(\Delta)}\,(z) \le \varepsilon\, C.$$

Similarly to [2], we get

$$\rho\left(\frac{2\,K_r}{C_2\, r^{d-\alpha} L(r)}, X_2(\Delta)\right) \le \varepsilon_1\, C + \varepsilon_1^{-2} \mathbf{Var}\,(X_{r,2} - X_2(\Delta)) + C\, r^{-\frac{\alpha(d-2\alpha)}{3(d-\alpha)}+\delta}.$$

Now we show how to estimate $\mathbf{Var}\,(X_{r,2} - X_2(\Delta))$.

Using the arguments from [2], by the isometry property of multiple stochastic integrals

$$R_r := \frac{\mathbb{E}\,|X_{r,2} - X_2(\Delta)|^2}{c_2^2(d,\alpha)} = \int_{\mathbb{R}^{2d}} \frac{|K_\Delta(\lambda_1+\lambda_2)|^2\,(Q_r(\lambda_1,\lambda_2)-1)^2}{\|\lambda_1\|^{d-\alpha}\,\|\lambda_2\|^{d-\alpha}}\,\mathrm{d}\lambda_1\,\mathrm{d}\lambda_2,$$

where

$$Q_r(\lambda_1,\lambda_2) := \frac{r^{(\alpha-d)}}{L(r)\ c_2(d,\alpha)} \left[\|\lambda_1\|^{d-\alpha}\,\|\lambda_2\|^{d-\alpha} f\left(\frac{\|\lambda_1\|}{r}\right) f\left(\frac{\|\lambda_2\|}{r}\right)\right]^{1/2}.$$

Let us rewrite the integral R_r as the sum of two integrals I_3 and I_4 with the integration regions $A(r) := \{(\lambda_1,\lambda_2) \in \mathbb{R}^{2d} :\ \max_{i=1,2}(||\lambda_i||) \le r^\gamma\}$ and $\mathbb{R}^{2d} \setminus A(r)$ respectively, where $\gamma \in (0,1)$. Our intention is to use the monotone equivalence property of regularly varying functions in the regions $A(r)$.

Following the steps from [2], we obtain for sufficiently large r

$$I_3 \le C\, g^2(r) \int_{A(r)\cap\mathbb{R}^{2d}} \frac{h_\tau^2\left(\frac{1}{\|\lambda_1\|}\right)\cdot \max\left(\|\lambda_1\|^{-\delta}, \|\lambda_1\|^{\delta}\right)|K_\Delta(\lambda_1+\lambda_2)|^2\,\mathrm{d}\lambda_1\mathrm{d}\lambda_2}{\|\lambda_1\|^{d-\alpha}\,\|\lambda_2\|^{d-\alpha}}$$
$$\le C\, g^2(r),$$

and

$$I_4 \le C r^{-(\gamma-\gamma_0)(d-2\alpha-2\delta)} + C\, r^{-\gamma_0(d+1-2\alpha-2\delta)},$$

where $\gamma > \gamma_0 > 0$. For any $\delta > 0$ we can estimate $g(r)$ as $g(r) \le C\, r^{\tau+\delta}$. Hence, by choosing $\varepsilon_1 := r^{-\beta}$, we obtain

$$\rho\left(\frac{2\,K_r}{C_2\, r^{d-\alpha}L(r)}, X_2(\Delta)\right) \le C\left(r^{-\frac{\alpha(d-2\alpha)}{3(d-\alpha)}+\delta} + r^{-\beta} + r^{2\tau+2\delta+2\beta}\right.$$
$$\left. r^{-(\gamma-\gamma_0)(d-2\alpha-2\delta)+2\beta} + r^{-\gamma_0(d+1-2\alpha-2\delta)+2\beta}\right).$$

Therefore, for any $\tilde{\varkappa}_1 \in (0, 3\varkappa_0)$ one can choose a sufficiently small $\delta > 0$ such that

$$\rho\left(\frac{2\,K_r}{C_2\, r^{d-\alpha}L(r)}, X_2(\Delta)\right) \le C r^{\delta}\left(r^{-\frac{\alpha(d-2\alpha)}{3(d-\alpha)}} + r^{-\frac{\tilde{\varkappa}_1}{3}}\right), \tag{6}$$

where

$$\varkappa_0 := \sup_{\substack{1>\gamma>\gamma_0>0\\ \beta>0}} \min\left(\beta, -2\tau-2\beta, (\gamma-\gamma_0)(d-2\alpha)-2\beta, \gamma_0(d+1-2\alpha)-2\beta\right).$$

Note, that for fixed $\gamma \in (0,1)$

$$\sup_{\gamma_0\in(0,\gamma)} \min\left((\gamma-\gamma_0)(d-2\alpha), \gamma_0(d+1-2\alpha)\right) = \frac{\gamma}{\frac{1}{d-2\alpha}+\frac{1}{d+1-2\alpha}},$$

and

$$\sup_{\gamma\in(0,1)} \frac{\gamma}{\frac{1}{d-2\alpha}+\frac{1}{d+1-2\alpha}} = \frac{1}{\frac{1}{d-2\alpha}+\frac{1}{d+1-2\alpha}}.$$

Let us denote

$$\varkappa_1 := \min\left(-2\tau, \frac{1}{\frac{1}{d-2\alpha}+\frac{1}{d+1-2\alpha}}\right).$$

Then $\varkappa_0 = \sup_{\beta>0}\min\left(\beta, \varkappa_1-2\beta\right) = \frac{\varkappa_1}{3}$.

Thus, $\tilde{\varkappa}_1$ can be selected from the interval $(0, \varkappa_1)$. Therefore, by (6) the upper bound on the rate of convergence is $o(r^{-\varkappa})$, where $\varkappa < \min\left(\frac{\alpha(d-2\alpha)}{3(d-\alpha)}, \frac{\varkappa_1}{3}\right)$, and the statement of the theorem follows. □

Remark 3. Note that Assumption S is weaker than Assumption 2 in [1]. It is formulated only in terms of the slowly varying function $L(\cdot)$. Contrary to Assumption 2 in [1] it does not require additional terms to bound deviations from $L(\cdot)$.

6. Conclusions

The results presented in the paper pose some new problems and provide a theoretical framework for studying more complex models. It would be interesting:

- to generalize the approaches in Section 3 and [11, 12, 21] and obtain non-central limit theorems for weighted functionals of Hermite polynomials of random fields when the Hermite rank $\kappa > 1$;
- to derive the rate of convergence for functionals of vector random fields under different assumptions on the covariance functions depicting long-range dependence (consult [17]);
- to obtain sharp rates of convergence at least for some cases when $\kappa > 1$.

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