LIMIT BEHAVIOR OF THE ROSENBLATT ORNSTEIN–UHLENBECK PROCESS WITH RESPECT TO THE HURST INDEX

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ABSTRACT. We study the convergence in distribution, as $H \to \frac{1}{2}$ and as $H \to 1$, of the integral $\int_{\mathbb{R}} f(u)dZ^H(u)$, where Z^H is a Rosenblatt process with self-similarity index $H \in \left(\frac{1}{2}, 1\right)$ and f is a suitable deterministic function. We focus our analysis on the case of the Rosenblatt Ornstein–Uhlenbeck process, which is the solution of the Langevin equation driven by the Rosenblatt process.

Key words and phrases. Wiener chaos, Rosenblatt process, cumulants, Hurst parameter.

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1. INTRODUCTION

The Rosenblatt process is a stochastic process in the second Wiener chaos, i. e. it can be expressed as a multiple integral of order two with respect to the Wiener process. It is a non-Gaussian self-similar process with stationary increments that exhibits long memory. The Rosenblatt process belongs to the class of so-called Hermite processes, which are self-similar processes with stationary increments in the qth Wiener chaos ($q \geq 1$) (the Rosenblatt process is obtained for $q = 2$ while for $q = 1$ we have the fractional Brownian motion, which is the only Gaussian Hermite process). The Rosenblatt process has been widely studied in the last decades, see e. g. the monographs [10] and [14] and the references therein.

There are several recent research works that investigate the asymptotic behavior in distribution of some fractional processes [1–3, 15] with respect to the Hurst parameter. In particular, in the case of the Rosenblatt process $(Z^H(t))_{t \geq 0}$ with self-similarity index $H \in (\frac{1}{2}, 1)$, it has been shown in [15] that Z^H converges weakly, as $H \to \frac{1}{2}$, in the space of continuous functions $C[0, T]$ (for every $T > 0$), to a Brownian motion while if $H \to 1$, it tends weakly to the stochastic process $(t \frac{1}{\sqrt{2}}(Z^2 - 1))_{t \geq 0}$, $Z^2 - 1$ being a so-called *centered chi-square random variable.* The case of the generalized Rosenblatt process has been considered in [2] while the case of the Rosenblatt sheet can be found in [1]. Hermite processes of higher order have been considered in [1, 3].

The purpose of this work is to investigate the asymptotic behavior in distribution, with respect to the Hurst parameter, of the Wiener integral with respect to the Rosenblatt process (or the Wiener–Rosenblatt integral). The Wiener–Rosenblatt integral $\int_{\mathbb{R}} f(u)dZ^H(u)$ has been introduced in [6], for a sufficiently regular deterministic function f. In a first part, we give the asymptotic behavior in distribution, as $H \to \frac{1}{2}$ and as $H \to 1$, of the random variable $\int_{\mathbb{R}} f(u)dZ^H(u)$, by assuming suitable integrability condition on f. We will then focus on the asymptotic behavior with respect to H of the Rosenblatt Ornstein–Uhlenbeck process (ROU for short) which constitutes the unique solution of the Langevin equation driven by the Rosenblatt process. The ROU process can be expressed in the form of a Wiener–Rosenblatt integral with a particular kernel f. In order to check that this kernel verifies the integrability conditions needed to apply our general result, we need to use some technical results, in particular the so-called power counting theorem from [13]. We will treat separately the cases of the non-stationary ROU (whose initial value does not depend on H) and of the stationary ROU (with initial value

depending on the Hurst parameter). We will prove that the (stationary) ROU converges weakly, as $H \to \frac{1}{2}$, to the (stationary) Gaussian Ornstein–Uhlenbeck process (solution of the Langevin equation driven by the Brownian motion) while as $H \to 1$, the ROU process converges weakly to a chi-square random variable multiplied by a deterministic process. Since we deal with processes in the second Wiener chaos, our proofs rely on the analysis of the asymptotic behavior of the cumulants of the random variables concerned (recall that the distribution of the elements of the second Wiener is completely determined by their cumulants, see [5] or [8]).

We organized our paper as follows. In Section 2 we recall some basic definitions for the Rosenblatt process and the Wiener–Rosenblatt integral and we also state a general result for the limit behavior in law of the Wiener–Rosenblatt integral as the Hurst index converges to its extreme values. In Section 3 we treat the particular case of the Rosenblatt Ornstein–Uhlenbeck process.

2. The Rosenblatt process and the Wiener–Rosenblatt integral

Below, in the first part, we present the definition and the basic properties of the Rosenblatt process and of the Wiener integral with respect to the Rosenblatt process. In the second part, we give a general result concerning the convergence in distribution with respect to the Hurst parameter of the Wiener–Rosenblatt integral. This general result will be applied in the next section in order to obtain the limit behavior of the Ornstein–Uhlenbeck process with Rosenblatt noise.

2.1. **Definition and basic properties.** Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $(B(y))_{y \in \mathbb{R}}$ be a Wiener process on Ω. We will denote by $(Z^H(t))_{t \geq 0}$ the Rosenblatt process with self-similarity index $H \in \left(\frac{1}{2}, 1\right)$. It is defined on Ω as a multiple stochastic integral of order 2 with respect to the Wiener process B via

$$Z^H(t) = c(H, 2) \int_{\mathbb{R}} \int_{\mathbb{R}} \left(\int_0^t (u - y_1)_+^{\frac{H}{2} - 1} (u - y_2)_+^{\frac{H}{2} - 1} du \right) dB(y_1) dB(y_2) = I_2(L_t^H), \quad t \geq 0, \tag{1}$$

where I_2 denotes the multiple stochastic integral of order two with respect to B (see the Appendix) and we denoted by L_t^H the kernel of the Rosenblatt process given by, for every $y_1, y_2 \in \mathbb{R}$,

$$L_t^H(y_1, y_2) = c(H, 2) \int_0^t (u - y_1)_+^{\frac{H}{2} - 1} (u - y_2)_+^{\frac{H}{2} - 1} du. \tag{2}$$

We denoted $x_+ = \max(x, 0)$. It is well-known (see e. g. [14]) that the kernel L_t^H belongs to $L^2(\mathbb{R}^2)$ for every $t \geq 0$ when $H > \frac{1}{2}$, which implies that the multiple integral of order two in (1) is well-defined. The strictly positive constant $c(H, 2)$ is chosen such that $\mathsf{E}(Z^H(1))^2 = 1$. Actually (see e. g. [14, Proposition 3.1]

$$c(H, 2)^2 = \frac{H(2H - 1)}{2\beta\left(\frac{H}{2}, 1 - H\right)^2}, \tag{3}$$

where β is Beta function $\beta(p, q) = \int_0^1 z^{p-1}(1 - z)^{q-1} dz, p, q > 0$.

The Rosenblatt process is a H-self-similar process with stationary increments. It exhibits long-range dependence and its sample paths are Hölder continuous of order δ for any $\delta \in (0, H)$. This process has been intensively studied in the last decades, see e. g. the monographs [10] and [14] and the references therein.

The Wiener integral with respect to the Rosenblatt process (or the *Wiener–Rosenblatt integral*) has been constructed in [6]. Let $f \in \mathcal{H}_H$ where $\mathcal{H}_H = \{f \colon \mathbb{R} \to \mathbb{R} : \|f\|_{\mathcal{H}_H} < \infty\}$

with

$$\|f\|_{\mathcal{H}_H}^2 := H(2H-1) \int_{\mathbb{R}} \int_{\mathbb{R}} f(u)f(v)|u-v|^{2H-2} du dv.$$

Then the Wiener integral of f with respect to Z^H is given by

$$\int_{\mathbb{R}} f(u) dZ^H(u) = I_2(J_H f) \tag{4}$$

where the kernel $J_H f$ has the expression, for every $y_1, y_2 \in \mathbb{R}$,

$$(J_H f)(y_1, y_2) = c(H, 2) \int_{\mathbb{R}} f(u)(u-y_1)_+^{\frac{H}{2}-1}(u-y_2)_+^{\frac{H}{2}-1} du. \tag{5}$$

The Wiener–Rosenblatt integral constitutes an isometry between $\mathcal{H}_H$ and $L^2(\Omega)$ since, for every $f, g \in \mathcal{H}_H$,

$$\mathsf{E}\left(\int_{\mathbb{R}} f(u) dZ^H(u) \int_{\mathbb{R}} g(u) dZ^H(u)\right) = \langle f, g \rangle_{\mathcal{H}_H} :=$$

$$:= H(2H-1) \int_{\mathbb{R}} \int_{\mathbb{R}} f(u)g(v)|u-v|^{2H-2} du dv. \tag{6}$$

The space $\mathcal{H}_H$ is not complete and may contains distributions. A subspace of functions included in $\mathcal{H}_H$ is the space $|\mathcal{H}_H|$ of measurable functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$\|f\|_{|\mathcal{H}_H|}^2 := \int_{\mathbb{R}} \int_{\mathbb{R}} |f(u)f(v)||u-v|^{2H-2} du dv < \infty. \tag{7}$$

2.2. **Asymptotic behavior of the Wiener–Rosenblatt integral.** Our purpose is to study the asymptotic behavior, as $H \to \frac{1}{2}$ and $H \to 1$, of the Wiener integral with respect ot the Rosenblatt process

$$\int_{\mathbb{R}} f(u) dZ^H(u)$$

where $(Z^H(t))_{t \geq 0}$ is a Rosenblatt process with self-similarity order $H \in (\frac{1}{2}, 1)$ and $f \in \mathcal{H}_H$.

The proof of the asymptotic behavior of the Wiener–Rosenblatt integral is based on the analysis of its cumulants. Since the random variable

$$\int_{\mathbb{R}} f(u) dZ^H(u)$$

belongs to the second Wiener chaos, see (4), its law is completely determined by its cumulants (or equivalently, by its moments). That is, if F, G are elements of the second Wiener chaos then F and G have the same law if and only if they have the same cumulants. Moreover, the convergence of the cumulants implies the convergence in distribution when we deal with sequences in the Wiener chaos of order two. Let us denote by $k_m(F)$, $m \geq 1$, the mth cumulant of a random variable F. It is defined as

$$k_m(F) = (-i)^n \frac{\partial^n}{\partial t^n} \ln \mathsf{E}(e^{itF})|_{t=0}.$$

We have the following link between the moments and the cumulants of F: for every $m \geq 1$,

$$k_m(F) = \sum_{\sigma=(a_1,\ldots,a_r)\in\mathcal{P}(\{1,\ldots,n\})} (-1)^{r-1}(r-1)! \mathsf{E} X^{|a_1|} \ldots \mathsf{E} X^{|a_r|} \tag{8}$$

if $F \in L^m(\Omega)$, where $\mathcal{P}(b)$ is the set of all partitions of b. In particular, for centered random variables F, we have $k_1(F) = \mathsf{E}F$, $k_2(F) = \mathsf{E}F^2$, $k_3(F) = \mathsf{E}F^3$, $k_4 = \mathsf{E}F^4 - (\mathsf{E}F^2)^2$.

In the particular situation where $G = I_2(f)$ is a multiple integral of order 2 with respect to a Wiener process $(B(y))_{y \in \mathbb{R}}$, its cumulants can be computed as (see e. g. [7, Proposition 7.2] or [14])

$$k_m(G) = 2^{m-1}(m-1)! \int_{\mathbb{R}^m} du_1 \dots du_m f(u_1, u_2) f(u_2, u_3) \dots f(u_{m-1}, u_m) f(u_m, u_1). \quad (9)$$

From the formula (9), we can obtain the following expression of the cumulants of the Wiener–Rosenblatt integral (see e. g. [12]).

Proposition 1. *We have, for $m \geq 2$,*

$$k_m \left(\int_{\mathbb{R}} f(u) dZ^H(u) \right) = c_{1,m} \int_{\mathbb{R}} \dots \int_{\mathbb{R}} du_1 \dots du_m f(u_1) \dots f(u_m) \times$$
$$\times |u_1 - u_2|^{H-1} |u_2 - u_3|^{H-1} \dots |u_{m-1} - u_m|^{H-1} |u_m - u_1|^{H-1} \quad (10)$$

with

$$c_{1,m} = 2^{\frac{m}{2}-1}(m-1)!(H(2H-1))^{\frac{m}{2}} \quad (11)$$

and $k_1 \left(\int_{\mathbb{R}} f(u) dZ^H(u) \right) = 0$.

We will treat separately the behavior of the Wiener–Rosenblatt integral as H is near $\frac{1}{2}$ and near 1. The limiting process will be different in these two cases.

2.2.1. *Convergence when $H \to 1$.* We have the following result.

Proposition 2. *Consider $f : \mathbb{R} \to \mathbb{R}$ such that for some $\varepsilon \in (0, \frac{1}{2})$*

$$\|f\|^2_{\left| \mathcal{H}_{\frac{1}{2}+\varepsilon} \right|} = \int_{\mathbb{R}} \int_{\mathbb{R}} du dv |f(u)||f(v)||u-v|^{2\varepsilon-1} < \infty \text{ and } f \in L^1(\mathbb{R}). \quad (12)$$

Then

$$\int_{\mathbb{R}} f(u) dZ^H(u) \xrightarrow[H \to 1]{(d)} \frac{1}{\sqrt{2}} \left(\int_{\mathbb{R}} f(u) du \right)(Z^2 - 1) \quad (13)$$

with $Z \sim N(0,1)$.

Proof. First notice that condition (12) implies that $f \in |\mathcal{H}_H|$ for $H \geq \frac{1}{2} + \varepsilon$. Indeed, by using the bound $\sup_{H \in [\frac{1}{2}+\varepsilon,1]} |x|^{2H-2} \leq 1 \vee |x|^{2\varepsilon-1}$, we get

$$\|f\|^2_{|\mathcal{H}_H|} \leq \int_{\mathbb{R}} \int_{\mathbb{R}} du dv |f(u)f(v)|(1 \vee |u-v|^{2\varepsilon-1}) \leq$$
$$\leq \left(\int_{\mathbb{R}} |f(u)| du \right)^2 + \int_{\mathbb{R}} \int_{\mathbb{R}} du dv |f(u)||f(v)||u-v|^{2\varepsilon-1} < \infty.$$

Consider the random variable

$$G := I_2 \left(\frac{1}{\sqrt{2}} \left(\int_{\mathbb{R}} f(u) du \right) 1^{\otimes 2}_{[0,1]} \right)$$

which has the same law as the right-hand side of (13). We have

$$k_m(G) = 2^{m-1}(m-1)! \left(\frac{1}{\sqrt{2}} \left(\int_{\mathbb{R}} f(u) du \right) \right)^m = 2^{\frac{m}{2}-1}(m-1)! \left(\int_{\mathbb{R}} f(u) du \right)^m.$$

On the other hand, since by (11)

$$c_{1,m} \xrightarrow[H \to 1]{} 2^{\frac{m}{2}-1}(m-1)!$$

the conclusion will follow if we show that

$$\int_{\mathbb{R}}\ldots\int_{\mathbb{R}}du_1\ldots du_m f(u_1)\ldots f(u_m)|u_1-u_2|^{H-1}|u_2-u_3|^{H-1}\ldots|u_{m-1}-u_m|^{H-1}|u_m-u_1|^{H-1}$$

converges, as $H \to 1$, to

$$\left(\int_{\mathbb{R}}f(u)du\right)^m.$$

Assume that the integrability condition (12) is satisfied. Consider the function g_H defined on $\mathbb{R}^m$ except the diagonals with values in $\mathbb{R}$, given by

$$g_H(u_1,\ldots,u_m) =$$
$$= f(u_1)\ldots f(u_m)|u_1-u_2|^{H-1}|u_2-u_3|^{H-1}\ldots|u_{m-1}-u_m|^{H-1}|u_m-u_1|^{H-1}.$$

Clearly $g_H(u_1,\ldots,u_m)$ converges to $f(u_1)\ldots f(u_m)$ for almost every $u_1,\ldots,u_m \in \mathbb{R}$. Also, using again the bound $\sup_{H\in[\frac{1}{2}+\varepsilon,1]}|x|^{H-1} \leq 1 \vee |x|^{-\frac{1}{2}+\varepsilon}$ for every $x \in \mathbb{R}$, we have

$$\sup_{H\in[\frac{1}{2}+\varepsilon,1]}|g_H(u_1,\ldots,u_m)| \leq$$

$$\leq |f(u_1)\ldots f(u_m)|\left(1 \vee \left(|u_1-u_2|^{-\frac{1}{2}+\varepsilon}\ldots|u_m-u_1|^{-\frac{1}{2}+\varepsilon}\right)\right) \leq$$

$$\leq |f(u_1)\ldots f(u_m)| + |f(u_1)\ldots f(u_m)|\ldots|u_1-u_2|^{-\frac{1}{2}+\varepsilon}\ldots|u_m-u_1|^{-\frac{1}{2}+\varepsilon}.$$

In order to apply the dominated convergence theorem, we need to show that the function

$$|f(u_1)\ldots f(u_m)| + |f(u_1)\ldots f(u_m)|\ldots|u_1-u_2|^{-\frac{1}{2}+\varepsilon}\ldots|u_m-u_1|^{-\frac{1}{2}+\varepsilon}$$

is integrable over $\mathbb{R}^m$. Since $f \in L^1(\mathbb{R})$, the first summand above is integrable over $\mathbb{R}^m$. On the other hand,

$$\int_{\mathbb{R}}\ldots\int_{\mathbb{R}}du_1\ldots du_m|f(u_1)\ldots f(u_m)|\ldots|u_1-u_2|^{-\frac{1}{2}+\varepsilon}\ldots|u_m-u_1|^{-\frac{1}{2}+\varepsilon}$$

represents, modulo a constant, the mth cumulant of the random variable $I_2(J_{\frac{1}{2}+\varepsilon}|f|)$, where $J_{\frac{1}{2}+\varepsilon}$ is given by (5). The fact that $\|f\|^2_{\left|\mathcal{H}_{\frac{1}{2}+\varepsilon}\right|} < \infty$ (see (10)) by (12) and the hypercontractivity property of multiple integrals (38) imply that the all the moments of $I_2(J_{\frac{1}{2}+\varepsilon}|f|)$ are finite and consequently, via (8), all the cumulants of $I_2(J_{\frac{1}{2}+\varepsilon}|f|)$ are finite. This implies the conclusion by using the dominated convergence theorem. $\quad\square$

Remark 1. For example, when f is bounded with compact support, then condition (12) is satisfied (in particular if $f = 1_{[0,t]}$ with $t > 0$ fixed). A detailed explanation can be found in the proof of Proposition 4 below.

2.2.2. *Convergence when* $H \to \frac{1}{2}$. Concerning the behavior of the Rosenblatt-Wiener integral when H approaches one half, we have the following result.

Proposition 3. *Assume $f \in \mathcal{H}_H$. Also assume that*

$$\sigma_f^2 = \lim_{H\to\frac{1}{2}}\|f\|^2_{\mathcal{H}_H} = \lim_{H\to\frac{1}{2}}H(2H-1)\int_{\mathbb{R}}\int_{\mathbb{R}}f(u)f(v)|u-v|^{2H-2}dudv \qquad (14)$$

exists and it is finite and

$$(2H-1)^2\int_{\mathbb{R}^4}du_1\ldots du_4 f(u_1)\ldots f(u_4)|u_1-u_2|^{H-1}|u_2-u_3|^{H-1} \times$$

$$\times |u_3-u_4|^{H-1}|u_4-u_1|^{H-1}\xrightarrow[H\to\frac{1}{2}]{} 0. \qquad (15)$$

Then

$$\int_{\mathbb{R}} f(u)dZ^H(u) \xrightarrow[H \to \frac{1}{2}]{(d)} N(0, \sigma_f^2).$$

Proof. We need to analyze the behavior of the cumulants of $\int_{\mathbb{R}} f(u)dZ^H(u)$ as H converges to $\frac{1}{2}$. Recall that $k_1\left(\int_{\mathbb{R}} f(u)dZ^H(u)\right) = \mathsf{E}\left(\int_{\mathbb{R}} f(u)dZ^H(u)\right) = 0$ and

$$k_2\left(\int_{\mathbb{R}} f(u)dZ^H(u)\right) = H(2H-1)\int_{\mathbb{R}}\int_{\mathbb{R}} du_1 du_2 f(u_1)f(u_2)|u_1 - u_2|^{2H-2} = \|f\|_{\mathcal{H}_H}^2.$$

By (14), we have $k_2\left(\int_{\mathbb{R}} f(u)dZ^H(u)\right) \xrightarrow[H \to \frac{1}{2}]{} \sigma_f^2$. On the other hand, due to condition (15),

$$k_4\left(\int_{\mathbb{R}} f(u)dZ^H(u)\right) \xrightarrow[H \to \frac{1}{2}]{} 0.$$

Since $\int_{\mathbb{R}} f(u)dZ^H(u)$ belongs to the second Wiener chaos, by the Fourth Moment Theorem (see [9], see also Theorem 1 in the Appendix) we conclude that $\int_{\mathbb{R}} f(u)dZ^H(u)$ converges in distribution as $H \to \frac{1}{2}$ to a Gaussian random variable with mean zero (the limit of its first cumulant) and variance σ_f^2 (the limit of its second cumulant). $\qquad\square$

We will see below that for certain kernels f the condition (14) is automatically satisfied. Recall that a sequence of functions $(f_n)_{n \geq 1}$ is an approximation of the identity as $n \to \infty$ if

- $f_n(t) \geq 0$ for every $t \in \mathbb{R}$.
- For every $\delta > 0$, $\int_{|t| \leq \delta} f_n(t)dt \xrightarrow[n \to \infty]{} 1$.
- For every $\delta > 0$, $\int_{|t| > \delta} f_n(t)dt \xrightarrow[n \to \infty]{} 0$.

Moreover, if $(f_n)_{n \geq 1}$ is an approximation of the identity, then for every $f \in L^p(\mathbb{R})$ with $p \in [1, \infty)$, the convolution $f * f_n$ converges in $L^p(\mathbb{R})$ to f.

Corollary 1. *Assume $f \in \mathcal{H}_H \cap L^2(\mathbb{R})$ with $\operatorname{supp}(f) \subset [0, \infty)$ and (15) holds. Then*

$$\int_{\mathbb{R}} f(u)dZ^H(u) \xrightarrow[H \to \frac{1}{2}]{(d)} N\left(0, \int_{\mathbb{R}} f^2(u)du\right).$$

Proof. By Proposition 2, it suffices to check that the limit (14) exists and it is equal to $\int_{\mathbb{R}} f^2(u)du$. We have

$$\|f\|_{\mathcal{H}_H}^2 = H(2H-1)\int_0^\infty \int_0^\infty dudv f(u)f(v)|u-v|^{2H-2} =$$
$$= 2H(2H-1)\int_0^\infty du f(u) \int_0^u dv f(u-v)v^{2H-2}.$$

Notice that the function $2H(2H-1)1_{[0,u]}(v)v^{2H-2}$ constitutes an approximation of the identity as $H \to \frac{1}{2}$. Therefore,

$$\|f\|_{\mathcal{H}_H}^2 \xrightarrow[H \to \frac{1}{2}]{} \int_{\mathbb{R}} f^2(u)du.$$

By condition (15) and Theorem 1, the conclusion follows. $\qquad\square$

Remark 2.   (i) The above result shows that, when $H \to \frac{1}{2}$, the Wiener–Rosenblatt integral $\int_{\mathbb{R}} f(u)dZ^H(u)$ converges in distribution to $\int_{\mathbb{R}} f(u)dW(u)$, where W is a Wiener process. This is a natural extension of the results in [1] or [15].

   (ii) The condition $\operatorname{supp}(f) \subset [0, \infty)$ in Corollary 1 cannot be omitted. If for example $\operatorname{supp}(f)$ is $\mathbb{R}$, the above proof does not work, since $2H(2H-1)1_{[0,\infty)}(v)v^{2H-2}$ is not an approximation of the unity.

3. Asymptotic behavior of the Rosenblatt Ornstein–Uhlenbeck process

The Rosenblatt Ornstein–Uhlenbeck (ROU) process is defined as the unique solution of the Langevin equation

$$X_t = \xi - \lambda \int_0^t X_s ds + \sigma Z^H(t), \quad t \geq 0, \tag{16}$$

where $\lambda, \sigma > 0$ and the initial condition ξ is a random variable in $L^2(\Omega)$. The case when the noise in (16) is the fractional Brownian motion has been considered in [4].

The unique solution to (16) can be expressed as

$$Y^H(t) = e^{-\lambda t}\left(\xi + \sigma \int_0^t e^{\lambda u} dZ^H(u)\right) \tag{17}$$

where the stochastic integral with respect to Z^H can be understood both in the Wiener or Riemann-Stieltjes sense.

The stationary Rosenblatt Ornstein–Uhlenbeck process is obtained by taking the initial condition $\xi = \sigma \int_{-\infty}^0 e^{-\lambda u} dZ^H(u)$ in (16). Then, the stationary ROU, which will be denoted in the sequel by $(X^H(t))_{t \geq 0}$, can be expressed as, for every $t \geq 0$,

$$X^H(t) = \sigma \int_{-\infty}^t e^{-\lambda(t-u)} dZ^H(u). \tag{18}$$

The process $(X^H(t))_{t \geq 0}$ is a stationary Gaussian process, H-self-similar with stationary increments. Moreover, it exhibits long-range dependence since $H > \frac{1}{2}$, see [4] or [6].

In this paragraph, our purpose is to analyze the asymptotic behavior, as $H \to 1$ and as $H \to \frac{1}{2}$, in the sense of the weak convergence, of the processes (17) and (18). The analysis of the limit behavior at the extreme critical values of the Hurst exponent is different for X^H and Y^H due to the fact that the the initial values depends on H in the case of X^H.

3.1. **Padded sets and the power counting theorem.** We need to recall some notation and results from [13] which are needed in order to check the integrability assumption from Proposition 3.

Consider a set $T = \{M_1, \ldots, M_m\}$ of linear functions on $\mathbb{R}^m$. The power counting theorem (see Theorem 1.1 and Corollary 1.1 in [13]) gives sufficient conditions for the integral

$$I = \int_{\mathbb{R}} \ldots \int_{\mathbb{R}} du_1 \ldots du_m f_1(M_1(u_1, \ldots, u_m)) \ldots f_m(M_m(u_1, \ldots, u_m)) \tag{19}$$

to be finite, where $f_i : \mathbb{R} \to \mathbb{R}$, $i = 1, \ldots, m$ are such that $|f_i|$ is bounded above on (a_i, b_i) $(0 < a_i < b_i < \infty)$ and

$$|f_i(y)| \leq c_i |y|^{\alpha_i} \text{ if } |y_i| < a_i \text{ and } |f_i(y)| \leq c_i |y|^{\beta_i} \text{ if } |y| > b_i.$$

For a subset $W \subset T$ we denote by $s_T(W) = span(W) \cap T$. A subset W of T is said to be *padded* if $s_T(W) = W$ and any functional $M \in W$ also belongs to $s_T(W \setminus \{M\})$. Denote by $\text{span}(W)$ the linear span generated by W and by $r(W)$ the number of linearly independent elements of W.

Then Theorem 1.1 in [13] says that the integral I (19) is finite if

$$d_0(W) = r(W) + \sum_{s_T(W)} \alpha_i > 0 \tag{20}$$

for any subset W of T with $s_T(W) = W$ and

$$d_\infty(W) = r(T) - r(W) + \sum_{T \setminus s_T(W)} \beta_i < 0 \tag{21}$$

for any proper subset W of T with $s_T(W) = W$, including the empty set. If $\alpha_i > -1$ then it suffices to check (20) for any padded subset $W \subset T$. Also, it suffices to verify (21) only for padded subsets of T if $\beta_i \geq -1$.

The condition (20) implies the integrability at the origin while (21) gives the integrability of I at infinity.

There is a similar result if one starts with a set T of affine functionals instead of linear functionals.

3.2. The (non-stationary) ROU process.

We first treat the case of the process (17) with initial condition not depending on the Hurst index. In the sequel, we fix $T > 0$ arbitrary chosen.

3.2.1. *Convergence when $H \to 1$.*

Proposition 4. *Assume that the initial condition ξ does not depend on H. Then the process $(Y^H(t))_{t \in [0,T]}$ converges weakly, in the space of continuous functions $C[0,T]$, to the stochastic process $(Y(t))_{t \in [0,T]}$ given by*

$$Y(t) = e^{-\lambda t}\xi + \sigma \frac{1}{\sqrt{2}}\left(\int_0^t e^{-\lambda(t-u)}du\right)(Z^2 - 1) = e^{-\lambda t}\xi + \frac{\sigma}{\sqrt{2\lambda}}\left(1 - e^{-\lambda t}\right)\left(Z^2 - 1\right) \quad (22)$$

with $Z \sim N(0,1)$.

Proof. We start by checking the convergence of the finite dimensional distribution of Y^H of those of Y. Take $\alpha_1, \ldots, \alpha_d \in \mathbb{R}$ and $t_1, \ldots, t_d \in [0,T]$. We will prove that $\sum_{j=1}^d \alpha_j Y^H(t_j)$ converges in distribution, as $H \to 1$ to the random variable $\sum_{j=1}^d \alpha_j Y(t_j)$.

We have, by the linearity of the Wiener–Rosenblatt integral,

$$\sum_{j=1}^d \alpha_j Y^H(t_j) = \int_{\mathbb{R}}\left(\sum_{j=1}^d \alpha_j 1_{[0,t_j]}(u)e^{-\lambda(t_j-u)}\right)dZ^H(u) = \int_{\mathbb{R}} f(u)dZ^H(u)$$

with

$$f(u) = \sum_{j=1}^d \alpha_j 1_{[0,t_j]}(u)e^{-\lambda(t_j-u)}. \quad (23)$$

In order to apply Proposition 2, we need to show that condition (12) holds true. Clearly f belongs to $L^1(\mathbb{R})$. Concerning the first part of (12), we have

$$\int_{\mathbb{R}}\int_{\mathbb{R}} dudv|f(u)f(v)||u-v|^{2\varepsilon-1} \leq$$

$$\leq \sum_{j,k=1}^d |\alpha_j\alpha_k|\int_0^{t_j} du \int_0^{t_k} dv e^{-\lambda(t_j-u)}e^{-\lambda(t_k-v)}|u-v|^{2\varepsilon-1} \leq$$

$$\leq \sum_{j,k=1}^d |\alpha_j\alpha_k|\int_0^T du \int_0^T dv e^{-\lambda(t_j-u)}e^{-\lambda(t_k-v)}|u-v|^{2\varepsilon-1} \leq$$

$$\leq \sum_{j,k=1}^d |\alpha_j\alpha_k|\int_0^T du \int_0^T dv |u-v|^{2\varepsilon-1} =$$

$$= \frac{1}{\varepsilon(\varepsilon+1)}T^{2\varepsilon+1}\sum_{j,k=1}^d |\alpha_j\alpha_k| < \infty.$$

Concerning the tightness, notice that for every $0 \leq s < t \leq T$ we have, since Y^H is a solution to (16)

$$\mathsf{E}\left|Y^H(t) - Y^H(s)\right|^2 \leq C|t-s|$$

and therefore, for every $p \geq 1$,

$$\mathsf{E}\big|Y^H(t) - Y^H(s)\big|^{2p} \leq C_p \Big(\mathsf{E}\big|Y^H(t) - Y^H(s)\big|^2\Big)^p \leq c|t - s|^p. \tag{24}$$

The tightness is obtained from (24) and e. g. Lemma 2.2 in [11].   □

Note that the limit process $(Y(t))_{t \in [0,T]}$ given by (22) is a second chaos stochastic process. Therefore, its finite dimensional distributions are characterized by the cumulants, i. e. for every $\alpha_1, \ldots, \alpha_d \in \mathbb{R}$ and $t_1, \ldots, t_d \in [0, T]$,

$$k_m \left(\sum_{j=1}^d \alpha_j Y(t_j) \right) = k_m \left(\frac{\sigma}{\sqrt{2\lambda}} 1_{[0,1]}^{\otimes 2} \sum_{j=1}^d \alpha_j (1 - e^{-\lambda t_j}) \right) =$$

$$= 2^{\frac{m}{2}-1}(m-1)! \frac{\sigma^m}{\lambda^m} \left(\sum_{j=1}^d \alpha_j (1 - e^{-\lambda t_j}) \right)^m,$$

for $m \geq 2$ and

$$k_1 \left(\sum_{j=1}^d \alpha_j Y(t_j) \right) = \mathsf{E} \left(\sum_{j=1}^d \alpha_j Y(t_j) \right) = \mathsf{E}(\xi) . \sum_{j=1}^d \alpha_j e^{-\lambda t_j}.$$

3.2.2. *Convergence when $H \to \frac{1}{2}$.* The (standard) Ornstein–Uhlenbeck process (denoted Y_0 in the sequel) is given by (17) with Z^H replaced by a Wiener process W. Thus it can be written as

$$Y_0(t) = e^{-\lambda t} \left(\xi + \sigma \int_0^t e^{\lambda u} dW(u) \right), \quad \text{for every } t \geq 0. \tag{25}$$

Consequently, $(Y_0(t))_{t \geq 0}$ is a Gaussian process with mean $\mathsf{E}Y_0(t) = e^{-\lambda t} \mathsf{E}\xi$ for any $t \geq 0$ and covariance function

$$Cov(Y_0(t), Y_0(s)) = \frac{\sigma^2}{2\lambda} \left(e^{-\lambda|t-s|} - e^{-\lambda(t+s)} \right)$$

for every $s, t \geq 0$.

The Ornstein–Uhlenbeck process will appear as limit of the ROU process as $H \to \frac{1}{2}$.

Proposition 5. *As $H \to \frac{1}{2}$, the process $Y^H(t)$ converges weakly to the Ornstein–Uhlenbeck process $(Y_0(t))_{t \in [0,T]}$.*

Proof. Consider $\lambda_1, \ldots, \lambda_d \in \mathbb{R}$ and $t_1, \ldots, t_d \in [0, T]$. We will apply Corollary 1. Clearly, as in the proof of Proposition 4, the function $f(u) = \sum_{j=1}^d \alpha_j 1_{[0,t_j]}(u) e^{-\lambda(t_j - u)}$ from (23) belongs to $\mathcal{H}_H \cap L^2(\mathbb{R})$.

We need to show that condition (15) is satisfied. We have

$$\int_{\mathbb{R}^4} du_1 \ldots du_4 f(u_1) \ldots f(u_4) |u_1 - u_2|^{H-1} |u_2 - u_3|^{H-1} |u_3 - u_4|^{H-1} |u_4 - u_1|^{H-1} \leq$$

$$\leq \sum_{j_1, \ldots, j_4 = 1}^d |\alpha_{j_1} \ldots \alpha_{j_4}| \int_0^T \ldots \int_0^T du_1 \ldots du_4 \times$$

$$\times |u_1 - u_2|^{H-1} |u_2 - u_3|^{H-1} |u_3 - u_4|^{H-1} |u_4 - u_1|^{H-1}.$$

Actually, the function

$$H \to \int_0^T \ldots \int_0^T du_1 \ldots du_4 |u_1 - u_2|^{H-1} |u_2 - u_3|^{H-1} |u_3 - u_4|^{H-1} |u_4 - u_1|^{H-1}$$

is finite and continuous on the set $(\frac{1}{4}, 1]$. This follows from Lemma 3.3 in [2] but also by applying the power counting theorem with $(\alpha_1, \ldots, \alpha_4) = (H - 1, \ldots, H - 1)$. Therefore

$$\sup_{H \in [\frac{1}{2}, 1]} \int_{\mathbb{R}} \ldots \int_{\mathbb{R}} du_1 \ldots du_4 f(u_1) \ldots f(u_4) \times$$

$$\times |u_1 - u_2|^{H-1} |u_2 - u_3|^{H-1} |u_3 - u_4|^{H-1} |u_4 - u_1|^{H-1} < \infty$$

and implies (15) and consequently, it gives the convergence of finite dimensional distributions of Y^H as $\to \frac{1}{2}$. The tighness follows from (24). □

3.3. Asymptotic behavior of the stationary Rosenblatt Ornstein–Uhlenbeck process.

Now, we analyze the asymptotic behavior of the process (18). The main idea is the same as in the previous section but we need to pay attention to the fact that the kernel of the stationary ROU process has the whole real line as support .

3.3.1. *Convergence when $H \to 1$.*

Proposition 6. *The stationary Rosenblatt Ornstein–Uhlenbeck process converges weakly, in the space of continuous functions $C[0, T]$, to the stochastic process $(X(t))_{t \in [0,T]}$ defined by, for every $t \in [0, T]$,*

$$X(t) = \sigma \left(\int_{-\infty}^{t} e^{-\lambda(t-u)} du \right)(Z^2 - 1) = \frac{\sigma}{\lambda}(Z^2 - 1). \tag{26}$$

Proof. We prove the convergence of the finite dimensional distributions of X^H to those of X when $H \to 1$. Consider $\alpha_1, \ldots, \alpha_d \in \mathbb{R}$ and $t_1, \ldots, t_d \in [0, T]$. We prove that

$$\sum_{j=1}^{d} \alpha_j X^H(t_j) \xrightarrow[H \to 1]{(d)} \sum_{j=1}^{d} \alpha_j X(t_j).$$

Notice that, by linearity

$$\sum_{j=1}^{d} \alpha_j X^H(t_j) = \int_{\mathbb{R}} g(u) dZ^H(u)$$

with

$$g(u) = \sigma \sum_{j=1}^{d} \alpha_j e^{-\lambda(t_j - u)} 1_{(-\infty, t_j)}(u). \tag{27}$$

We need to show (12). First, notice that g belongs to $L^1(\mathbb{R})$ because

$$\int_{\mathbb{R}} |g(u)| du \le \sigma \sum_{j=1}^{d} |\alpha_j| \int_{-\infty}^{t_j} e^{-\lambda(t_j - u)} du = \sigma \sum_{j=1}^{d} |\alpha_j| \int_{0}^{\infty} e^{-\lambda u} du = \frac{\sigma}{\lambda} \sum_{j=1}^{d} |\alpha_j| < \infty. \tag{28}$$

Concerning the first part of (12), we have, with g from (27),

$$\int_{\mathbb{R}} \int_{\mathbb{R}} du dv |g(u) g(v)| |u - v|^{2\varepsilon - 1} \le$$

$$\le \sigma^2 \sum_{j,k=1}^{d} |\alpha_j \alpha_k| \int_{-\infty}^{t_j} du \int_{-\infty}^{t_k} dv e^{-\lambda(t_j - u)} e^{-\lambda(t_k - v)} |u - v|^{2\varepsilon - 1} \le$$

$$\le \sigma^2 \sum_{j,k=1}^{d} |\alpha_j \alpha_k| \int_{0}^{\infty} du \int_{0}^{\infty} dv e^{-\lambda(u+v)} |u - v - (t_j - t_k)|^{2\varepsilon - 1}.$$

The fact that the above integral is finite follows from the computation of the term I given by (33) below.

Again the tightness is obtained since X^H obviously satisfies (24). □

3.3.2. *Convergence when $H \to \frac{1}{2}$.* We will denote by $(X_0(t))_{t\geq 0}$ the stationary Ornstein–Uhlenbeck process. Recall that the stationary Ornstein–Uhlenbeck process is obtained from (25) by taking the initial condition $\xi = \sigma \int_{-\infty}^0 e^{-\lambda u} dW(u)$ where $(W(u))_{u\in\mathbb{R}}$ is a Wiener process on the whole real line. Thus

$$X_0(t) = \sigma \int_{-\infty}^t e^{-\lambda(t-u)} dW(u) \tag{29}$$

with $\sigma, \lambda > 0$. The process $(X_0(t))_{t\geq 0}$ is a centered Gaussian process, with stationary increments, with covariance function

$$\mathsf{E}X_0(t)X_0(s) = \frac{\sigma^2}{2\lambda} e^{-\lambda|t-s|} \tag{30}$$

for every $s, t \geq 0$. It follows from (30) that X_0 is a stationary Gaussian process with stationary increments.

Proposition 7. *The process $(X^H(t))_{t\in[0,T]}$ converges weakly, in the space of continuous functions $C[0,T]$, to the stationary Ornstein–Uhlenbeck process $(X_0(t))_{t\in[0,T]}$.*

Proof. Consider $\alpha_1, \ldots, \alpha_d \in \mathbb{R}$ and $t_1, \ldots, t_d \in [0,T]$. In order to show that the random variable $\sum_{j=1}^d \alpha_j X^H(t_j)$ converges in distribution to $\sum_{j=1}^d \alpha_j X_0(t_j)$ as $H \to \frac{1}{2}$, we use Proposition 3. In order to apply this result, we first notice that the function g given by (27) is in $\mathcal{H}_H$ (acutally the computations below will show that it also belongs to $|\mathcal{H}_H|$). We also need to verify (14) and (15) in Proposition 3. Let us start by checking (14). We have to prove that

$$\mathsf{E}\left(\sum_{j=1}^d \alpha_j X^H(t_j)\right)^2 \xrightarrow[H\to\frac{1}{2}]{} \mathsf{E}\left(\sum_{j=1}^d \alpha_j X_0(t_j)\right)^2. \tag{31}$$

First, notice that by (30),

$$\mathsf{E}\left(\sum_{j=1}^d \alpha_j X_0(t_j)\right)^2 = \sum_{j,k=1}^d \alpha_j \alpha_k \mathsf{E}X_0(t_j)X_0(t_k) =$$

$$= \frac{\sigma^2}{2\lambda} \sum_{j,k=1}^d \alpha_j \alpha_k e^{-\lambda|t_j - t_k|}.$$

On the other hand,

$$\mathsf{E}\left(\sum_{j=1}^d \alpha_j X^H(t_j)\right)^2 =$$

$$= \sigma^2 H(2H-1) \sum_{j,k=1}^d \alpha_j \alpha_k \int_{-\infty}^{t_j} du \int_{-\infty}^{t_k} dv e^{-\lambda(t_j-u)} e^{-\lambda(t_k-v)} |u-v|^{2H-2} =$$

$$= \sigma^2 H(2H-1) \sum_{j,k=1}^d \alpha_j \alpha_k \int_0^\infty du \int_0^\infty dv e^{-\lambda u} e^{-\lambda v} |u - v - (t_j - t_k)|^{2H-2}. \tag{32}$$

We need to compute the integral

$$I = H(2H-1) \int_0^\infty du \int_0^\infty dv e^{-\lambda u} e^{-\lambda v} |u - v - K|^{2H-2} \tag{33}$$

and we can assume by symmetry that $K \geq 0$. We have

$$I = H(2H-1) \int_0^\infty dv e^{-\lambda v} \int_{v+K}^\infty du e^{-\lambda u} (u - v - K)^{2H-2} +$$

$$+ H(2H - 1) \int_0^\infty dv e^{-\lambda v} \int_0^{v+K} du e^{-\lambda u} (v + K - u)^{2H-2} :=$$
$$:= I_1 + I_2.$$

Let us regard the summand I_1. By the change of variables $\tilde{u} = u - (v + K)$,

$$I_1 = H(2H - 1) \int_0^\infty dv e^{-\lambda v} \int_0^\infty du e^{-\lambda(u+v+k)} u^{2H-2} =$$

$$= H(2H - 1) e^{-\lambda K} \int_0^\infty dv e^{-2\lambda v} \lambda^{1-2H} \int_0^\infty dz e^{-z} z^{2H-2} =$$

$$= H(2H - 1) e^{-\lambda K} \lambda^{1-2H} \frac{1}{2\lambda} \int_0^\infty dz e^{-z} z^{2H-2} = H(2H - 1) e^{-\lambda K} \frac{1}{2\lambda^{2H}} \Gamma(2H)$$

where Γ is the gamma function. By integrating by parts

$$H(2H - 1) \int_0^\infty dz e^{-z} z^{2H-2} \xrightarrow[H \to \frac{1}{2}]{} 1$$

so

$$I_1 \xrightarrow[H \to \frac{1}{2}]{} \frac{e^{-\lambda K}}{4\lambda}. \tag{34}$$

Concerning the summand I_2, with $\tilde{u} = v + K - u$,

$$I_2 = H(2H - 1) \int_0^\infty dv e^{-\lambda v} \int_0^{v+K} du e^{-\lambda(v+K-u)} u^{2H-2} =$$

$$= H(2H - 1) e^{-\lambda K} \int_0^\infty du e^{\lambda u} u^{2H-2} \int_{(u-K)\vee 0}^\infty dv e^{-2\lambda v} =$$

$$= H(2H - 1) e^{-\lambda K} \int_0^K du e^{\lambda u} u^{2H-2} \int_0^\infty dv e^{-2\lambda v} +$$

$$+ H(2H - 1) e^{-\lambda K} \int_K^\infty du e^{\lambda u} u^{2H-2} \int_{u-K}^\infty dv e^{-2\lambda v} =$$

$$= H(2H - 1) e^{-\lambda K} \frac{1}{2\lambda} \int_0^K du e^{\lambda u} u^{2H-2} + H(2H - 1) e^{\lambda K} \frac{1}{2\lambda} \int_K^\infty du e^{-\lambda u} u^{2H-2}.$$

By using the integration by parts, we obtain

$$I_2 = \frac{H}{2\lambda} e^{-\lambda K} \left[e^{\lambda K} K^{2H-1} - \lambda \int_0^K e^{\lambda u} u^{2H-1} du \right] +$$

$$+ \frac{H}{2\lambda} e^{\lambda K} \left[-e^{-\lambda K} K^{2H-1} + \lambda \int_K^\infty e^{-\lambda u} u^{2H-1} du \right].$$

Since

$$\lambda \int_0^K e^{\lambda u} u^{2H-1} du \xrightarrow[H \to \frac{1}{2}]{} e^{\lambda K} - 1 \quad \text{and} \quad \lambda \int_K^\infty e^{-\lambda u} u^{2H-1} du \xrightarrow[H \to \frac{1}{2}]{} e^{-\lambda K}$$

we get

$$I_2 \xrightarrow[H \to \frac{1}{2}]{} \frac{e^{-\lambda K}}{4\lambda}. \tag{35}$$

From (34) and (35), the integral I from (33) verifies

$$I \xrightarrow[H \to \frac{1}{2}]{} \frac{e^{-\lambda K}}{2\lambda}. \tag{36}$$

Relation (36), together with (32), will imply that (31), i. e. the assumption (14) is verified.

Let us now check the assumption (15). With g from (27),

$$\int_{\mathbb{R}} \cdots \int_{\mathbb{R}} du_1 \ldots du_4 |g(u_1) \ldots g(u_m)||u_1 - u_2|^{H-1} \ldots |u_4 - u_1|^{-H-1} \leq$$

$$\leq \sum_{j_1,j_2,\ldots,j_4=1}^{d} |\alpha_{j_1} \ldots \alpha_{j_4}| \int_{-\infty}^{t_{j_1}} du_1 \ldots \int_{-\infty}^{t_{j_4}} du_m e^{-\lambda(t_{j_1} - u_1)} \ldots e^{-\lambda(t_{j_4} - u_4)} \times$$

$$\times |u_1 - u_2|^{H-1} \ldots |u_4 - u_1|^{-H-1} =$$

$$= \sum_{j_1,j_2,\ldots,j_4=1}^{d} |\alpha_{j_1} \ldots \alpha_{j_4}| \int_{0}^{\infty} du_1 \ldots \int_{0}^{\infty} du_4 e^{-\lambda(u_1 + \ldots + u_4)} \times$$

$$\times |u_1 - u_2 - (t_{j_1} - t_{j_2})|^{H-1} \ldots |u_4 - u_1 - (t_{j_4} - t_{j_1})|^{H-1} \leq$$

$$\leq e^{\frac{\lambda}{2}(|t_{j_1} - t_{j_2}| + \ldots + |t_{j_4} - t_{j_1}|)} \sum_{j_1,j_2,\ldots,j_4=1}^{d} |\alpha_{j_1} \ldots \alpha_{j_4}| \int_{0}^{\infty} du_1 \ldots \int_{0}^{\infty} du_4 \times$$

$$\times e^{-\frac{\lambda}{2}(|u_1 - u_2 - (t_{j_1} - t_{j_2})| + \ldots + |u_4 - u_1 - (t_{j_4} - t_{j_1})|)} \times$$

$$\times \left(1 \vee |u_1 - u_2 - (t_{j_1} - t_{j_2})|^{H-1}\right) \ldots \left(1 \vee |u_4 - u_1 - (t_{j_4} - t_{j_1})|^{H-1}\right).$$

Now, we will consider the set T' of affine functionals on $\mathbb{R}^4$ given by

$$T' = \{u_1 - u_2 - (t_{j_1} - t_{j_2}), \ldots, u_4 - u_1 - (t_{j_4} - t_{j_1})\}.$$

As before, T' is the only paddet subset of T'.

We apply the power counting theorem with

$$(\alpha_1, \ldots, \alpha_4) = (H - 1, \ldots, H - 1) \quad \text{and} \quad (\beta_1, \ldots, \beta_4) = (-\gamma, \ldots, -\gamma)$$

with $\gamma \in (1 - \frac{1}{4}, 1) = (\frac{3}{4}, 1)$. We have

$$d_0(T') = 4 - 1 + 4(H - 1) > 0 \quad \text{if } H > \frac{1}{4}$$

and

$$d_\infty(\emptyset) = 4 - 1 - 4\gamma < 0 \quad \text{if } \gamma > 1 - \frac{1}{4} = \frac{3}{4}.$$

Therefore, the function

$$H \to \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} du_1 \ldots du_4 |g(u_1) \ldots g(u_m)||u_1 - u_2|^{H-1} \ldots |u_4 - u_1|^{H-1}$$

is finite and continuous on the set $D = \{H \in (0,1], H > \frac{1}{4}\}$ which implies that condition (15) is satisfied. The conclusion follows from Proposition 3. $\qquad\square$

4. Appendix: Multiple stochastic integrals and the Fourth Moment Theorem

Here, we shall only recall some elementary facts; our main reference is [16]. Consider $\mathcal{H}$ a real separable infinite-dimensional Hilbert space with its associated inner product $\langle.,.\rangle_{\mathcal{H}}$, and $(B(\varphi), \varphi \in \mathcal{H})$ an isonormal Gaussian process on a probability space $(\Omega, \mathfrak{F}, \mathbb{P})$, which is a centered Gaussian family of random variables such that

$$\mathsf{E}\big(B(\varphi)B(\psi)\big) = \langle\varphi, \psi\rangle_{\mathcal{H}},$$

for every $\varphi, \psi \in \mathcal{H}$. Denote by I_q the qth multiple stochastic integral with respect to B. This I_q is actually an isometry between the Hilbert space $\mathcal{H}^{\odot q}$ (symmetric tensor product) equipped with the scaled norm $\frac{1}{\sqrt{q!}}\|\cdot\|_{\mathcal{H}^{\otimes q}}$ and the Wiener chaos of order

q, which is defined as the closed linear span of the random variables $H_q(B(\varphi))$ where $\varphi \in \mathcal{H}$, $\|\varphi\|_{\mathcal{H}} = 1$ and H_q is the Hermite polynomial of degree $q \geq 1$ defined by:

$$H_q(x) = (-1)^q \exp\left(\frac{x^2}{2}\right) \frac{\mathrm{d}^q}{\mathrm{d}x^q}\left(\exp\left(-\frac{x^2}{2}\right)\right), \; x \in \mathbb{R}. \tag{37}$$

We recall the hypercontractivity property of multiple stochastic integrals. If $Y = I_2(f)$, with $f \in \mathcal{H}^{\otimes 2}$, then (see relation (2.7.2) in [8])

$$\mathsf{E}|Y|^q \leq (q-1)^q \mathsf{E}\big|Y^2\big|^{\frac{q}{2}} \tag{38}$$

for every $q > 2$. We will use the following famous result initially proven in [9] that characterizes the convergence in distribution of a sequence of multiple integrals toward the Gaussian law.

Theorem 1. *Fix $n \geq 2$ and let $(F_k, k \geq 1)$, $F_k = I_n(f_k)$ (with $f_k \in \mathcal{H}^{\odot n}$ for every $k \geq 1$), be a sequence of square-integrable random variables in the nth Wiener chaos such that $\mathsf{E}\big[F_k^2\big] \to 1$ as $k \to \infty$. The following are equivalent:*

    (1) the sequence $(F_k)_{k \geq 0}$ converges in distribution to the normal law $\mathcal{N}(0,1)$;

    (2) $\mathsf{E}\big[F_k^4\big] \to 3$ as $k \to \infty$ (or equivalently $k_4(F_k) \to 0$ as $k \to \infty$).

Other equivalent condition can be stated in term of the Malliavin derivatives of F_k or contractions of the kernels f_k, see [8].

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ГРАНИЧНА ПОВЕДІНКА РОЗЕНБЛАТТІВСЬКОГО ПРОЦЕСУ ОРНШТЕЙНА – УЛЕНБЕКА ВІДНОСНО ІНДЕКСУ ХЮРСТА

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Анотація. Ми досліджуємо збіжність за розподілом при $H \to \frac{1}{2}$ та при $H \to 1$ інтеграла $\int_{\mathbb{R}} f(u) dZ^H(u)$, де Z^H — процес Розенблатта з індексом самоподібності $H \in \left(\frac{1}{2}, 1\right)$, а f — невипадкова функція. Ми аналізуємо випадок розенблаттівського процесу Орнштейна – Уленбека, що є розв'язком рівняння Ланжевена, керованого процесом Розенблатта.

ПРЕДЕЛЬНОЕ ПОВЕДЕНИЕ РОЗЕНБЛАТТОВСКОГО ПРОЦЕССА ОРНШТЕЙНА – УЛЕНБЕКА ПО ОТНОШЕНИЮ К ИНДЕКСУ ХЮРСТА

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Аннотация. Мы исследуем сходимость по распределению при $H \to \frac{1}{2}$ и при $H \to 1$ интеграла $\int_{\mathbb{R}} f(u) dZ^H(u)$, где Z^H — процесс Розенблатта с индексом самоподобия $H \in \left(\frac{1}{2}, 1\right)$, а f — неслучайная функция. Мы анализируем случай розенблаттовского процесса Орнштейна – Уленбека, который является решением уравнения Ланжевена, управляемого процессом Розенблатта.