EXISTENCE AND UNIQUENESS OF MILD SOLUTION
TO STOCHASTIC HEAT EQUATION
WITH WHITE AND FRACTIONAL NOISES

YU. MISHURA, K. RALCHENKO, G. SHEVCHENKO

ABSTRACT. We prove the existence and uniqueness of a mild solution for a class of non-autonomous parabolic mixed stochastic partial differential equations defined on a bounded open subset $D \subset \mathbb{R}^d$ and involving standard and fractional $L^2(D)$-valued Brownian motions. We assume that the coefficients are homogeneous, Lipschitz continuous and the coefficient at the fractional Brownian motion is an affine function.

Key words and phrases. Fractional Brownian motion, stochastic partial differential equation, Green's function.

1. INTRODUCTION

Let $(\Omega, \mathcal{F}, \mathsf{P})$ be a complete probability space. For a fixed $T > 0$ let $\mathbb{F} = \{\mathcal{F}\}_{t \in [0,T]}$ be a filtration satisfying the standard assumptions. Let $\beta \in (0,1)$ be fixed throughout the paper. Assume that $D \subset \mathbb{R}^d$ is a bounded domain with boundary ∂D of class $C^{2+\beta}$.

We consider the following stochastic partial differential equation with boundary conditions

$$du(x,t) = \Big(\mathrm{div}\big(k(x,t)\nabla u(x,t)\big) + f\big(u(x,t)\big)\Big)dt + g\big(u(x,t)\big)W(x,dt)$$
$$+ h\big(u(x,t)\big)W^H(x,dt), \qquad (x,t) \in D \times (0,T], \tag{1}$$

$$u(x,0) = \varphi(x), \qquad x \in D, \tag{2}$$

$$\frac{\partial u(x,t)}{\partial n(k)} = 0, \qquad (x,t) \in \partial D \times (0,T]. \tag{3}$$

Here W is an $L^2(D)$-valued Wiener process and W^H is an $L^2(D)$-valued fractional Brownian motion with the Hurst index $H \in (1/2, 1)$. Furthermore, $k = \{k_{i,j}\} \colon D \to \mathbb{R}^{d \times d}$ is a matrix-valued field, consequently,

$$\mathrm{div}\big(k(x,t)\nabla u(x,t)\big) = \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\Big(k_{i,j}(x,t)\frac{\partial}{\partial x_j}u(x,t)\Big);$$

$n(k)(x) \coloneqq k(x,t)n(x)$ denotes the conormal vector-field, and the relation (3) stands for conormal derivative of u relative to k, that is

$$\frac{\partial u(x,t)}{\partial n(k)} = \sum_{i,j=1}^{d} k_{i,j}(x,t)n_i(x)\frac{\partial}{\partial x_j}u(x,t),$$

where $n(x) \in \mathbb{R}^d$ is an outer normal vector to ∂D. We are interested in the existence and uniqueness of a mild solution to (1)–(3). The precise statement of the problem and the definition of a mild solution will be given in Sections 2 and 3.

In the pure Wiener case, where $h = 0$, the problem was investigated by Sanz-Solé and Vuillermot [11], who introduced three different notions for solutions (namely, variation solutions of the first and the second kind and mild solutions) and showed their indistinguishability. Also, they proved the existence, the uniqueness and the pointwise boundedness of the moments along with the spatial Sobolev regularity of such solutions. Later their results for mild solutions were improved in several directions by Veraar [15, Sec. 8]. In particular, he proved the existence and uniqueness of a mild solution in the case of random coefficients, depending also on (x, t).

The pure "fractional" case, where $g = 0$, was studied in [10, 12]. In [10] the existence, uniqueness and indistinguishability of two types of variational solutions were proved, assuming that the coefficients f and h are Lipschitz continuous and the derivative of h is Hölder continuous. In [12] the authors proved the existence of a mild solution and established its relation with the variational solution of type II from [10] and the Hölder continuity of its sample paths. When h is an affine function, they also proved the uniqueness of a mild solution and the indistinguishability of mild and variational solutions.

In this paper we will consider mild formulation of the stochastic heat equation (1) with white and fractional noises:

$$u(\cdot, t) = U(t, 0)\varphi(\cdot) + \int_0^t U(t, s)\big(f(u(\cdot, s))\, ds + $$
$$+ g(u(\cdot, s))dB(\cdot, s) + h(u(\cdot, s))dB^H(\cdot, s), \tag{4}$$

where $U(t, s) = \exp\left\{\int_s^t A_u du\right\}$ is the evolution family on $L^2(D)$ corresponding to the elliptic operator $A_t\psi(\cdot) = \operatorname{div}\big(k(\cdot, t)\nabla\psi(\cdot)\big)$ with Neumann boundary conditions (3); the precise formulations will be given later. We show that equation (1) has a unique mild solution. The conditions on the coefficients are similar to those of [11, 12]. In particular, the functions f and g are assumed to be Lipschitz continuous, and h is assumed to be an affine function. In order to analyze the equation with two different noises (and, consequently, with two different types of stochastic integrals), we replace the fractional Brownian motion by a smooth process, transforming the equation (1) into a stochastic partial differential equation with random drift driven by a Wiener process. This approach was developed for ordinary stochastic differential equations involving both Wiener process and fractional Brownian motion in the article [8]. In [13] it was applied to mixed stochastic delay equations.

We organize this article in the following way. In Section 2, we formulate the assumptions, define $L^2(D)$-valued Wiener and fractional Brownian processes and introduce the corresponding stochastic integrals. Also, this section contains some properties of Green's function associated with our equation. In Section 3 we define a mild solution and prove its existence and uniqueness. In Appendix, we collect auxiliary estimates for solutions that are used for the proving of main result.

2. Preliminaries

2.1. Assumptions on the coefficients and on the initial value.

(A1) Assumptions on k and n:
  (i) $k_{i,j} = k_{j,i}$ for all $i, j = 1, \ldots, d$;
  (ii) $k_{i,j} \in C^{\beta, \beta'}(\overline{D} \times [0, T])$ for some $\beta' \in (\frac{1}{2}, 1]$ and for all $i, j = 1, \ldots, d$;
  (iii) $\frac{\partial}{\partial x_l} k_{i,j} \in C^{\beta, \beta/2}(\overline{D} \times [0, T])$ for all $i, j, l = 1, \ldots, d$;

(iv) there exists $\underline{k} > 0$ such that

$$\sum_{i,j=1}^{d} k_{i,j}(x,t)q_i q_j \geq \underline{k}|q|^2,$$

for all $x \in \overline{D}$, $t \in [0,T]$, $q \in \mathbb{R}^d$ (here $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^d$);

(v) $(x,t) \mapsto \sum_{i=1}^{d} k_{i,j}(x,t)n_i(x) \in C^{1+\beta,(1+\beta)/2}(\partial D \times [0,T])$ for each j;

(vi) the conormal vector-field $(x,t) \mapsto n(k)(x,t) = k(x,t)n(x)$ is outward point-ing, nowhere tangent to ∂D for every t.

(A2) The initial condition satisfies $\varphi \in C^{2+\beta}(\overline{D})$ and the conormal boundary condi-tion (3) relative to k.

(A3) $f, g \colon \mathbb{R} \to \mathbb{R}$ are Lipschitz continuous functions.

(A4) $h \colon \mathbb{R} \to \mathbb{R}$ is an affine function.

2.2. **Norms and spaces.** Let $\|\cdot\|_2$ and $\|\cdot\|_\infty$ be the norms in $L^2(D)$ and $L^\infty(D)$ respectively. For $\alpha \in (0,1)$ denote by $\mathcal{B}^{\alpha,2}(0,T;L^2(D))$ the Banach space of Lebesgue-measurable mappings $u \colon [0,T] \to L^2(D)$ endowed with the norm

$$\|u\|^2_{\alpha,2,T} := \left(\sup_{t \in [0,T]} \|u(t)\|_2 \right)^2 + \int_0^T \left(\int_0^t \frac{\|u(t) - u(s)\|_2}{(t-s)^{\alpha+1}} \, ds \right)^2 dt < \infty.$$

Denote also for $u \colon [0,T] \to L^2(D)$

$$\|u\|_{\alpha,1,T} := \int_0^T \left(\frac{\|u(t)\|_2}{t^\alpha} + \int_0^t \frac{\|u(t) - u(s)\|_2}{(t-s)^{\alpha+1}} \, ds \right) dt.$$

For $f \colon [0,T] \to \mathbb{R}$ and $\alpha \in (0,1)$ define a seminorm

$$\|f\|_{\alpha,0;t} = \sup_{0 \leq u < v < t} \left(\frac{|f(v) - f(u)|}{(v-u)^{1-\alpha}} + \int_u^v \frac{|f(u) - f(z)|}{(z-u)^{2-\alpha}} \, dz \right).$$

2.3. **Green's function.** Let $G \colon \{(x,t,y,s) : 0 \leq s < t \leq T, x,y \in \overline{D}\} \to \mathbb{R}$ be the par-abolic Green's function associated with the principal part of (1). It is known from [3, 4] that under assumptions (A1) and (A2) G is a continuous function, twice continuously differentiable in x, once continuously differentiable in t. For every $(y,s) \in D \times (0,T]$ it is a classical solution to the linear initial-boundary value problem

$$\partial_t G(x,t;y,s) = \operatorname{div}\big(k(x,t)\nabla_x G(x,t;y,s)\big), \qquad (x,t) \in D \times (0,T],$$

$$\frac{\partial G(x,t;y,s)}{\partial n(k)} = 0, \qquad (x,t) \in \partial D \times (0,T],$$

$$\int_D G(\cdot,s;y,s)\varphi(y)\,dy := \lim_{t \downarrow s} \int_D G(\cdot,t;y,s)\varphi(y)\,dy = \varphi(\cdot).$$

Moreover, G satisfies the heat kernel estimates

$$\left| \partial_x^\gamma \partial_t^\delta G(x,t;y,s) \right| \leq C(t-s)^{-\frac{1}{2}(d+|\gamma|+2\delta)} \exp\left\{ -C\frac{|x-y|^2}{t-s} \right\}$$

for $\gamma = (\gamma_1, \ldots, \gamma_d)$, $\gamma_1, \ldots, \gamma_d, \delta \in \mathbb{N} \cup \{0\}$, and $|\gamma| + 2\delta \leq 2$ with $|\gamma| = \sum_{j=1}^d \gamma_j$. In particular, for $|\gamma| = \delta = 0$, we have

$$|G(x,t;y,s)| \leq C(t-s)^{-\frac{d}{2}} \exp\left\{ -C\frac{|x-y|^2}{t-s} \right\}. \tag{5}$$

We shall refer to (5) as the *Gaussian property* of G.

The evolution family corresponding to the operator $A_t\psi(\cdot) = \text{div}\left(k(\cdot,t)\nabla\psi(\cdot)\right)$ with Neumann boundary conditions (3) is defined as

$$U(t,s)\psi(x) = \int_D G(x,t;y,s)\psi(y)dy. \tag{6}$$

From the Gaussian estimates (5) it can be easily shown that the family $U(t,s)$ is of contractive type on $L^2(D)$. Indeed, denoting by ζ a standard Gaussian vector in $\mathbb{R}^d$, we have from (5) for $\psi \in L^2(D)$

$$|U(t,s)\psi(x)| \leq C\mathsf{E}\big|\psi(x + c\sqrt{t-s}\zeta)\mathbb{1}_D(x + c\sqrt{t-s}\zeta)\big|$$

with some $c > 0$, whence

$$\|U(t,s)\psi\|_2^2 \leq C\int_D \left(\mathsf{E}\big|\psi(x + c\sqrt{t-s}\zeta)\big|\mathbb{1}_D(x + c\sqrt{t-s}\zeta)\right)^2 dx$$

$$\leq C\mathsf{E}\int_D \psi(x + c\sqrt{t-s}\zeta)^2\mathbb{1}_D(x + c\sqrt{t-s}\zeta)dx \leq C\mathsf{E}\|\psi\|_2^2.$$

In the proposition below we collect other useful estimates for G, see equations (3.4), (3.5) and (3.36) in [12].

Proposition 2.1 [12]. *Under assumption (A1), for all $x,y \in D$ and $\delta \in (\frac{d}{d+2}, 1)$, G satisfies the following inequalities.*

(i) For all $0 < r < v < t < T$ and some $t^* \in (r,v)$,

$$|G(x,t;y,v) - G(x,t;y,r)| \leq (t-v)^{-\delta}(v-r)^\delta(t-t^*)^{-d/2}\exp\left\{-C\frac{|x-y|^2}{t-t^*}\right\}. \tag{7}$$

(ii) For all $0 < v < s < t < T$ and some $v^* \in (s,t)$,

$$|G(x,t;y,v) - G(x,s;y,v)| \leq (t-s)^\delta(s-v)^{-\delta}(v^*-v)^{-d/2}\exp\left\{-C\frac{|x-y|^2}{v^*-v}\right\}. \tag{8}$$

(iii) For all $0 < r < v < s < t < T$ and some $v^*, r^* \in (s,t)$,

$$|G(x,t;y,v) - G(x,s;y,v) - G(x,t;y,r) - G(x,s;y,r)| \leq$$

$$\leq (t-s)^\delta(s-v)^{-1}(v-r)^{1-\delta} \times$$

$$\times \left((v^*-v)^{-d/2}\exp\left\{-C\frac{|x-y|^2}{v^*-v}\right\} + (r^*-r)^{-d/2}\exp\left\{-C\frac{|x-y|^2}{r^*-r}\right\}\right). \tag{9}$$

2.4. $L^2(D)$-valued Wiener and fractional Brownian processes. Let $\{\lambda_j, j \in \mathbb{N}\}$ and $\{\mu_j, j \in \mathbb{N}\}$ be the sequences of positive real numbers and $\{e_i, i \in \mathbb{N}\}$ be an orthonormal basis of $L^2(D)$. Assume that

(A5) $\displaystyle\sup_j\|e_j\|_\infty < \infty, \quad \sum_{j=1}^\infty \lambda_j < \infty, \quad \text{and} \quad \sum_{j=1}^\infty \mu_j^{1/2} < \infty.$

Let $B_j = \{B_j(t), t \geq 0\}$, $j \in \mathbb{N}$, be a sequence of one-dimensional, independent Brownian motions defined on $(\Omega, \mathcal{F}, \mathbb{F}, \mathsf{P})$. Let us define $L^2(D)$-valued Wiener process $W = \{W(\cdot, t), t \geq 0\}$ by

$$W(\cdot, t) = \sum_{j=1}^\infty \lambda_j^{1/2}e_j(\cdot)B_j(t),$$

where the series converges in $L^2(\Omega, \mathcal{F}, \mathsf{P})$, see, e.g., [1, Section 4.1] or [2, Section 3.5].

Similarly, let $B_j^H = \{B_j^H(t), t \geq 0\}$, $j \in \mathbb{N}$, be a sequence of one-dimensional, independent fractional Brownian motions with the Hurst parameter $H \in (0,1)$, defined

on $(\Omega, \mathcal{F}, \mathbb{F}, \mathsf{P})$ and starting at the origin. Following [6], define $L^2(D)$-valued fractional Brownian process $W^H = \{W^H(\cdot, t), t \geq 0\}$ by

$$W^H(\cdot, t) = \sum_{j=1}^{\infty} \mu_j^{1/2} e_j(\cdot) B_j^H(t), \tag{10}$$

where the series converges a. s. in $L^2(D)$.

Remark 2.1. The assumption $\sum_{j=1}^{\infty} \mu_j < \infty$ is sufficient for the series (10) to converge in $L_2(D)$. The stronger condition $\sum_{j=1}^{\infty} \mu_j^{1/2} < \infty$ is needed for the definition of a stochastic integral with respect to this process, see Subsection 2.5.2 below.

2.5. **Stochastic integration with respect to $L^2(D)$-valued processes.** Let $\Phi = \{\Phi(t), t \in [0, T]\}$ be an adapted stochastic process taking values in the space of linear operators on $L^2(D)$, $\psi = \{\psi(t), t \in [0, T]\}$ be an adapted process in $L^\infty(D)$.

2.5.1. *Integration with respect to W.* Define the stochastic integral with respect to $L_2(D)$-valued Wiener process W by

$$\int_0^t \Phi(s)\psi(s)\, dW(s) := \sum_{j=1}^{\infty} \lambda_j^{1/2} \int_0^t \Phi(s)(\psi(s)e_j)\, dB_j(s),$$

where the integrals with respect to B_j, $j \in \mathbb{N}$, are Itô integrals, see [1, 2, 16]. The Itô isometry of the form

$$\mathsf{E}\left\|\int_0^t \Phi(s)\psi(s)\, dW(s)\right\|_2^2 = \mathsf{E}\sum_{j=1}^{\infty} \lambda_j \int_0^t \|\Phi(s)(\psi(s)e_j)\|_2^2\, ds,$$

holds if the right-hand side is finite [16, Section 4]. Moreover, one has the following version of the Burkholder–Davis–Gundy inequality from [1, Theorem 4.36] (see also [2, Lemma 3.24]): for all $p \geq 2$ there exists $C_p > 0$ such that for all $t \in [0, T]$,

$$\mathsf{E}\sup_{0 \leq s \leq t}\left\|\int_0^s \Phi(v)\psi(v)\, dW(v)\right\|_2^p \leq C_p \mathsf{E}\left[\sum_{j=1}^{\infty} \lambda_j \int_0^t \|\Phi(s)(\psi(s)e_j)\|_2^2\, ds\right]^{p/2}.$$

This result can be generalized to the case of stochastic convolutions (see [5, 14]). For simplicity we formulate it for the evolution family $U(t, s)$ defined by (6). Define

$$S(\cdot, t) = \int_0^t U(t, s)\psi(s)\, dW(s) = \sum_{j=1}^{\infty} \lambda_j^{1/2} \int_0^t \int_D G(\cdot, t, y, s)\psi(y, s)e_j(y)dy\, dB_j(s).$$

Then for every $p \geq 2$ there exists $C_p' > 0$ such that for all $t \in [0, T]$,

$$\mathsf{E}\sup_{0 \leq s \leq t} \|S(\cdot, s)\|_2^p \leq C_p' \mathsf{E}\left[\int_0^t \|\psi(\cdot, s)\|_2^2\, ds\right]^{p/2}. \tag{11}$$

2.5.2. *Integration with respect to W^H.* In order to introduce the integral with respect to $L_2(D)$-valued fractional Brownian process W^H with $H > 1/2$, we need to recall the definition of generalized Lebesgue–Stieltjes integral. Let $f, g\colon [a, b] \to \mathbb{R}$ be two functions and $\alpha \in (0, 1)$. Then the Riemann–Liouville left- and right-sided fractional derivatives are defined by

$$\mathcal{D}_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(1-\alpha)}\left(\frac{f(x)}{(x-a)^{\alpha}} + \alpha \int_a^x \frac{f(x) - f(y)}{(x-y)^{\alpha+1}}\, dy\right),$$

$$\mathcal{D}_{b-}^{1-\alpha} g(x) = \frac{1}{\Gamma(\alpha)}\left(\frac{g(x)}{(b-x)^{1-\alpha}} + (1-\alpha) \int_x^b \frac{g(x) - g(y)}{(y-x)^{2-\alpha}}\, dy\right).$$

Assume that $\mathcal{D}_{a+}^{\alpha} f \in L_1[a,b]$, $\mathcal{D}_{b-}^{1-\alpha} g_{b-} \in L_\infty[a,b]$, where $g_{b-}(x) = g(b-) - g(x)$. Under these assumptions, the generalized Lebesgue–Stieltjes integral $\int_a^b f(x)dg(x)$ is defined by

$$\int_a^b f(x)dg(x) = \int_a^b \mathcal{D}_{a+}^{\alpha} f(x) \, \mathcal{D}_{b-}^{1-\alpha} g_{b-}(x) \, dx, \tag{12}$$

see, e.g., [7, Section 2.1] for details. It is not hard to see that this integral admits the bound

$$\left| \int_a^b f(x)dg(x) \right| \leq C_\alpha \|g\|_{\alpha,0;b} \int_a^b \left(\frac{|f(x)|}{(x-a)^\alpha} + \int_a^x \frac{|f(x)-f(y)|}{(x-y)^{\alpha+1}} \, dy \right) dx, \tag{13}$$

Fix $H \in (1/2, 1)$, $\alpha \in (1-H, 1/2)$, and let Φ, ψ be as above. Assume that

$$\sup_{j \in \mathbb{N}} \|\Phi(s)(\psi(s)e_j)\|_{\alpha,1,b} < \infty, \quad b, s \in \mathbb{R}^+.$$

Following [6], we introduce the integral with respect to $L^2(D)$-valued fractional Brownian process by

$$\int_a^b \Phi(s)\psi(s) \, dB^H(s) := \sum_{j=1}^\infty \mu_j^{1/2} \int_a^b \Phi(s)(\psi(s)e_j) \, dB_j^H(s),$$

where the integrals with respect to B_j^H, $j \in \mathbb{N}$, are pathwise generalized Lebesgue–Stieltjes integrals defined by (12). One can easily derive from (13) the following inequality (see [6, Equation (2.16)])

$$\left\| \int_a^b \Phi(s)\psi(s) \, dB^H(s) \right\|_2 \leq C_\alpha \xi_{\alpha,H,b} \sup_{j \in \mathbb{N}} \int_a^b \left(\frac{\|\Phi(s)(\psi(s)e_j)\|_2}{(s-a)^\alpha} + \right.$$
$$\left. + \int_a^s \frac{\|(\Phi(s)(\psi(s)e_j) - \Phi(v)(\psi(v)e_j)\|_2}{(s-v)^{\alpha+1}} \, dv \right) ds, \tag{14}$$

where

$$\xi_{\alpha,H,b} := \sum_{j=1}^\infty \mu_j^{1/2} \|B_j^H\|_{\alpha,0;b}. \tag{15}$$

Note that the value $\mathsf{E}\|B_j^H\|_{\alpha,0;b}$ is finite by [9, Lemma 7.5], moreover, it does not depend on j, since B_j^H's are equally distributed. Hence, using the monotone convergence theorem and assumption (A5), we get

$$\mathsf{E} \sum_{j=1}^\infty \mu_j^{1/2} \|B_j^H\|_{\alpha,0;b} = \sum_{j=1}^\infty \mu_j^{1/2} \mathsf{E}\|B_j^H\|_{\alpha,0;b} < \infty.$$

Therefore the random variable $\xi_{\alpha,H,b}$ is finite a.s.

3. Existence and uniqueness of mild solution

In this section, we consider unique solvability of the problem (1)–(3). We understand its solution in a mild sense. Recall that we consider $H \in (1/2, 1)$.

Definition 3.1. $L^2(D)$-valued random field $\{u(\cdot, t), t \in [0,T]\}$ is a *mild solution* to the problem (1)–(3) if the following two conditions are satisfied:

(1) $u \in \mathcal{B}^{\alpha,2}(0, T; L^2(D))$ a.s. for some $\alpha \in (1-H, 1/2)$,
(2) the relation (4), equivalently,

$$u(\cdot, t) = \int_D G(\cdot, t; y, 0)\varphi(y) \, dy + \int_0^t \int_D G(\cdot, t; y, s) f(u(y, s)) \, dy \, ds +$$
$$+ \sum_{j=1}^\infty \lambda_j^{1/2} \int_0^t \int_D G(\cdot, t; y, s) g(u(y, s)) e_j(y) \, dy \, dB_j(s) +$$

$$+ \sum_{j=1}^{\infty} \mu_j^{1/2} \int_0^t \int_D G(\cdot, t; y, s) h(u(y, s)) e_j(y) \, dy \, dB_j^H(s) \tag{16}$$

holds a. s. for every $t \in [0, T]$ as an equality in $L^2(D)$.

Here the integrals w. r. t. B_j, $j \in \mathbb{N}$, are Itô integrals, and the integrals w. r. t. B_j^H, $j \in \mathbb{N}$, are path-wise generalized Lebesgue–Stieltjes integrals.

For clarity, in the following we will write the integrals with respect to W and W^H in their full form, as in (16).

Theorem 3.1. *Assume that Hypotheses* (A1)–(A5) *hold,* $H \in \left(\frac{d+1}{d+2}, 1 \right)$. *Then the problem* (1)–(3) *has a unique mild solution.*

The proof will be divided into several logical steps. First fix some $\alpha \in \left(1 - H, \frac{1}{d+2} \right)$.

3.1. Construction of approximations. Fix $N \geq 1$. Let

$$\tau_N = \inf\{t : \xi_{\alpha, H, t} \geq N\} \wedge T,$$

where $\xi_{\alpha, H, t}$ is defined in (15). Put $B_j^{H,N}(t) = B_j^H(t \wedge \tau_N)$, $t \in [0, T]$, $j \in \mathbb{N}$. For each $n, j \in \mathbb{N}$ define a smooth approximation of $B_j^{H,N}$ by

$$B_j^{H,N,n}(t) = n \int_{(t-1/n)\vee 0}^t B_j^{H,N}(s) \, ds, \tag{17}$$

see [8]. Consider the equation

$$u_{N,n}(\cdot, t) = \int_D G(\cdot, t; y, 0) \varphi(y) \, dy + \int_0^t \int_D G(\cdot, t; y, s) f(u_{N,n}(y, s)) \, dy \, ds +$$

$$+ \sum_{j=1}^{\infty} \lambda_j^{1/2} \int_0^t \int_D G(\cdot, t; y, s) g(u_{N,n}(y, s)) e_j(y) \, dy \, dB_j(s) +$$

$$+ \sum_{j=1}^{\infty} \mu_j^{1/2} \int_0^t \int_D G(\cdot, t; y, s) h(u_{N,n}(y, s)) e_j(y) \, dy \, \frac{d}{ds} B_j^{H,N,n}(s) \, ds, \tag{18}$$

or

$$u_{N,n}(\cdot, t) = \int_D G(\cdot, t; y, 0) \varphi(y) \, dy + \int_0^t \int_D G(\cdot, t; y, s) b^{N,n}(u_{N,n}(y, s), y, \omega, s) \, dy \, ds +$$

$$+ \sum_{j=1}^{\infty} \lambda_j^{1/2} \int_0^t \int_D G(\cdot, t; y, s) g(u_{N,n}(y, s)) e_j(y) \, dy \, dB_j(s),$$

where

$$b^{N,n}(u, \omega, x, s) = f(u) + h(u) \sum_{j=1}^{\infty} \mu_j^{1/2} e_j(x) \frac{d}{ds} B_j^{H,N,n}(s)$$

is a random drift depending also on (y, s). In other words, $u_{N,n}$ is a mild solution of the equation

$$du(x, t) = \left(\operatorname{div}\left(k(x, t) \nabla u(x, t) \right) + b^{N,n}\left(u(x, t), x, \omega, t \right) \right) dt + g\left(u(x, t) \right) W(x, dt),$$

$(x, t) \in D \times [0, T]$, with initial-boundary conditions (2)–(3). Such equations were studied in [15, Section 8].

By assumption (A5),

$$\left| \sum_{j=1}^{\infty} \mu_j^{1/2} e_j(y) \frac{d}{ds} B_j^{H,N,n}(s) \right| \leq C \sum_{j=1}^{\infty} \mu_j^{1/2} \left| \frac{d}{ds} B_j^{H,N,n}(s) \right| =$$

$$= Cn \sum_{j=1}^{\infty} \mu_j^{1/2} \left| B_j^{H,N}(s) - B_j^{H,N}\big((s-\tfrac{1}{n}) \vee 0\big) \right| \leq$$

$$\leq Cn^\alpha \sum_{j=1}^{\infty} \mu_j^{1/2} \left\| B_j^{H,N} \right\|_{0,\alpha,s} \leq Cn^\alpha N.$$

Therefore, the function $b^{N,n}$ satisfies the following conditions: for all $y \in D$, $\omega \in \Omega$, $s \in [0,T]$ and $u, v \in \mathbb{R}$,

$$\left| b^{N,n}(u,y,\omega,s) - b^{N,n}(v,y,\omega,s) \right| \leq C|u-v|, \tag{19}$$

$$\left| b^{N,n}(u,y,\omega,s) \right| \leq C(1+|u|). \tag{20}$$

Then, by [15, Example 8.2], there exists a unique mild solution $u_{N,n}$ with paths in $C\big([0,T]; L^2(D)\big)$ a. s. Moreover, $u_{N,n}$ belongs to $C^{\beta_1,\beta_2}\big(\overline{D} \times [0,T]\big)$ for all $\beta_1 \in (0,1)$, $\beta_2 \in (0,1/2)$.

3.2. **Convergence of approximations.** Let us prove that, for a fixed $N \geq 1$, the sequence $\{u_{N,n}, n \geq 1\}$ is fundamental in probability in the norm $\|\cdot\|_{\alpha,2,T}$. For all $\varepsilon > 0$, $R \geq 1$ and $n, m \in \mathbb{N}$, we have

$$\mathsf{P}\Big(\|u_{N,n} - u_{N,m}\|_{\alpha,2,T} > \varepsilon \Big) \leq$$

$$\leq \mathsf{P}\Big(\|u_{N,n} - u_{N,m}\|_{\alpha,2,T} > \varepsilon, \|u_{N,n}\|_{\alpha,2,T} \leq R, \|u_{N,m}\|_{\alpha,2,T} \leq R \Big) +$$

$$+ \mathsf{P}\Big(\|u_{N,n}\|_{\alpha,2,T} > R \Big) + \mathsf{P}\Big(\|u_{N,m}\|_{\alpha,2,T} > R \Big). \tag{21}$$

By [8, Proposition 2.1], for any $j \in \mathbb{N}$, $\left\| B_j^{H,N,n} - B_j^{H,N} \right\|_{\alpha,0;T} \to 0$, $n \to \infty$, a. s. Then, in view of the boundedness,

$$\mathsf{E}\left\| B_j^{H,N,n} - B_j^{H,N} \right\|_{\alpha,0;T}^2 \to 0, \quad n \to \infty. \tag{22}$$

Since $B_j^{H,N,n} - B_j^{H,N}$, $j \in \mathbb{N}$, are identically distributed, we see that

$$\mathsf{E}\sum_{j=1}^{\infty} \mu_j^{1/2} \left\| B_j^{H,N,n} - B_j^{H,N} \right\|_{\alpha,0;T}^2 \to 0, \quad n \to \infty,$$

by (A5). Then, by the Cauchy–Schwarz inequality,

$$\mathsf{E}\left(\sum_{j=1}^{\infty} \mu_j^{1/2} \left\| B_j^{H,N,n} - B_j^{H,N} \right\|_{\alpha,0;T} \right)^2 \leq$$

$$\leq \mathsf{E}\left[\sum_{j=1}^{\infty} \mu_j^{1/2} \left\| B_j^{H,N,n} - B_j^{H,N} \right\|_{\alpha,0;T}^2 \right] \left(\sum_{j=1}^{\infty} \mu_j^{1/2} \right) \to 0, \quad n \to \infty. \tag{23}$$

Therefore, using Lemma A.3 and Markov's inequality, we see that for all $\varepsilon > 0$, $R \geq 1$,

$$\mathsf{P}\Big(\|u_{N,n} - u_{N,m}\|_{\alpha,2,T} > \varepsilon, \|u_{N,n}\|_{\alpha,2,T} \leq R, \|u_{N,m}\|_{\alpha,2,T} \leq R \Big) \to 0, \quad n \to \infty,$$

and

$$\limsup_{n,m\to\infty} \mathsf{P}\Big(\|u_{N,n} - u_{N,m}\|_{\alpha,2,T} > \varepsilon \Big) \leq 2 \sup_{n\in\mathbb{N}} \mathsf{P}\Big(\|u_{N,n}\|_{\alpha,2,T} > R \Big),$$

by (21).

Moreover, it follows from the convergence (22) that $\sup_{n \in \mathbb{N}} \mathsf{E} \left\| B_j^{H,N,n} \right\|_{\alpha,0;T}^2 < \infty$. Hence, Lemma A.2 and Markov's inequality imply that

$$\sup_{n \in \mathbb{N}} \mathsf{P}\Big(\|u_{N,n}\|_{\alpha,2,T} > R \Big) \to 0, \quad R \to \infty.$$

Therefore, $\|u_{N,n} - u_{N,m}\|_{\alpha,2,T} \to 0$, $n, m \to \infty$, in probability. Consequently, there exists a random process u_N such that $\|u_{N,n} - u_N\|_{\alpha,2,T} \to 0$, $n \to \infty$, in probability. Then there exists an a.s. convergent subsequence, and without loss of generality we can assume that

$$\|u_{N,n} - u_N\|_{\alpha,2,T} \to 0, \quad n \to \infty, \text{ a.s.} \tag{24}$$

3.3. The limit provides a solution. We have

$$\|u_{N,n}(\cdot, t) - u_N(\cdot, t)\|_2 \le C\Big(\left\| \tilde{I}_{\Delta f}(\cdot, t) \right\|_2^2 + \left\| \tilde{I}_{\Delta g}(\cdot, t) \right\|_2^2 + \left\| \tilde{I}_{\Delta h}(\cdot, t) \right\|_2^2 + \left\| \tilde{I}_{\Delta Z}(\cdot, t) \right\|_2^2 \Big),$$

where

$$\tilde{I}_{\Delta f}(x, t) := \int_0^t \int_D G(x, t; y, s)\big(f(u_{N,n}(y, s)) - f(u_N(y, s)) \big)\, dy\, ds,$$

$$\tilde{I}_{\Delta g}(x, t) := \sum_{j=1}^\infty \lambda_j^{1/2} \int_0^t \int_D G(x, t; y, s)\big(g(u_{N,n}(y, s)) - g(u_N(y, s)) \big) e_j(y)\, dy\, dB_j(s),$$

$$\tilde{I}_{\Delta h}(x, t) := \sum_{j=1}^\infty \mu_j^{1/2} \int_0^t \int_D G(x, t; y, s)\big(h(u_{N,n}(y, s)) - h(u_N(y, s)) \big) e_j(y)\, dy\, dB_j^{H,N,n}(s),$$

$$\tilde{I}_{\Delta Z}(x, t) := \sum_{j=1}^\infty \mu_j^{1/2} \int_0^t \int_D G(x, t; y, s) h(u_N(y, s)) e_j(y)\, dy\, \big(B_j^{H,N,n} - B_j^{H,N} \big)(ds).$$

Consequently, in order to prove that u_N satisfies (16) with B_j^H replaced by $B_j^{H,N}$, we need to show that these four integrals converge to zero. Note that they can be bounded exactly in the same way as the corresponding integrals in Lemma A.3 of Appendix A. Denoting

$$\mathbb{1}_T = \mathbb{1}_{\left\{ \|u_{N,n}\|_{\alpha,2,T} \le R, \|u_N\|_{\alpha,2,T} \le R \right\}},$$

we will obtain

$$\left\| \tilde{I}_{\Delta f}(\cdot, T) \right\|^2 \le C \int_0^T \|u_{N,n} - u_N\|_{\alpha,2,s}^2\, ds,$$

$$\left\| \tilde{I}_{\Delta h}(\cdot, T) \right\|^2 \le C_N \int_0^T \|u_{N,n} - u_N\|_{\alpha,2,s}^2\, ds,$$

$$\left\| \tilde{I}_{\Delta Z}(\cdot, T) \right\| \mathbb{1}_T \le CR \sum_{j=1}^\infty \mu_j^{1/2} \left\| B_j^{H,N,n} - B_j^{H,N} \right\|_{\alpha,0;T},$$

and

$$\mathsf{E}\left[\left\| \tilde{I}_{\Delta g}(\cdot, T) \right\|_2^2 \mathbb{1}_T \right] \le C \int_0^T \mathsf{E}\left[\|u_{N,n} - u_N\|_{\alpha,2,s}^2 \mathbb{1}_s \right] ds.$$

Taking into account (23) and (24), we get that $\mathsf{E}\|u_{N,n}(\cdot, T) - u_N(\cdot, T)\|_2^2 \mathbb{1}_T \to 0$ as $n \to \infty$, and, consequently, $\|u_{N,n}(\cdot, T) - u_N(\cdot, T)\|_2 \mathbb{1}_T \to 0$ in probability. Thanks to the convergence $\|u_{N,n} - u_N\|_{\alpha,2,T} \to 0$, $n \to \infty$, the event $\left\{ \|u_N\|_{\alpha,2,T} \le R \right\}$ implies $\left\{ \|u_{N,n}\|_{\alpha,2,T} \le R \right\}$ for n large enough, therefore we have the convergence of the integrals in probability on $\left\{ \|u_N\|_{\alpha,2,T} \le R \right\}$ and arbitrary $R \ge 1$, therefore on Ω.

3.4. **Letting $N \to \infty$ and uniqueness.** From Lemma A.3 and Remark A.1 after it, it is obvious that the processes u_N and u_M with $M \geq N$ coincide almost surely on the set $A_{N,T} = \{\xi_{\alpha,H,T} \leq N\}$. Therefore, there exists a process u such that for each $N \geq 1$, $u_N = u$ a. s. on $A_{N,T}$. Consequently, u satisfies (16) on each of the sets $A_{N,T}$, $N \geq 1$, hence, almost surely.

Finally, the uniqueness also follows from Lemma A.3: each solution to (16) must coincide with u on each of the sets $A_{N,T}$, hence, almost surely.

Appendix A. Auxiliary results

Let u be a mild solution, defined by (16). Introduce the following notation:

$$I_0(x,t) := \int_D G(x,t;y,0)\varphi(y)\,dy,$$

$$I_f(x,t) := \int_0^t \int_D G(x,t;y,s)f(u(y,s))\,dy\,ds,$$

$$I_g(x,t) := \sum_{j=1}^{\infty} \lambda_j^{1/2} \int_0^t \int_D G(x,t;y,s)g(u(y,s))e_j(y)\,dy\,dB_j(s),$$

$$I_h(x,t) := \sum_{j=1}^{\infty} \mu_j^{1/2} \int_0^t \int_D G(x,t;y,s)h(u(y,s))e_j(y)\,dy\,dB_j^H(s).$$

Also, let

$$J_a(t) := \int_0^t \left(\int_0^s \frac{\|I_a(\cdot,s) - I_a(\cdot,v)\|_2}{(s-v)^{\alpha+1}}\,dv \right)^2 ds, \quad a \in \{0,f,g,h\}. \tag{25}$$

Then

$$\|I_a\|_{\alpha,2,t} = \sup_{s\in[0,t]} \|I_a(\cdot,s)\|_2^2 + J_a(t), \quad a \in \{0,f,g,h\}.$$

Lemma A.1. *Let $N \geq 1$. Define*

$$A_{N,t} := \{\xi_{\alpha,H,t} \leq N\}, \quad t \in [0,T],$$

where $\xi_{\alpha,H,t}$ is given by (15). Then under assumptions of Theorem 3.1,

$$\|u\|_{\alpha,2,t}^2 \leq C_N \left(1 + \int_0^t \|u\|_{\alpha,2,s}^2\,ds + \sup_{s\in[0,t]} \|I_g(\cdot,s)\|_2^2 + J_g(t) \right)$$

for all $\omega \in A_{N,t}$.

Proof. Fix $\omega \in A_{N,t}$. It follows from (16) that

$$\|u(\cdot,t)\|_2^2 \leq C \Big(\|I_0(\cdot,t)\|_2^2 + \|I_f(\cdot,t)\|_2^2 + \|I_g(\cdot,t)\|_2^2 + \|I_h(\cdot,t)\|_2^2 \Big). \tag{26}$$

Evidently,

$$\|I_0(\cdot,s)\|_2^2 = \int_D \left(\int_D G(x,s;y,0)\varphi(y)\,dy \right)^2 dx \leq C, \tag{27}$$

since φ is bounded and the Gaussian property (5) holds. Using the Schwarz inequality and (5), we also get

$$\|I_f(\cdot,t)\|_2^2 = \int_D \left(\int_0^t \int_D |G(x,t;y,s)f(u(y,s))|\,dy\,ds \right)^2 dx \leq$$

$$\leq C \int_D \left(\int_0^t \int_D |G(x,t;y,s)|\,dy\,ds \right) \left(\int_0^t \int_D |G(x,t;y,s)||f(u(y,s))|^2\,dy\,ds \right) dx \leq$$

$$\leq C \int_D \int_0^t \int_D |G(x,t;y,s)||f(u(y,s))|^2 dy\, ds\, dx \leq$$

$$\leq C \int_0^t \int_D \left(\int_D |G(x,t;y,s)|\, dx\right)\left(1+|u(y,s)|^2\right) dy\, ds \leq$$

$$\leq C \int_0^t \int_D \left(1+|u(y,s)|^2\right) dy\, ds \leq C\left(1+\int_0^t \|u(\cdot,s)\|_2^2 ds\right). \qquad (28)$$

Define

$$a_{j,t}(u)(x,s) := \int_D G(x,t;y,s)h(u(y,s))e_j(y)\, dy. \qquad (29)$$

Then applying (14), we obtain

$$\|I_h(\cdot,t)\|_2 = \left\|\sum_{j=1}^\infty \mu_j^{1/2} \int_0^t a_{j,t}(u)(\cdot,s)\, dB_j^H(s)\right\|_2 \leq$$

$$\leq C\xi_{\alpha,H,t} \sup_{j\in\mathbb{N}} \int_0^t \left(\frac{\|a_{j,t}(u)(\cdot,s)\|_2}{s^\alpha} + \int_0^s \frac{\|a_{j,t}(u)(\cdot,s)-a_{j,t}(u)(\cdot,v)\|_2}{(s-v)^{\alpha+1}}\, dv\right) ds.$$

Taking into account that $\sup_{j\in\mathbb{N}}\|e_j\|_\infty < \infty$, one can derive the following bound similarly to (28):

$$\sup_{j\in\mathbb{N}}\|a_{j,t}(u)(\cdot,s)\|_2 \leq C(1+\|u(\cdot,s)\|_2). \qquad (30)$$

Further, by the assumption (A4),

$$|a_{j,t}(u)(x,s)-a_{j,t}(u)(x,v)| \leq C\int_D |G(x,t;y,s)||u(y,s)-u(y,v)|\, dy +$$

$$+ C\int_D |G(x,t;y,s)-G(x,t;y,v)|(1+|u(y,v)|)\, dy.$$

Applying the Schwarz inequality, we obtain

$$|a_{j,t}(u)(x,s)-a_{j,t}(u)(x,v)|^2 \leq C\int_D |G(x,t;y,s)||u(y,s)-u(y,v)|^2 dy +$$

$$+ C\int_D |G(x,t;y,s)-G(x,t;y,v)|(1+|u(y,v)|)^2 dy,$$

since the integrals $\int_D |G(x,t;y,s)| dy$ and $\int_D |G(x,t;y,s)-G(x,t;y,v)| dy$ are bounded uniformly in s,v due to the Gaussian property (5). Then applying the bounds (5) and (7) and integrating the preceding estimate w.r.t. $x\in D$, we get

$$\sup_{j\in\mathbb{N}}\|a_{j,t}(u)(\cdot,s)-a_{j,t}(u)(\cdot,v)\|_2^2 \leq C\Big(\|u(\cdot,s)-u(\cdot,v)\|_2^2 +$$

$$+ (t-s)^{-\delta}(s-v)^\delta\Big(1+\|u(\cdot,v)\|_2^2\Big)\Big),$$

whence

$$\sup_{j\in\mathbb{N}}\|a_{j,t}(u)(\cdot,s)-a_{j,t}(u)(\cdot,v)\|_2 \leq C\Big(\|u(\cdot,s)-u(\cdot,v)\|_2 +$$

$$+ (t-s)^{-\frac{\delta}{2}}(s-v)^{\frac{\delta}{2}}(1+\|u(\cdot,v)\|_2)\Big) \qquad (31)$$

for any $\delta\in\left(\frac{d}{d+2},1\right)$. Therefore,

$$\|I_h(\cdot,t)\|_2 \leq C\xi_{\alpha,H,t}\left(1+\int_0^t \frac{\|u(\cdot,s)\|_2}{s^\alpha}\, ds + \int_0^t \int_0^s \frac{\|u(\cdot,s)-u(\cdot,v)\|_2}{(s-v)^{\alpha+1}}\, dv\, ds +\right.$$

$$\left.+ \int_0^t (t-s)^{-\delta/2}\int_0^s (s-v)^{\delta/2-\alpha-1}(1+\|u(\cdot,v)\|_2)\, dv\, ds\right), \qquad (32)$$

where $\delta \in \left(\frac{d}{d+2}, 1\right)$ is arbitrary. Since $\alpha < 1/2$, we can choose $\delta > 2\alpha$. Then by changing the order of integration, the last term can be calculated as follows

$$
\int_0^t (t-s)^{-\delta/2} \int_0^s (s-v)^{\delta/2-\alpha-1}(1 + \|u(\cdot,v)\|_2)\, dv\, ds =
$$

$$
= \int_0^t \left(\int_v^t (t-s)^{-\delta/2}(s-v)^{\delta/2-\alpha-1} ds \right)(1 + \|u(\cdot,v)\|_2)\, dv =
$$

$$
= \int_0^t (t-v)^{-\alpha} \left(\int_0^1 (1-z)^{-\delta/2} z^{\delta/2-\alpha-1} dz \right)(1 + \|u(\cdot,v)\|_2)\, dv =
$$

$$
= B\left(\tfrac{\delta}{2} - \alpha, 1 - \tfrac{\delta}{2}\right) \int_0^t (t-v)^{-\alpha}(1 + \|u(\cdot,v)\|_2)\, dv. \tag{33}
$$

We arrive at

$$
\|I_h(\cdot,t)\|_2 \le CN\left(1 + \int_0^t \left(\frac{1}{s^\alpha} + \frac{1}{(t-s)^\alpha}\right)\|u(\cdot,s)\|_2\, ds + \right.
$$

$$
\left. + \int_0^t \int_0^s \frac{\|u(\cdot,s) - u(\cdot,v)\|_2}{(s-v)^{\alpha+1}}\, dv\, ds\right).
$$

Applying the Schwarz inequality relative to the measure ds on $[0,t]$ to both integrals, we deduce that

$$
\|I_h(\cdot,t)\|_2^2 \le CN^2\left(1 + \int_0^t \|u(\cdot,s)\|_2^2\, ds + \int_0^t \left(\int_0^s \frac{\|u(\cdot,s) - u(\cdot,v)\|_2}{(s-v)^{\alpha+1}}\, dv\right)^2 ds\right). \tag{34}
$$

Combining (26)–(28) and (34), we obtain

$$
\|u(\cdot,t)\|_2^2 \le CN^2\left(1 + \int_0^t \|u(\cdot,s)\|_2^2\, ds + \int_0^t \left(\int_0^s \frac{\|u(\cdot,s) - u(\cdot,v)\|_2}{(s-v)^{\alpha+1}}\, dv\right)^2 ds\right) +
$$

$$
+ C\|I_g(\cdot,t)\|_2^2 \le
$$

$$
\le CN^2\left(1 + \int_0^t \sup_{v\in[0,s]} \|u(\cdot,v)\|_2^2\, ds + \int_0^t \left(\int_0^s \frac{\|u(\cdot,s) - u(\cdot,v)\|_2}{(s-v)^{\alpha+1}}\, dv\right)^2 ds\right) +
$$

$$
+ C\|I_g(\cdot,t)\|_2^2.
$$

By definition of the norm $\|\cdot\|_{\alpha,2,t}$,

$$
\|u(\cdot,t)\|_{\alpha,2,t}^2 \le CN^2\left(1 + \int_0^t \sup_{v\in[0,s]} \|u(\cdot,v)\|_2^2\, ds + \int_0^t \left(\int_0^s \frac{\|u(\cdot,s) - u(\cdot,v)\|_2}{(s-v)^{\alpha+1}}\, dv\right)^2 ds\right) +
$$

$$
+ C \sup_{s\in[0,t]} \|I_g(\cdot,s)\|_2^2. \tag{35}
$$

Obviously,

$$
\int_0^t \left(\int_0^s \frac{\|u(\cdot,s) - u(\cdot,v)\|_2}{(s-v)^{\alpha+1}}\, dv\right)^2 ds \le C\big(J_0(t) + J_f(t) + J_g(t) + J_h(t)\big), \tag{36}
$$

where the terms in the right-hand side are defined in (25).

Using the boundedness of φ and (8), we get for any $\delta \in (\frac{d}{d+2}, 1)$ and some $v^* \in (v, s)$,

$$
\|I_0(\cdot,s) - I_0(\cdot,v)\|_2^2 \le C\int_D \left(\int_D |G(x,s;y,0) - G(x,v;y,0)|\, dy\right)^2 dx \le
$$

$$
\le C\int_D \left((s-v)^\delta v^{-\delta} \int_D (v^*)^{-d/2} \exp\left\{-C\frac{|x-y|^2}{v^*}\right\} dy\right)^2 dx \le
$$

$$\leq C(s-v)^{2\delta}v^{-2\delta},$$

whence

$$J_0(t) \leq C \int_0^t \left(\int_0^s (s-v)^{\delta-\alpha-1} v^{-\delta} \, dv \right)^2 ds \leq C, \tag{37}$$

since we can choose $\delta > \alpha$.

In order to estimate $J_f(t)$, we write

$$I_f(x,s) - I_f(x,v) = \int_v^s \int_D G(x,s;y,z) f(u(y,z)) \, dy \, dz +$$

$$+ \int_0^v \int_D \big(G(x,s;y,z) - G(x,v;y,z) \big) f(u(y,z)) \, dy \, dz =:$$

$$=: K_f'(x,s,v) + K_f''(x,s,v).$$

Then,

$$J_f(t) \leq \int_0^t \left(\int_0^s \Big(\big\| K_f'(\cdot,s,v) \big\|_2 + \big\| K_f''(\cdot,s,v) \big\|_2 \Big)(s-v)^{-\alpha-1} \, dv \right)^2 ds.$$

It is not hard to show that

$$\big\| K_f'(\cdot,s,v) \big\|_2 \leq C(s-v)^{1/2} \left(\int_v^s \Big(1 + \|u(\cdot,z)\|_2^2 \Big) dz \right)^{1/2}, \tag{38}$$

$$\big\| K_f''(\cdot,s,v) \big\|_2 \leq C(s-v)^{\delta/2} \left(\int_0^v (v-z)^{-\delta} \Big(1 + \|u(\cdot,z)\|_2^2 \Big) dz \right)^{1/2}, \tag{39}$$

for every $\delta \in \left(\frac{d}{d+2}, 1 \right)$. Indeed, (38) follows by applying the Schwarz inequality w.r.t. the finite measure $|G(x,s;y,z)| dy \, dz$ on $D \times [v,s]$ and using the Gaussian property along with assumption (A3). Inequality (39) is derived by applying the Schwarz inequality w.r.t. the measure $|G(x,s;y,z) - G(x,v;y,z)| dy \, dz$ on $D \times [0,v]$ and then (8). Further, (38) implies

$$\int_0^t \left(\int_0^s \big\| K_f'(\cdot,s,v) \big\|_2 (s-v)^{-\alpha-1} \, dv \right)^2 ds \leq$$

$$\leq C \int_0^t \left(\int_0^s (s-v)^{-1/2-\alpha} \left(\int_v^s \Big(1 + \|u(\cdot,z)\|_2^2 \Big) dz \right)^{1/2} dv \right)^2 ds \leq$$

$$\leq C \int_0^t \left(1 + \sup_{z \in [0,s]} \|u(\cdot,z)\|_2^2 \right) \left(\int_0^s (s-v)^{-1/2-\alpha} dv \right)^2 ds \leq$$

$$\leq C \int_0^t \left(1 + \sup_{z \in [0,s]} \|u(\cdot,z)\|_2^2 \right) ds,$$

since $\alpha < 1/2$. Applying the bound (39) and the Schwarz inequality, we obtain

$$\int_0^t \left(\int_0^s \big\| K_f''(\cdot,s,v) \big\|_2 (s-v)^{-\alpha-1} \, dv \right)^2 ds \leq$$

$$\leq C \int_0^t \left(\int_0^s (s-v)^{\delta/2-\alpha-1} \left(\int_0^v (v-z)^{-\delta} \Big(1 + \|u(\cdot,z)\|_2^2 \Big) dz \right)^{1/2} dv \right)^2 ds \leq$$

$$\leq C \int_0^t \left(\int_0^s (s-v)^{\delta/2-\alpha-1} dv \right) \times$$

$$\times \left(\int_0^s (s-v)^{\delta/2-\alpha-1} \int_0^v (v-z)^{-\delta} \Big(1 + \|u(\cdot,z)\|_2^2 \Big) dz \, dv \right) ds \leq$$

$$\leq C \int_0^t \int_0^s (s-v)^{\delta/2-\alpha-1} \int_0^v (v-z)^{-\delta}\Big(1+\|u(\cdot,z)\|_2^2\Big) dz\, dv\, ds \leq$$

$$\leq C \int_0^t \left(1+ \sup_{z\in[0,s]} \|u(\cdot,z)\|_2^2\right) \int_0^s (s-v)^{\delta/2-\alpha-1} v^{1-\delta}\, dv\, ds \leq$$

$$\leq C \int_0^t \left(1+ \sup_{z\in[0,s]} \|u(\cdot,z)\|_2^2\right) ds,$$

where the last inequality follows from

$$\int_0^s (s-v)^{\delta/2-\alpha-1} v^{1-\delta}\, dv = \mathrm{B}(\delta/2-\alpha, 2-\delta) s^{1-\delta/2-\alpha} \leq \mathrm{B}(\delta/2-\alpha, 2-\delta) T^{1-\delta/2-\alpha},$$

since $\delta/2 + \alpha < 1$. Combining the above estimates, we arrive at

$$J_f(t) \leq C\left(1+ \int_0^t \|u\|_{\alpha,2,s}^2 ds\right). \tag{40}$$

It remains to estimate J_h. We start by writing

$$I_h(x,s) - I_h(x,v) = \sum_{j=1}^\infty \mu_j^{1/2} \int_v^s a_{j,s}(u)(x,z)\, dB_j^H(z) +$$

$$+ \sum_{j=1}^\infty \mu_j^{1/2} \int_0^v (a_{j,s}(u)(x,z) - a_{j,v}(u)(x,z)) dB_j^H(z) =$$

$$= K_h'(x,s,v) + K_h''(x,s,v),$$

where $a_{j,s}(u)(x,z)$ is defined by (29). Then

$$J_h(t) \leq \int_0^t \left(\int_0^s (\|K_h'(\cdot,s,v)\|_2 + \|K_h''(\cdot,s,v)\|_2)(s-v)^{-\alpha-1}\, dv\right)^2 ds. \tag{41}$$

Applying (14), (30) and (31), we get

$$\|K_h'(\cdot,s,v)\|_2 \leq C\xi_{\alpha,H,t} \sup_{j\in\mathbb{N}} \int_v^s \left(\frac{\|a_{j,s}(u)(\cdot,z)\|_2}{(z-v)^\alpha} + \right.$$

$$+ \int_v^z \frac{\|a_{j,s}(u)(\cdot,z) - a_{j,s}(u)(\cdot,r)\|_2}{(z-r)^{\alpha+1}}\, dr\Big) dz \leq$$

$$\leq C\xi_{\alpha,H,t} \sup_{j\in\mathbb{N}} \left(\int_v^s \frac{1+\|u(\cdot,z)\|_2}{(z-v)^\alpha} dz + \int_v^s \int_v^z \frac{\|u(\cdot,z)-u(\cdot,r)\|_2}{(z-r)^{\alpha+1}}\, dr\, dz + \right.$$

$$+ \int_v^s \int_v^z (s-z)^{-\delta/2}(z-r)^{\delta/2-\alpha-1}(1+\|u(\cdot,r)\|_2) dr\, dz\Big)$$

for any $\delta \in \left(\frac{d}{d+2}, 1\right)$. The last term is computed similarly to (33) (recall that $\delta > 2\alpha$):

$$\int_v^s \int_v^z (s-z)^{-\delta/2}(z-r)^{\delta/2-\alpha-1}(1+\|u(\cdot,r)\|_2) dr\, dz =$$

$$= \mathrm{B}\big(\tfrac{\delta}{2}-\alpha, 1-\tfrac{\delta}{2}\big) \int_s^v (s-r)^{-\alpha}(1+\|u(\cdot,r)\|_2)\, dr.$$

Thus,

$$\|K_h'(\cdot,s,v)\|_2 \leq C\xi_{\alpha,H,t}\left(\int_v^s \left(\frac{1}{(z-v)^\alpha} + \frac{1}{(s-z)^\alpha}\right)(1+\|u(\cdot,z)\|_2)\, dz + \right.$$

$$+ \int_v^s \int_v^z \frac{\|u(\cdot,z)-u(\cdot,r)\|_2}{(z-r)^{\alpha+1}}\, dr\, dz\Big). \tag{42}$$

Denoting $a_{j,s,v}^*(u)(x,z) = a_{j,s}(u)(x,z) - a_{j,v}(u)(x,z)$, we can write by (14),

$$\|K_h''(\cdot,s,v)\|_2 \le C\xi_{\alpha,H,t} \sup_{j\in\mathbb{N}} \int_0^v \left(\frac{\|a_{j,s,v}^*(u)(\cdot,z)\|_2}{z^\alpha} + \right.$$
$$\left. + \int_0^z \frac{\|a_{j,s,v}^*(u)(\cdot,z) - a_{j,s,v}^*(u)(\cdot,r)\|_2}{(z-r)^{\alpha+1}} \, dr \right) dz.$$

Similarly to (30) and (31), we can prove the inequalities

$$\left\|a_{j,s,v}^*(u)(\cdot,z)\right\|_2 \le C(s-v)^{\frac{\delta}{2}}(v-z)^{-\frac{\delta}{2}}(1+\|u(\cdot,z)\|_2), \tag{43}$$

and

$$\left\|a_{j,s,v}^*(u)(\cdot,z) - a_{j,s,v}^*(u)(\cdot,r)\right\|_2 \le C(s-v)^{\frac{\delta}{2}}\left((v-z)^{-\frac{\delta}{2}}\|u(\cdot,z)-u(\cdot,r)\|_2 + \right.$$
$$\left. + (v-z)^{-\frac{1}{2}}(z-r)^{-\frac{1}{2}(1-\delta)}(1+\|u(\cdot,r)\|_2)\right) \tag{44}$$

for any $\delta \in \left(\frac{d}{d+2}, 1\right)$. For (43) the key estimate is (8). For (44) one should apply (8) along with (9).

Then

$$\|K_h''(\cdot,s,v)\|_2 \le C\xi_{\alpha,H,t}(s-v)^{\delta/2}\left(\int_0^v (v-z)^{-\delta/2}\left(\frac{1}{z^\alpha} + \frac{1}{(v-z)^\alpha}\right)(1+\|u(\cdot,z)\|_2)\, dz + \right.$$
$$\left. + \int_0^v (v-z)^{-\delta/2}\int_v^z \frac{\|u(\cdot,z)-u(\cdot,r)\|_2}{(z-r)^{\alpha+1}}\, dr\, dz\right). \tag{45}$$

Thus, combining (41), (42) and (45), we get

$$J_h(t) \le CN^2 \int_0^t \left[(L_1(s))^2 + (L_2(s))^2 + (L_3(s))^2 + (L_4(s))^2\right] ds,$$

where

$$L_1(s) = \int_0^s (s-v)^{-\alpha-1}\int_v^s \left(\frac{1}{(z-v)^\alpha} + \frac{1}{(s-z)^\alpha}\right)(1+\|u(\cdot,z)\|_2)\, dz\, dv,$$

$$L_2(s) = \int_0^s (s-v)^{-\alpha-1}\int_v^s \int_v^z \frac{\|u(\cdot,z)-u(\cdot,r)\|_2}{(z-r)^{\alpha+1}}\, dr\, dz\, dv,$$

$$L_3(s) = \int_0^s (s-v)^{\delta/2-\alpha-1}\int_0^v (v-z)^{-\delta/2}\left(\frac{1}{z^\alpha} + \frac{1}{(v-z)^\alpha}\right)(1+\|u(\cdot,z)\|_2)\, dz\, dv,$$

$$L_4(s) = \int_0^s (s-v)^{\delta/2-\alpha-1}\int_0^v (v-z)^{-\delta/2}\int_v^z \frac{\|u(\cdot,z)-u(\cdot,r)\|_2}{(z-r)^{\alpha+1}}\, dr\, dz\, dv.$$

Similarly to Equations (3.42)–(3.45) of [12], one can estimate the integrals $\int_0^t (L_i(s))^2\, ds$, $i = 1, 2, 3, 4$. This leads to the inequality

$$J_h(t) \le CN^2\left(1 + \int_0^t \|u\|_{\alpha,2,s}^2 ds\right). \tag{46}$$

Finally, combining (36), (37), (40) and (46), we get

$$\int_0^t \left(\int_0^s \frac{\|u(\cdot,s)-u(\cdot,v)\|_2}{(s-v)^{\alpha+1}}\, dv\right)^2 ds \le CN^2\left(1 + \int_0^t \|u\|_{\alpha,2,s}^2 ds\right) + CJ_g(t).$$

Inserting this bound into (35), we conclude the proof. $\square$

Lemma A.2. *Let $N \ge 1$. Define a stopping time*

$$\tau_N = \inf\{t : \xi_{\alpha,H,t} \ge N\} \wedge T$$

and a stopped process $u^N(\cdot, t) = u(\cdot, t \wedge \tau_N)$. Then

$$\mathsf{E}\Big[\big\|u^N\big\|^2_{\alpha,2,T}\Big] \leq C_N.$$

Proof. By Lemma A.1,

$$\mathsf{E}\Big[\big\|u^N\big\|^2_{\alpha,2,t}\Big] \leq C_N\left(1 + \int_0^t \mathsf{E}\Big[\big\|u^N\big\|^2_{\alpha,2,s}\Big]ds + \mathsf{E}\left[\sup_{s\in[0,t]}\big\|I_g^N(\cdot,s)\big\|^2_2\right] + \mathsf{E}\big[J_g^N(t)\big]\right), \tag{47}$$

where

$$I_g^N(x,t) = \sum_{j=1}^{\infty} \lambda_j^{1/2} \int_0^t \int_D G(x,t;y,s)g\big(u^N(y,s)\big)e_j(y)\,dy\,dB_j(s),$$

$$J_g^N(t) = \int_0^t \left(\int_0^s \frac{\big\|I_g^N(\cdot,s) - I_g^N(\cdot,v)\big\|_2}{(s-v)^{\alpha+1}}\,dv\right)^2 ds.$$

Using the inequality (11) and then the assumption (A3), we get

$$\mathsf{E}\sup_{s\in[0,t]}\big\|I_g^N(\cdot,s)\big\|^2_2 \leq C\mathsf{E}\int_0^t \big\|g\big(u^N(\cdot,s)\big)\big\|^2_2\,ds \leq C\mathsf{E}\int_0^t\left(1 + \big\|u^N(\cdot,s)\big\|^2_2\right)ds \leq$$

$$\leq C\left(1 + \mathsf{E}\int_0^t \sup_{z\in[0,s]}\big\|u^N(\cdot,z)\big\|^2_2\,ds\right) \leq$$

$$\leq C\left(1 + \int_0^t \mathsf{E}\Big[\big\|u^N\big\|^2_{\alpha,2,s}\Big]ds\right). \tag{48}$$

By the Schwarz inequality, we have

$$\mathsf{E}J_g^N(t) \leq \mathsf{E}\int_0^t\left(\int_0^s \big\|I_g^N(\cdot,s) - I_g^N(\cdot,v)\big\|^2_2(s-v)^{-3/2-\alpha}\,dv\right)\left(\int_0^s (s-v)^{-1/2-\alpha}\,dv\right)ds \leq$$

$$\leq C\int_0^t\int_0^s \mathsf{E}\big\|I_g^N(\cdot,s) - I_g^N(\cdot,v)\big\|^2_2(s-v)^{-3/2-\alpha}\,dv\,ds \leq$$

$$\leq C\int_0^t\int_0^s\left(\mathsf{E}\big\|K_g'(\cdot,s,v)\big\|^2_2 + \mathsf{E}\big\|K_g''(\cdot,s,v)\big\|^2_2\right)(s-v)^{-3/2-\alpha}\,dv\,ds,$$

where

$$K_g'(x,s,v) = \sum_{j=1}^{\infty}\lambda_j^{1/2}\int_v^s\int_D G(x,s;y,z)g\big(u^N(y,z)\big)e_j(y)\,dy\,dB_j(z),$$

$$K_g''(x,s,v) = \sum_{j=1}^{\infty}\lambda_j^{1/2}\int_0^v\int_D \big(G(x,s;y,z) - G(x,v;y,z)\big)g\big(u^N(y,z)\big)e_j(y)\,dy\,dB_j(z).$$

By Ito's isometry,

$$\mathsf{E}\big\|K_g'(\cdot,s,v)\big\|^2_2 = \mathsf{E}\int_D\left(\sum_{j=1}^{\infty}\lambda_j^{1/2}\int_v^s\int_D G(x,s;y,z)g\big(u^N(y,z)\big)e_j(y)\,dy\,dB_j(z)\right)^2 dx =$$

$$= \mathsf{E}\int_D\sum_{j=1}^{\infty}\lambda_j\int_v^s\left(\int_D G(x,s;y,z)g\big(u^N(y,z)\big)e_j(y)\,dy\right)^2 dz\,dx \leq$$

$$\leq \left(\sum_{j=1}^{\infty}\lambda_j\|e_j\|_{\infty}\right)\mathsf{E}\int_v^s\int_D\left(\int_D \big|G(x,s;y,z)g\big(u^N(y,z)\big)\big|\,dy\right)^2 dx\,dz. \tag{49}$$

Then using the assumption (A5) and bounding the inner integrals similarly to (28), we arrive at

$$\mathsf{E}\big\|K'_g(\cdot, s, v)\big\|_2^2 \le C(s - v)\Big(1 + \mathsf{E}\big\|u^N\big\|_{\alpha,2,s}^2\Big).$$

Therefore,

$$\int_0^t \int_0^s \mathsf{E}\big\|K'_g(\cdot, s, v)\big\|_2 (s - v)^{-3/2-\alpha}\, dv\, ds \le$$
$$\le C \int_0^t \Big(1 + \mathsf{E}\big\|u^N\big\|_{\alpha,2,s}^2\Big) \int_0^s (s - v)^{-1/2-\alpha}\, dv\, ds \le C\Big(1 + \int_0^t \mathsf{E}\Big[\big\|u^N\big\|_{\alpha,2,s}^2\Big] ds\Big).$$

Similarly to (49),

$$\mathsf{E}\big\|K''_g(\cdot, s, v)\big\|_2^2 \le C\mathsf{E} \int_0^v \int_D \Big(\int_D |G(x, s; y, z) - G(x, v; y, z)|\big|g\big(u^N(y, z)\big)\big|\, dy\Big)^2 dx\, dz.$$

One can bound the integral in the right-hand side in the same way as K''_f in (39):

$$\mathsf{E}\big\|K''_g(\cdot, s, v)\big\|_2^2 \le C\mathsf{E}(s - v)^\delta \int_0^v (v - z)^{-\delta}\Big(1 + \big\|u^N(\cdot, z)\big\|_2^2\Big) dz.$$

Further,

$$\mathsf{E}\big\|K''_g(\cdot, s, v)\big\|_2^2 \le C(s - v)^\delta \mathsf{E}\Big(1 + \sup_{z \in [0,s]}\big\|u^N(\cdot, z)\big\|_2^2\Big) \int_0^v (v - z)^{-\delta} dz \le$$
$$\le C(s - v)^\delta \Big(1 + \mathsf{E}\big\|u^N\big\|_{\alpha,2,s}^2\Big).$$

Hence, choosing $\delta > \frac{1}{2} + \alpha$, we obtain

$$\int_0^t \int_0^s \mathsf{E}\big\|K''_g(\cdot, s, v)\big\|_2 (s - v)^{-3/2-\alpha}\, dv\, ds \le$$
$$\le C \int_0^t \Big(1 + \mathsf{E}\big\|u^N\big\|_{\alpha,2,s}^2\Big) \int_0^s (s - v)^{\delta-\frac{3}{2}-\alpha}\, dv\, ds \le C\Big(1 + \int_0^t \mathsf{E}\Big[\big\|u^N\big\|_{\alpha,2,s}^2\Big] ds\Big).$$

Thus

$$\mathsf{E}J^N_g(t) \le C\Big(1 + \int_0^t \mathsf{E}\Big[\big\|u^N\big\|_{\alpha,2,s}^2\Big] ds\Big). \tag{50}$$

Combining (47), (48), and (50), we get

$$\mathsf{E}\Big[\big\|u^N\big\|_{\alpha,2,t}^2\Big] \le C_N\Big(1 + \int_0^t \mathsf{E}\Big[\big\|u^N\big\|_{\alpha,2,s}^2\Big] ds\Big),$$

and the proof follows from Gronwall's lemma. $\square$

Lemma A.3. *Let the assumptions of Theorem 3.1 hold. Then*

$$\mathsf{E}\|u_{N,n} - u_{N,m}\|_{\alpha,2,T}^2 \mathbb{1}_{A_T^{N,R}} \le C_{N,R}\mathsf{E}\left(\sum_{j=1}^\infty \mu_j^{1/2}\big\|B_j^{H,N,n} - B_j^{H,N,m}\big\|_{\alpha,0;T}\right)^2$$

where $A_T^{N,R} = \big\{\|u_{N,n}\|_{\alpha,2,T} \le R, \|u_{N,m}\|_{\alpha,2,T} \le R\big\}$, $B_j^{H,N,n}$ *and* $u_{N,n}$ *are defined by* (17)–(18).

Proof. The proof will be similar to that of Lemmas A.1–A.2, so we omit some details. Denote $\mathbb{1}_t = \mathbb{1}_{A_t^{N,R}}$, $u = u_{N,n}$, $\tilde{u} = u_{N,m}$, $Z_j = B_j^{H,N,n}$, $\widetilde{Z}_j = B_j^{H,N,m}$, $\Delta Z_j = Z_j - \widetilde{Z}_j$, $\eta_{\alpha,H,t} = \sum_{j=1}^{\infty} \mu_j^{1/2} \|\Delta Z_j\|_{\alpha,0;T}$. We have

$$\|u - \tilde{u}\|_{\alpha,2,t}^2 \le C \sup_{s \in [0,t]} \Big(\|I_{\Delta f}(\cdot,s)\|_2^2 + \|I_{\Delta g}(\cdot,s)\|_2^2 + \|I_{\Delta h}(\cdot,s)\|_2^2 + \|I_{\Delta Z}(\cdot,s)\|_2^2 \Big) +$$
$$+ J_{\Delta f}(t) + J_{\Delta g}(t) + J_{\Delta h}(t) + J_{\Delta Z}(t),$$

where

$$I_{\Delta f}(x,t) := \int_0^t \int_D G(x,t;y,s)\big(f(u(y,s)) - f(\tilde{u}(y,s))\big)\, dy\, ds,$$

$$I_{\Delta g}(x,t) := \sum_{j=1}^{\infty} \lambda_j^{1/2} \int_0^t \int_D G(x,t;y,s)\big(g(u(y,s)) - g(\tilde{u}(y,s))\big)e_j(y)\, dy\, dB_j(s),$$

$$I_{\Delta h}(x,t) := \sum_{j=1}^{\infty} \mu_j^{1/2} \int_0^t \int_D G(x,t;y,s)\big(h(u(y,s)) - h(\tilde{u}(y,s))\big)e_j(y)\, dy\, dZ_j(s),$$

$$I_{\Delta Z}(x,t) := \sum_{j=1}^{\infty} \mu_j^{1/2} \int_0^t \int_D G(x,t;y,s)h(\tilde{u}(y,s))e_j(y)\, dy\, d\big(\Delta Z_j(s)\big),$$

$$J_a(t) := \int_0^t \left(\int_0^s \frac{\|I_a(\cdot,s) - I_a(\cdot,v)\|_2}{(s-v)^{\alpha+1}}\, dv \right)^2 ds, \quad a \in \{\Delta f, \Delta g, \Delta h, \Delta Z\}.$$

Similarly to (32),

$$\|I_{\Delta Z}(\cdot,t)\|_2 \le C\eta_{\alpha,H,t}\left(1 + \int_0^t \frac{\|\tilde{u}(\cdot,s)\|_2}{s^\alpha}\, ds + \int_0^t \int_0^s \frac{\|\tilde{u}(\cdot,s) - \tilde{u}(\cdot,v)\|_2}{(s-v)^{\alpha+1}}\, dv\, ds + \right.$$
$$\left. + \int_0^t (t-s)^{-\delta/2} \int_0^s (s-v)^{\delta/2-\alpha-1}(1 + \|\tilde{u}(\cdot,v)\|_2)\, dv\, ds \right),$$

for any $\delta \in \left(\frac{d}{d+2}, 1 \right)$. Arguing as in the proof of Lemma A.1 for the term I_h, we obtain

$$\|I_{\Delta Z}(\cdot,t)\|_2^2 \le C\eta_{\alpha,H,t}^2 \left(1 + \int_0^t \|\tilde{u}(\cdot,s)\|_2^2\, ds + \int_0^t \left(\int_0^s \frac{\|\tilde{u}(\cdot,s) - \tilde{u}(\cdot,v)\|_2}{(s-v)^{\alpha+1}}\, dv \right)^2 ds \right) \le$$
$$\le C\eta_{\alpha,H,t}^2 \left(1 + \|u\|_{\alpha,2,t}^2 \right).$$

Therefore

$$\|I_{\Delta Z}(\cdot,t)\|_2^2 \mathbb{1}_t \le CR^2 \eta_{\alpha,H,t}^2.$$

The term $J_{\Delta Z}$ can be bounded analogously to (46):

$$J_{\Delta Z}(t)\mathbb{1}_t \le C\eta_{\alpha,H,t}^2 \left(1 + \int_0^t \|\tilde{u}\|_{\alpha,2,s}^2\, ds \right) \mathbb{1}_t \le CR^2 \eta_{\alpha,H,t}^2.$$

The terms $I_{\Delta f}$ and $J_{\Delta f}$ can be estimated in the same way as the terms I_f and J_f in the proof of Lemma A.1, using the Lipschitz condition (A3) instead of the inequality $|f(u)| \le C(1 + |u|)$. This leads to the bounds

$$\sup_{s \in [0,t]} \|I_{\Delta f}(\cdot,t)\|_2^2 \le C \int_0^t \|u - \tilde{u}\|_{\alpha,2,s}^2\, ds \quad \text{and} \quad J_{\Delta f}(t) \le C \int_0^t \|u - \tilde{u}\|_{\alpha,2,s}^2\, ds.$$

In order to estimate $I_{\Delta h}$, we can use the same arguments as for I_h in the proof of Lemma A.1, using the bounds

$$\sup_{j \in \mathbb{N}} \|a_{j,t}(u)(\cdot,s) - a_{j,t}(\tilde{u})(\cdot,s)\|_2 \le C(\|u(\cdot,s) - \tilde{u}(\cdot,s)\|_2)$$

and

$$\sup_{j \in \mathbb{N}} \|a_{j,t}(u)(\cdot, s) - a_{j,t}(\tilde{u})(\cdot, s) - a_{j,t}(u)(\cdot, v) + a_{j,t}(\tilde{u})(\cdot, v)\|_2 \le$$

$$\le C(t-s)^{-\delta}(s-v)^\delta \|u(\cdot, v) - \tilde{u}(\cdot, v)\|_2 +$$
$$+ C\|u(\cdot, s) - \tilde{u}(\cdot, s) - u(\cdot, v) + \tilde{u}(\cdot, v)\|_2, \tag{51}$$

instead of (30) and (31) respectively. These bounds can be established similarly to (30) and (31), see [12, Lemma 3.3] for their proofs. Mention that for (51) we need the assumption that h is an affine function. We will obtain

$$\sup_{s \in [0,t]} \|I_{\Delta h}(\cdot, t)\|_2^2 \le C_N \int_0^t \|u - \tilde{u}\|_{\alpha,2,s}^2 ds.$$

Finally, the bound

$$J_{\Delta h}(t) \le C_N \int_0^t \|u - \tilde{u}\|_{\alpha,2,s}^2 ds$$

can be proved similarly to (46), inequalities (43) and (44) are replaced by (3.37) and (3.38) from [12] respectively. Note that for this term we need to choose $\delta \in \left(\frac{d}{d+2}, 1 - 2\alpha \right)$, this leads to the restriction $\alpha < \frac{1}{d+2}$. We refer to the proof of part (b) of Theorem 2.3 in [12] for the details on the estimation of $I_{\Delta h}$ and $J_{\Delta h}$.

Thus, we see that

$$\|u - \tilde{u}\|_{\alpha,2,t}^2 \mathbb{1}_t \le C_{N,R} \left[\eta_{\alpha,H,t}^2 + \int_0^t \|u - \tilde{u}\|_{\alpha,2,s}^2 \mathbb{1}_s ds + \sup_{s \in [0,t]} \|I_{\Delta g}(\cdot, t)\|_2^2 \mathbb{1}_t + J_{\Delta g}(t) \mathbb{1}_t \right].$$

Similarly to (48) (using the Lipschitz continuity of g instead of the linear growth condition), we get

$$\mathsf{E} \left[\sup_{s \in [0,t]} \|I_{\Delta g}(\cdot, s)\|_2^2 \mathbb{1}_t \right] \le C \int_0^t \mathsf{E} \left[\|u - \tilde{u}\|_{\alpha,2,s}^2 \mathbb{1}_s \right] ds,$$

$$\mathsf{E}[J_{\Delta g}(t) \mathbb{1}_t] \le C \int_0^t \mathsf{E} \left[\|u - \tilde{u}\|_{\alpha,2,s}^2 \mathbb{1}_s \right] ds.$$

Then

$$\mathsf{E} \left[\|u - \tilde{u}\|_{\alpha,2,t}^2 \mathbb{1}_t \right] \le C_{N,R} \left(\mathsf{E}\eta_{\alpha,H,t}^2 + \int_0^t \mathsf{E} \left[\|u - \tilde{u}\|_{\alpha,2,s}^2 \mathbb{1}_s \right] ds \right),$$

and the result follows from Gronwall's lemma. $\qquad\square$

Remark A.1. It follows from the above proof that the statement of Lemma A.3 remains true, if we replace the sequences $\left\{ B_j^{H,N,n}, j \in \mathbb{N} \right\}$ and $\left\{ B_j^{H,N,m}, j \in \mathbb{N} \right\}$ by other two sequences $\{Z_j, j \in \mathbb{N}\}$ and $\left\{ \widetilde{Z}_j, j \in \mathbb{N} \right\}$ of independent identically distributed Hölder continuous processes, satisfying the assumptions

$$\sum_{j=1}^\infty \mu_j^{1/2} \|Z_j\|_{\alpha,0;T} \le N \quad \text{and} \quad \sum_{j=1}^\infty \mu_j^{1/2} \left\| \widetilde{Z}_j \right\|_{\alpha,0;T} \le N.$$

More precisely, if u and $\tilde{u}$ are two processes, satisfying (16) with B_j^H replaced by Z_j and $\widetilde{Z}_j$ respectively, then we have

$$\mathsf{E}\|u - \tilde{u}\|_{\alpha,2,T}^2 \mathbb{1}_{A_T^R} \le C_{N,R} \mathsf{E} \left(\sum_{j=1}^\infty \mu_j^{1/2} \left\| Z_j - \widetilde{Z}_j \right\|_{\alpha,0;T} \right)^2,$$

where $A_T^R = \left\{ \|u\|_{\alpha,2,T} \leq R, \|\tilde{u}\|_{\alpha,2,T} \leq R \right\}$. For example, we may consider the processes $Z_j = B_j^{H,N,n}$ and $\widetilde{Z}_j = B_j^{H,N}$ from the proof of Theorem 3.1 and the corresponding solutions $u = u_{N,n}$ and $\tilde{u} = u_N$.

REFERENCES

1. G. Da Prato, J. Zabczyk, *Stochastic equations in infinite dimensions*, second ed., Encyclopedia of Mathematics and its Applications, vol. 152, Cambridge University Press, Cambridge, 2014.

2. J. Duan, W. Wang, *Effective dynamics of stochastic partial differential equations*, Elsevier Insights, Elsevier, Amsterdam, 2014.

3. S. D. Eidel'man, S. D. Ivasišen, *Investigation of the Green's matrix of a homogeneous parabolic boundary value problem*, Trudy Moskov. Mat. Obšč., **23** (1970), 179–234, English transl. in Trans. Moscow Math. Soc. 23 (1970), 179–242 (1972).

4. S. D. Eidelman, N. V. Zhitarashu, *Parabolic boundary value problems*, Operator Theory: Advances and Applications, vol. 101, Birkhäuser Verlag, Basel, 1998, Translated from the Russian original by Gennady Pasechnik and Andrei Iacob.

5. P. Kotelenez, *A stopped Doob inequality for stochastic convolution integrals and stochastic evolution equations.* Stochastic Anal. Appl., **2** (1984), no. 3, 245–265.

6. B. Maslowski, D. Nualart, *Evolution equations driven by a fractional Brownian motion*, J. Funct. Anal., **202** (2003), no. 1, 277–305.

7. Y. Mishura, *Stochastic calculus for fractional Brownian motion and related processes*, vol. 1929, Springer Science & Business Media, 2008.

8. Y. Mishura, G. Shevchenko, *Mixed stochastic differential equations with long-range dependence: Existence, uniqueness and convergence of solutions*, Comput. Math. Appl., **64** (2012), no. 10, 3217–3227.

9. D. Nualart, A. Răşcanu, *Differential equations driven by fractional Brownian motion*, Collect. Math., **53** (2002), no. 1, 55–81.

10. D. Nualart, P.-A. Vuillermot, *Variational solutions for partial differential equations driven by a fractional noise*, J. Funct. Anal., **232** (2006), no. 2, 390–454.

11. M. Sanz-Solé, P.-A. Vuillermot, *Equivalence and Hölder-Sobolev regularity of solutions for a class of non-autonomous stochastic partial differential equations*, Ann. Inst. H. Poincaré Probab. Statist., **39** (2003), no. 4, 703–742.

12. M. Sanz-Solé, P. A. Vuillermot, *Mild solutions for a class of fractional SPDEs and their sample paths*, J. Evol. Equ., **9** (2009), no. 2, 235–265.

13. G. Shevchenko, *Mixed stochastic delay differential equations*, Theory Probab. Math. Statist., **89** (2014), 181–195.

14. L. Tubaro, *An estimate of Burkholder type for stochastic processes defined by the stochastic integral*, Stochastic Anal. Appl., **2** (1984), no. 2, 187–192.

15. M. C. Veraar, *Non-autonomous stochastic evolution equations and applications to stochastic partial differential equations*, J. Evol. Equ., **10** (2010), no. 1, 85–127.

16. J. Zabczyk, *A mini course on stochastic partial differential equations*, Stochastic climate models (Chorin, 1999), Progr. Probab., vol. 49, Birkhäuser, Basel, 2001, 257–284.

Department of Probability Theory, Statistics and Actuarial Mathematics, Taras Shevchenko National University of Kyiv, 64 Volodymyrska, 01601 Kyiv, Ukraine
E-mail address: `myus@univ.kiev.ua`

Department of Probability Theory, Statistics and Actuarial Mathematics, Taras Shevchenko National University of Kyiv, 64 Volodymyrska, 01601 Kyiv, Ukraine
E-mail address: `k.ralchenko@gmail.com`

Department of Probability Theory, Statistics and Actuarial Mathematics, Taras Shevchenko National University of Kyiv, 64 Volodymyrska, 01601 Kyiv, Ukraine
E-mail address: `zhora@univ.kiev.ua`

ІСНУВАННЯ ТА ЄДИНІСТЬ М'ЯКОГО РОЗВ'ЯЗКУ СТОХАСТИЧНОГО РІВНЯННЯ ТЕПЛОПРОВІДНОСТІ З БІЛИМ ТА ДРОБОВИМ ШУМАМИ

Ю. МІШУРА, К. РАЛЬЧЕНКО, Г. ШЕВЧЕНКО

Анотація. Доведено існування та єдиність м'якого розв'язку для класу неавтономних параболічних змішаних стохастичних диференціальних рівнянь із частинними похідними, які визначені на обмеженій відкритій підмножині $D \subset \mathbb{R}^d$ та містять стандартний і дробовий $L^2(D)$-значні броунівські рухи. Припускається, що коефіцієнти є однорідними, ліпшицевими, а коефіцієнт при дробовому броунівському русі є афінною функцією.

СУЩЕСТВОВАНИЕ И ЕДИНСТВЕННОСТЬ МЯГКОГО РЕШЕНИЯ СТОХАСТИЧЕСКОГО УРАВНЕНИЯ ТЕПЛОПРОВОДНОСТИ С БЕЛЫМ И ДРОБНЫМ ШУМАМИ

Ю. МИШУРА, К. РАЛЬЧЕНКО, Г. ШЕВЧЕНКО

Аннотация. Доказаны существование и единственность мягкого решения для класса неавтономных параболических смешанных стохастических дифференциальных уравнений в частных производных, которые определены на ограниченном открытом подмножестве $D \subset \mathbb{R}^d$ и содержат стандартное и дробное $L^2(D)$-значные броуновские движения. Предполагается, что коэффициенты — однородные и липшицевые, а коэффициент при дробном броуновском движении — афинная функция.