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ORDER OF APPROXIMATION IN THE CENTRAL LIMIT THEOREM FOR ASSOCIATED RANDOM VARIABLES AND A MODERATE DEVIATION RESULT

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Abstract. An estimate of the order of approximation in the central limit theorem for strictly stationary associated random variables with finite moments of order $q > 2$ is obtained. A moderate deviation result is also obtained. We have a refinement of recent results in Çağin et al. [2]. The order of approximation obtained here is an improvement over the corresponding result in Wood [12].

Key words and phrases. Associated random variables, central limit theorem, rate of convergence, Berry–Esséen type bound, moderate deviations.

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1. Introduction

A set of random variables (rvs) $\{X_1, X_2, \ldots, X_k\}$ is said to be associated if for each pair of coordinate-wise nondecreasing functions $f, g : \mathbb{R}^k \to \mathbb{R}$

$$\mathrm{Cov}(f(X_1, X_2, \ldots, X_k), g(X_1, X_2, \ldots, X_k)) \geq 0$$

whenever the covariance exists.

A sequence $\{X_n\}$ of rvs is associated if for every $n \in N$ the family $X_1, X_2, \ldots, X_n$ is associated.

In this paper we consider a strictly stationary sequence of centered square integrable associated rvs $\{X_n\}$. Central limit theorem (CLT) for $\{X_n\}$ was proved by Newman [7] and a Berry–Esséen type theorem giving an estimate of the order of approximation in the CLT was proved by Wood [12]. In the case of finite third absolute moment $\mathsf{E}|X_1|^3$ Wood's result gives an estimate of the order $O(n^{-1/5})$. Birkel [1] obtained a rate of the order $O(n^{-1/2}\log^2 n)$ under the strong additional assumption that the Cox–Grimmett coefficients $u(n)$ decrease exponentially. Birkel also provided an interesting example to show the reasonableness of the assumptions to obtain the above order of approximation. In that example he showed that the above rate cannot be obtained if $u(n)$ decreases only as a power. Thus there is a huge gap between the results of Wood and Birkel. In a recent paper Çağin et al. [2] obtained another estimate of the order of approximation in the CLT for associated rvs and also obtained a moderate deviation type result. However their estimate in the case of finite third absolute moment $\mathsf{E}|X_1|^3$ is quite complicated.

Large deviation probability and moderate deviation probability investigations received much attention due to their importance in statistical inference and applied probability. We refer to monographs by Varadhan [10], Dembo and Zeitouni [3] and Hollander [5] and recent papers by Wang [11] and Çağin et al. [2] for other references. These investigations are also useful in the construction of certain counter examples (see, for example, Tikhomirov [9] and Birkel [1]).

We give an estimate of the order of approximation in the CLT which is a refined version of the Theorem 4.2 in Çağin et al. [2] and also prove the corresponding moderate deviation result. In the case of X_n with finite third absolute moment, when Cox-Grimmett coefficients $u(n)$ are of order $n^{-\delta}$, the order of approximation in the CLT is proved to go to zero as $n^{-3/8}$ as $\delta \to \infty$. The main steps in the proof are the classical decomposition of the partial sum $S_n = \sum_{j=1}^{n} X_j$ into blocks (of size $p_n = [n^{1-\alpha}], 0 < \alpha < 1$), coupling them (see Section 2) with the blocks variables with the same distributions but independent and use the inequality due to Newman [7]. Our approach is similar to that in Çağin et al. However the estimate of the order of approximation we obtain does not depend on the value of α. The refinement is in terms of the assumptions, bound and simplification of the steps. This helps us to get a moderate deviation type result too under assumptions milder than those in Çağin et al. [2] and also get an order of approximation in the CLT which is an improvement over the corresponding result in Wood [12].

The paper is organized as follows. In Section 2 we introduce notation and give some lemmas. In Section 3 we shall have a set of propositions that will be used in later sections. Order of approximation in the CLT is investigated in Section 4. Finally a moderate deviation type result is discussed in Section 5.

2. Notation

Let $\{X_n\}$ be a strictly stationary sequence of centered square integrable associated rvs. Set $\mathsf{E}(X_1^2) = \sigma_1^2$, $c_j = \mathrm{Cov}(X_1, X_{1+j})$, $S_n = \sum_{j=1}^{n} X_j$, $\mathsf{E}S_n^2 = s_n^2$ and $\sigma^2 = \sigma_1^2 + 2\sum_{j=1}^{\infty} c_j > 0$. We assume that $\sum_{j=1}^{\infty} c_j < \infty$. Then (see Theorem 4.1, p. 104, Oliveira [7])

$$\frac{S_n}{\sigma\sqrt{n}} \xrightarrow{D} Z_1 \sim N(0,1) \tag{2.1}$$

where $N(0,1)$ denotes the standard normal distribution. The standard proof of this result involves writing S_n as the sum of blocks of fixed size, approximating the distribution of S_n by the distribution of corresponding sum of coupling block rvs (to be defined shortly) and appealing to the CLT for the coupling block rvs. We need more notation to explain this. Define the initial blocks

$$Y_{j,n} = \sum_{i=(j-1)p_n+1}^{jp_n} X_i, \quad j = 1, 2, \dots, m_n$$

and

$$Y_{m_n+1,n} = \sum_{i=m_n p_n+1}^{n} X_i$$

where $m_n = [n/p_n]$, $p_n < n/2$ and $[r]$ denotes the largest integer not exceeding r. Clearly

$$S_n = \sum_{j=1}^{m_n} Y_{j,n} + Y_{m_n+1,n}.$$

We note that $Y_{j,n}$, $j = 1, 2, \dots, m_n$ are identically distributed. Further $n - m_n p_n$ does not exceed p_n. We next define independent coupling blocks $Y^*_{j,n}, j = 1, 2, \dots, m_n$, where $Y^*_{j,n} \overset{D}{=} Y_{j,n}$. Note that since the X_k are strictly stationary, the rvs $Y^*_{j,n}$ are identically distributed.

Set $p_n = [n^{1-\alpha}]$ where $0 < \alpha < 1$.

In what follows limits are taken as $n \to \infty$ and statements hold for sufficiently large values of n. We introduce the following assumptions on the covariances c_j and moments of X_k:

Assumption A_1: $\mathsf{E}|X_k|^q < \infty$ for some $q > 2$.

Assumption A_2: $\left|\frac{s_n^2}{n\sigma^2} - 1\right| = O(n^{-\theta})$ for some $\theta > 0$ where $s_n^2 = \mathsf{E}S_n^2$.

Assumption A_3: $u(n) = \sum_{j=n}^{\infty} c_j < C_1 n^{-\delta}$, where $\delta > 0$.

Remark 2.1. (i) If $k_n \to \infty$ such that $\frac{k_n}{n} \to 0$ then the assumption A_2 implies $\left| \frac{s_n^2}{n\sigma^2} - \frac{s_{k_n}^2}{k_n \sigma^2} \right| = O(k_n^{-\theta})$.

(ii) By the assumption that $\sum_{i=1}^{\infty} c_i < \infty$ and the fact that $\sigma^2 - \frac{s_n^2}{n} = 2u(n) + \frac{2}{n} \sum_{j=1}^{n-1} jc_j$ it follows that if the assumption A_2 holds for some $\theta > 0$ then the assumption A_3 holds for $\delta = \theta$ and conversely.

(iii) Under the assumption A_1 there exist positive constants A and B such that for all the positive integers n, $A\, n^{1/2} < s_n < B\, n^{1/2}$ and $A\, n^{q/2} < \mathsf{E}|S_n|^q < B\, n^{q/2}$. (see (2.16) in Birkel [1]).

Here and elsewhere $C_1, C_2, \ldots$ are positive constants independent of n. Further $\eta_1, \eta_2, \ldots$ are constants with absolute values ≤ 1. The following result is known.

Lemma 2.2 (Newman's inequality [7]). *Suppose $U_1, U_2, \ldots, U_n$ are associated rvs with finite variances. Then for any real numbers $t_1, t_2, \ldots, t_n$*

$$\left| \mathsf{E}\left(\exp^{i \sum_{j=1}^{n} t_j\, U_j}\right) - \prod_{j=1}^{n} \mathsf{E}\left(\exp^{it_j\, U_j}\right) \right| \leq \sum_{i,j=1, j>i}^{n} |t_i||t_j| \operatorname{Cov}(U_i, U_j).$$

Remark 2.3. If $\{X_n\}$ is a sequence of associated rvs, the block rvs $Y_{1,n}, Y_{2,n}, \ldots, Y_{m_n,n}$ are associated. Further the characteristic functions satisfy

$$\mathsf{E}\left(e^{i \sum_{j=1}^{m_n} t_j Y_{j,n}^*}\right) = \prod_{j=1}^{m_n} \mathsf{E}\left(e^{it_j Y_{j,n}^*}\right) = \prod_{j=1}^{m_n} \mathsf{E}\left(e^{it_j Y_{j,n}}\right) \tag{2.2}$$

because $Y_{j,n}^*$ are independent and $Y_{j,n}^* \stackrel{D}{=} Y_{j,n}$.

Before we close this section we give a well known result concerning the expansion of a characteristic function in terms of its moments. Let f be the characteristic function corresponding to a distribution function F and let $m^{(k)} = \int x^k dF(x)$ and $\mu^{(r)} = \int |x|^r df(x)$, $k = 0, 1, \ldots, r \geq 0$. In the following Lemma we quote from the differentiability properties discussed on page 212, Loève [6] the portion that is relevant for our purpose.

Lemma 2.4. *If $\mu^{(n+\delta)} < \infty$ for a $\delta \geq 0$, then for every $k \leq n$*

$$f^{(k)}(u) = i^k \int e^{iux}\, x^k\, dF(x), \quad u \in \mathbb{R},$$

and $f^{(k)}$ is continuous and bounded by $\mu^{(k)}$; moreover

$$f(u) = \sum_{k=0}^{n-1} m^{(k)} \frac{(iu)^k}{k!} + \rho_n(u), \quad u \in \mathbb{R},$$

where

$$\rho_n(u) = u^n \int_0^1 \frac{(1-t)^{n-1}}{(n-1)!} f^{(n)}(tu)\, dt = m^{(n)} \frac{(iu)^n}{n!} + o(u^n) = \zeta\, \mu^{(n)} \frac{|u|^n}{n!}$$

with $|\zeta| \leq 1$, and if $0 < \delta \leq 1$, then

$$\rho_n(u) = m^{(n)} \frac{(iu)^n}{n!} + 2^{1-\delta}\, \zeta'\, \mu^{(n+\delta)} \frac{|u|^{n+\delta}}{(1+\delta)(2+\delta)\cdots(n+\delta)} \tag{2.3}$$

with $|\zeta'| \leq 1$.

In the next section we use the notation $T_1 = a_n n^{\frac{\alpha}{2}}$ and $T_2 = b_n n^{\frac{\alpha}{2}}$ where $a_n = (\log n)^a$ and $b_n = (\log n)^b$ with $a < b < 0$.

3. Some preliminary results

In this section we discuss some preliminary results that will be used later and these are of independent interest too. The following result notes that while dealing with asymptotic properties of $S_n/(\sigma s_n)$ it is adequate to consider the sum $\sum_{j=1}^{m_n} Y_{j,n}$.

Proposition 3.1. *Suppose the assumption A_1 holds. Then for $\mu_n = n^{-3\alpha/8}$, $\quad 0 < \alpha < 1$*

$$\mathsf{P}(|Y_{m_n+1,n}| > \mu_n s_n) < C_2 n^{-q\alpha/8}.$$

To see this note that because of stationarity of $\{X_n\}$

$$\mathsf{P}(|Y_{m_n+1,n}| > \mu_n s_n) < \frac{\mathsf{E}|S_{n-m_n p_n}|^q}{\mu_n^q s_n^q}.$$

The result follows now from Remark 2.1 and the assumption A_1.

Remark 3.2. This is an improvement of the result in Step 3 of the Theorem 3.1 in Çağin et al. [2] in view of their restrictions on α and q.

Next we approximate the distribution of the sum of the original rvs by that of the coupling blocks; i.e., the distribution of $\sum_{j=1}^{m_n} Y_{j,n}$ by that of $\sum_{j=1}^{m_n} Y_{j,n}^*$. The method of approximation is based on the celebrated Berry–Esséen inequality and Newman's inequality for associated rvs.

Proposition 3.3. *Suppose the assumptions A_1 and A_2 hold. Then for $\theta > 0$ given in A_2*

$$\sup_{x\in\mathbb{R}} \left| \mathsf{P}\left(\sum_{j=1}^{m_n} Y_{j,n} \le x s_n\right) - \mathsf{P}\left(\sum_{j=1}^{m_n} Y_{j,n}^* \le x s_n\right)\right| <$$
$$< C_3 \frac{b_n^2}{n^{\theta-\alpha(1+\theta)}} I\left(\frac{2\theta}{3+2\theta} \le \alpha < \frac{\theta}{1+\theta}\right) + C_4 \frac{1}{b_n n^{\alpha/2}} I\left(\alpha < \frac{2\theta}{3+2\theta}\right).$$

Proof. By the Berry–Esséen inequality and (2.2) we have

$$\sup_{x\in\mathbb{R}} \left| \mathsf{P}\left(\sum_{j=1}^{m_n} Y_{j,n} \le x s_n\right) - \mathsf{P}\left(\sum_{j=1}^{m_n} Y_{j,n}^* \le x s_n\right)\right| <$$
$$< C_5 \int_{-T_2}^{T_2} \frac{1}{|t|} \left| \mathsf{E}\left(e^{i\frac{t}{s_n}\sum_{j=1}^{m_n} Y_{j,n}}\right) - \prod_{j=1}^{m_n} \mathsf{E}\left(e^{i\frac{t}{s_n} Y_{j,n}^*}\right)\right| dt + \frac{C_6}{T_2} =$$
$$= C_5 \int_{-T_2}^{T_2} \frac{1}{|t|} \left| \mathsf{E}\left(e^{i\frac{t}{s_n}\sum_{j=1}^{m_n} Y_{j,n}}\right) - \prod_{j=1}^{m_n} \mathsf{E}\left(e^{i\frac{t}{s_n} Y_{j,n}}\right)\right| dt + \frac{C_6}{T_2}. \tag{3.1}$$

By the Lemma 2.2 with $U_j = Y_{j,n}$, $j = 1, 2, \dots, m_n$, we have

$$\left| \mathsf{E}\left(e^{i\frac{t}{s_n}\sum_{j=1}^{m_n} Y_{j,n}}\right) - \prod_{j=1}^{m_n} \mathsf{E}\left(e^{i\frac{t}{s_n} Y_{j,n}}\right)\right| \le \frac{t^2}{s_n^2} \sum_{j,k=1,j>k}^{m_n} \mathrm{Cov}(Y_{j,n}, Y_{k,n}) =$$
$$= \frac{t^2 m_n p_n}{2 s_n^2} \left| \frac{s_{m_n p_n}^2}{m_n p_n} - \frac{s_{p_n}^2}{p_n} \right|.$$

In view of the Remark 2.1 we then have from (3.1)

$$\sup_{x\in\mathbb{R}}\left|\mathsf{P}\left(\sum_{j=1}^{m_n} Y_{j,n} \le x s_n\right) - \mathsf{P}\left(\sum_{j=1}^{m_n} Y^*_{j,n} \le x s_n\right)\right| < \frac{C_7}{p_n^{\theta}}\int_{-T_2}^{T_2}|t|dt + \frac{C_6}{T_2} = = \frac{C_7\, T_n^2}{p_n^{\theta}} + \frac{C_6}{T_2}.$$

Recalling that $p_n = [n^{1-\alpha}]$, $T_2 = b_n\, n^{\alpha/2}$ we note that the right side above goes to zero only for $\alpha < \theta/(1+\theta)$. Further $(1-\alpha)\theta \le 3\alpha/2$ if and only if $\alpha \ge 2\theta/(3+2\theta)$. Hence

$$\sup_{x\in\mathbb{R}}\left|\mathsf{P}\left(\sum_{j=1}^{m_n} Y_{j,n} \le x s_n\right) - \mathsf{P}\left(\sum_{j=1}^{m_n} Y^*_{j,n} \le x s_n\right)\right| \le \le C_6\,\frac{1}{b_n n^{\alpha/2}}\, I\left(\alpha < \frac{2\theta}{3+2\theta}\right) + C_8\,\frac{b_n^2}{n^{(1-\alpha)\theta-\alpha}}\, I\left(\frac{2\theta}{3+2\theta} \le \alpha < \frac{\theta}{1+\theta}\right).$$

This completes the proof of the Proposition 3.3. □

Remark 3.4. In the proof of the Theorem 4.1 in Çağin et al. [2] the above bound was obtained separately for the odd numbered blocks and the even numbered blocks. Further, the bound obtained above goes to zero faster than their corresponding bound.

Our next result is concerned with the approximation of the characteristic function of the sum of coupling blocks by the characteristic function of an appropriate normal variable.

Proposition 3.5. *Denote* $\varphi_j(t) = \mathsf{E}\left(e^{itY^*_{j,n}}\right)$. *Then under the assumptions* A_1 *and* A_2, *for* $|t| < T_2$

$$\left|\prod_{j=1}^{m_n}\varphi_j\left(\frac{t}{s_n}\right) - e^{-\frac{m_n t^2 s_{p_n}^2}{2s_n^2}}\right| \le C_9\,\frac{m_n|t|^q p_n^{q/2}}{s_n^q} e^{-\frac{m_n\, t^2 s_{p_n}^2}{2s_n^2}}$$

for $2 < q < 3$. *The same inequality holds with* $q = 3$ *in the case* $q \ge 3$.

Proof. Let us first consider the case $2 < q < 3$. We now apply the Lemma 2.4 to the characteristic function of $Y_{j,n}$ and use, in particular, the expansion at (2.3) taking $n = 2$, and $q = 2+\delta < 3$. Note that since $Y^*_{j,n} \overset{D}{=} Y_{j,n}$

$$\varphi_j(t/s_n) = 1 - \frac{t^2 s_{p_n}^2}{2s_n^2} + 2^{1-\delta}\,\zeta'\,\frac{|t|^q}{q(q-1)s_n^q}\mathsf{E}|Y_{j,n}|^q,$$

i. e.,

$$\varphi_j(t/s_n) = 1 - \frac{t^2 s_{p_n}^2}{2s_n^2} + \eta_1\,\frac{|t|^q}{s_n^q}\mathsf{E}|Y_{j,n}|^q.$$

For $|t| < T_1 = a_n n^{\alpha/2}$, with $a_n = (\log n)^a$, $a < 0$,

$$\frac{t^2\, s_{p_n}^2}{s_n^2} < C_{10}\, a_n^2 \to 0.$$

Further

$$\frac{|t|^q}{s_n^q}\mathsf{E}|Y_{j,n}|^q < C_{11}\, a_n^q \to 0.$$

Hence $|\varphi_j(t/s_n) - 1| \to 0$ and therefore $\varphi_j(t/s_n)$ is bounded away from 0 for $|t| < T_1$ so that we can take its logarithm. Then for each j we have by the Lemma 2.4

$$\log \varphi_j(t/s_n) = -\frac{t^2 s_{p_n}^2}{2s_n^2} + \eta_1 \frac{|t|^q}{s_n^q}\mathsf{E}|Y_{j,n}|^q + \eta_2 \left[-\frac{t^2 s_{p_n}^2}{2s_n^2} + \eta_1 \frac{|t|^q}{s_n^q}\mathsf{E}|Y_{j,n}|^q\right]^2 =$$

$$= -\frac{t^2 s_{p_n}^2}{2s_n^2} + \eta_3 \frac{|t|^q}{s_n^q}\mathsf{E}|Y_{j,n}|^q.$$

Then using the fact $|e^x - 1| < |x|\, e^{|x|}$ we get

$$\left|\prod_{j=1}^{m_n} \varphi_j(t/s_n) - e^{-\frac{m_n\, t^2 s_{p_n}^2}{2s_n^2}}\right| \le e^{-\frac{m_n t^2 s_{p_n}^2}{2s_n^2}}\ \frac{m_n |t|^q \mathsf{E}|Y_{j,n}|^q}{2\, s_n^2}\ e^{\frac{m_n |t|^q \mathsf{E}|Y_{j,n}|^q}{s_n^q}}.$$

Note that $|t|^{q-2}\mathsf{E}|Y_{1,n}|^q\ s_n^{-(q-2)}\ s_{p_n}^{-2} < a_n^{q-2} \to 0$ so that

$$m_n |t|^q\ \mathsf{E}|Y_{1,n}|\ s_n^{-q} < (1/4)\ m_n t^2 s_{p_n}^2\ s_n^{-2}$$

and hence

$$\left|\prod_{j=1}^{m_n} \varphi_j(t/s_n) - e^{-\frac{m_n\, t^2 s_{p_n}^2}{2s_n^2}}\right| < C_{12}\ \frac{m_n\ |t|^q\ \mathsf{E}|Y_{1,n}|^q}{s_n^q}\ \ e^{-\frac{m_n\ t^2\ s_{p_n}^2}{4\ s_n^2}} \tag{3.2}$$

for $|t| < T_1$. We shall prove that the relation (3.2) holds for $T_1 \le |t| < T_2$ also.

Let W_j, $j = 1, 2, \ldots, m_n$ be rvs such that for each j, W_j is independent of $Y^*_{j,n}$ and distributed as $Y^*_{j,n}$. Then

$$\mathsf{E}(W_j - Y^*_{j,n}) = 0, \quad \mathsf{E}(W_j - Y^*_{j,n})^2 = 2s_{p_n}^2 \quad \text{and} \quad \mathsf{E}|W_j - Y^*_{j,n}|^q \le 2^q \mathsf{E}|Y^*_{j,n}|^q.$$

Further

$$|\varphi_j(t/s_n)|^2 = \mathsf{E}\Big(e^{i\frac{t}{s_n}(W_j - Y^*_{j,n})}\Big) =$$

$$= 1 - \frac{t^2 s_{p_n}^2}{s_n^2} + \eta_4 \frac{2^q |t|^q \mathsf{E}|Y^*_{j,n}|^q}{s_n^q}.$$

Note that for $|t| < T_2 = b_n\ n^{\alpha/2}$ by the Lemma 2.2,

$$\left|\eta_4 \frac{2^q |t|^q \mathsf{E}|Y^*_{j,n}|^q}{s_n^q}\right| < C_{13} \frac{t^2 s_{p_n}^2}{s_n^2}\left[\frac{T_2}{n^{\alpha/2}}\right]^{q-2} <$$

$$< C_{14} \frac{t^2 s_{p_n}^2 b_n^{q-2}}{s_n^2} < \frac{3t^2 s_{p_n}^2}{4s_n^2},$$

since $b_n \to 0$. Hence

$$|\varphi_j(t/s_n)|^2 < 1 - \frac{t^2 s_{p_n}^2}{4s_n^2}.$$

Since $t^2 s_{p_n}^2 s_n^{-2} \to 0$, using the fact $1 - u < e^{-u}$ for $u > 0$ we have

$$|\varphi_j(t/s_n)|^2 < \exp\big(-(1/4) t^2 s_{p_n}^2\ s_n^{-2}\big).$$

Thus for $|t| < T_2$

$$\left|\prod_{j=1}^{m_n} \varphi_j(t/s_n) - e^{-\frac{m_n\ t^2 s_{p_n}^2}{2s_n^2}}\right| < 2e^{-\frac{m_n\ t^2 s_{p_n}^2}{4\ s_n^2}}. \tag{3.3}$$

Now to complete the proof of the claim that (3.2) also holds for $T_1 \le |t| < T_2$, consider

$$C_{12} \frac{m_n |t|^q \mathsf{E}|Y^*_{1,n}|^q}{s_n^q} > C_{12} \frac{m_n T_1^q [\mathsf{E} S_{p_n}^2]^{q/2}}{s_n^q} > C_{15} n^\alpha a_n^q \to \infty.$$

Hence for n large

$$C_{12}\frac{m_n|t|^q\mathsf{E}|Y_{1,n}^*|^q}{s_n^q} > 2,$$

and the claim that (3.2) holds for $T_1 < |t| < T_2$ also follows from (3.3).

The result of the Proposition then follows from (3.2) and the Remark 2.1 in the case $2 < q < 3$.

In the case $q \geq 3$ we can expand $\log \varphi_j(t/s_n)$ using the third moment also and similar calculations lead to the same bound as above and hence the Proposition holds true for $q \geq 3$. □

Remark 3.6. The above proof is similar to that in the Theorem 4.1 in Çağin et al. [2] but has greater clarity. Further the final bound is a bit different because we use different values of Ts.

Corollary 3.7. *Suppose the assumptions A_1 and A_2 hold. Then for the choice of α in the definition of p_n*

$$\int_{-T_2}^{T_2} \frac{1}{|t|}\left|\prod_{j=1}^{m_n} \varphi_j\left(\frac{t}{s_n}\right) - e^{-\frac{m_n\, t^2 s_{p_n}^2}{2s_n^2}}\right| dt \leq C_{16}\, \frac{1}{n^{\alpha(q-2)/2}}$$

in the case $2 < q < 3$. Further the above inequality holds with $q = 3$ giving the bound $C_{16}n^{-\alpha/2}$ in the case $q \geq 3$.

Note that $m_n\ |t|^q p_n^{q/2}\ s_n^{-q} \sim |t|^q\ n^{-\alpha(q-2)/2}$. Here we use the fact that the normal distribution has finite moments.

The final result of this section is to approximate the normal distribution with the characteristic function $\exp\bigl(-(1/2)\ m_n t^2 s_{p_n}^2\ s_n^{-2}\bigr)$ by the standard normal distribution.

Proposition 3.8. *Suppose the assumptions A_1 and A_2 hold. Let $G_n(x)$ be the distribution function with the characteristic function exp$(-\frac{m_n t^2 s_{p_n}^2}{2s_n^2})$ and Φ be the standard normal distribution function. Then*

$$\sup_{x\in\mathbb{R}} |G_n(x) - \Phi(x)| < C_{17}\, \frac{1}{b_n\ n^{\alpha/2}}\ I\left(\alpha \leq \frac{2\theta}{1+2\theta}\right) + C_{18}\, \frac{1}{n^{(1-\alpha)\theta}}\ I\left(\alpha > \frac{2\theta}{1+2\theta}\right).$$

Proof. By the Berry–Esséen inequality

$$\sup_{x\in\mathbb{R}} |G(x) - \Phi(x)| \leq C_{19} \int_{-T_2}^{T_2} \frac{1}{|t|}\left|e^{-\frac{m_n t^2 s_{p_n}^2}{2s_n^2}} - e^{-t^2/2}\right| dt + C_{20}\, \frac{1}{T_2}.$$

Using again the fact that $|e^a - 1| \leq |a|e^{|a|}$ and recalling that $m_n\ s_{p_n}^2\ s_n^{-2} \to 1$ we have for large n

$$\begin{aligned}\frac{1}{|t|}\left|e^{-\frac{m_n t^2 s_{p_n}^2}{2s_n^2}} - e^{-t^2/2}\right| &\leq e^{-\frac{t^2}{2}}\left|\frac{t}{2}\left(\frac{m_n s_{p_n}^2}{s_n^2} - 1\right)\right| e^{\frac{t^2}{2}\left|\frac{m_n s_{p_n}^2}{s_n^2}-1\right|} \leq \\ &\leq \left|\frac{m_n s_{p_n}^2}{s_n^2} - 1\right| \frac{|t|}{2} e^{-\frac{t^2}{4}}. \qquad (3.4)\end{aligned}$$

Since the normal distribution has all the moments finite

$$\begin{aligned}\sup_{x\in\mathbb{R}} |G(x) - \Phi(x)| &\leq C_{19}\left|\frac{m_n s_{p_n}^2}{s_n^2} - 1\right| + C_{20}\frac{1}{T_2} < \\ &< C_{21}\frac{1}{n^{(1-\alpha)\theta}} + C_{20}\frac{1}{n^{\alpha/2} b_n}.\end{aligned}$$

Note that $n^{-(1-\alpha)\theta} \to 0$ faster than $b_n^{-1}n^{-\alpha/2}$ for $\alpha < 2\theta/(1+2\theta)$ while for $\alpha > 2\theta/(1+2\theta)$, $b_n^{-1}n^{-\alpha/2} \to 0$ faster than $n^{-(1-\alpha)\theta}$ giving us the stated bound. □

Remark 3.9. The bound obtained here uses a better bound in (3.4) than the bound used in the line 8 of the page 291 in Çağin et al. [2].

4. Order of approximation in the CLT

We now obtain an estimate of the order of approximation in the CLT which is a refined version of the result in Çağin et al. [2]. The refinement is in terms of the assumptions, bound and simplification of the steps. It also provides a better bound than the bound of order $n^{-1/5}$ obtained from Wood's result under the assumption of finiteness of third absolute moments. See Corollary 4.14 in Oliveira [8].

Theorem 4.1. *Let the assumptions A_1 and A_2 hold. Then*

$$\sup_{x\in\mathbb{R}} |\mathsf{P}(S_n \le xs_n) - \Phi(x)| < C_{22}\Big[n^{-\frac{\theta(q-2)}{q+2\theta}} I(2 < q \le 8/3) + + n^{-\frac{q\theta}{q+8+8\theta}} I(8/3 \le q < 3) + n^{-\frac{3\theta}{11+8\theta}} I(q \ge 3)\Big].$$

In particular when $q = 3$ the bound becomes $C_{22}n^{-\frac{3\theta}{11+8\theta}}$.

Proof. Recall $\mu_n = n^{-3\alpha/8}$. Then by the Proposition 3.1, after making elementary adjustments, we get

$$\begin{aligned}\sup_{x\in\mathbb{R}} |\mathsf{P}(S_n \le xs_n) - \Phi(x)| &\le \sup_{x\in\mathbb{R}}\left|\mathsf{P}\left(\sum_{j=1}^{m_n} Y_{j,n} \le xs_n\right) - \Phi(x)\right| + \\ &+ \mathsf{P}(|Y_{m_n+1,n}| > \mu_n s_n) + 2\sup_{x\in\mathbb{R}} |\Phi(x+\mu_n) - \Phi(x)| < \\ &< \sup_{x\in\mathbb{R}}\left|\mathsf{P}\left(\sum_{j=1}^{m_n} Y_{j,n} \le xs_n\right) - \Phi(x)\right| + C_{23}\frac{1}{n^{q\alpha/8}} + C_{24}\frac{1}{n^{3\alpha/8}}. \qquad (4.1)\end{aligned}$$

Further by the Berry–Esséen inequality

$$\begin{aligned}\sup_{x\in\mathbb{R}}\left|\mathsf{P}\left(\sum_{j=1}^{m_n} Y_{j,n} \le xs_n\right) - \Phi(x)\right| &\le \sup_{x\in\mathbb{R}}\left|\mathsf{P}\left(\sum_{j=1}^{m_n} Y_{j,n} \le xs_n\right) - \mathsf{P}\left(\sum_{j=1}^{m_n} Y^*_{j,n} \le xs_n\right)\right| + \\ &+ \sup_{x\in\mathbb{R}}\left|\mathsf{P}\left(\sum_{j=1}^{m_n} Y^*_{j,n} \le xs_n\right) - \Phi(x)\right| < \\ &< \sup_{x\in\mathbb{R}}\left|\mathsf{P}\left(\sum_{j=1}^{m_n} Y_{j,n} \le xs_n\right) - \mathsf{P}\left(\sum_{j=1}^{m_n} Y^*_{j,n} \le xs_n\right)\right| + \\ &+ C_{25}\int_{-T_2}^{T_2} \frac{1}{|t|}\left|\mathsf{E}\left(e^{i\frac{t}{s_n}\sum_{j=1}^{m_n} Y^*_{j,n}}\right) - e^{-\frac{t^2}{2}}\right| dt + C_{26}\,\frac{1}{T_2} < \\ &< \sup_{x\in\mathbb{R}}\left|\mathsf{P}\left(\sum_{j=1}^{m_n} Y_{j,n} \le xs_n\right) - \mathsf{P}\left(\sum_{j=1}^{m_n} Y^*_{j,n} \le xs_n\right)\right| + C_{25}(I_1 + I_2) + C_{26}\frac{1}{T_2}, \quad (4.2)\end{aligned}$$

where

$$I_1 = \int_{-T_2}^{T_2} \frac{1}{|t|}\left|\prod_{j=1}^{m_n} \varphi_j\left(\frac{t}{s_n}\right) - e^{-\frac{m_n t^2 s_{p_n}^2}{2s_n^2}}\right| dt$$

and

$$I_2 = \int_{-T_2}^{T_2} \frac{1}{|t|} \left| e^{-\frac{m_n t^2 s_{pn}^2}{2s_n^2}} - e^{-t^2/2} \right| dt.$$

The bounds for the expressions on the right side of (4.1) and (4.2) are obtained from the Propositions 3.1, 3.3, 3.8, Corollary 3.7 and the value of T_2.

To obtain the final bound we compare

$$R_1 = n^{-q\alpha/8}, \quad R_2 = b_n^2 n^{\alpha(1+\theta)-\theta} I\left(\frac{2\theta}{3+2\theta} \le \alpha < \frac{\theta}{1+\theta}\right),$$
$$R_3 = b_n^{-1} n^{-\alpha/2} I\left(0 < \alpha \le \frac{2\theta}{1+2\theta}\right), \quad R_4 = n^{-3\alpha/8},$$
$$R_5 = b_n^{-1} n^{-\alpha/2} I\left(0 < \alpha < \frac{2\theta}{3+2\theta}\right),$$
$$R_6 = n^{-(1-\alpha)\theta} I\left(\frac{2\theta}{1+2\theta} < \alpha < 1\right) \quad \text{and} \quad R_7 = n^{-\alpha(q-2)/2}$$

for various values of α and q. We consider the cases $2 < q \le 3$ and $q > 3$ separately.

Let us say $c_n > 0$ dominates over d_n if $\frac{d_n}{c_n} \to 0$ as $n \to \infty$. The bound contains terms some of which dominate over others. In the case $2 < q \le 3$ the domination depends on the value of q in the ranges $2 < q < 8/3$ and $8/3 \le q \le 3$ and the choice of α. Let us first consider the case $2 < q < 8/3$. Then, after some tedious but elementary analysis, we find that R_7 dominates over all the other R_j for the value of α in the intervals $\left(0, \frac{2\theta}{q+2\theta}\right]$ and $\left(\frac{\theta}{1+\theta}, \frac{2\theta}{q-2+2\theta}\right]$, while R_2 dominates over all the other R_j for the values of $\alpha \in \left(\frac{2\theta}{q+2\theta}, \frac{\theta}{1+\theta}\right]$ and R_6 dominates over all the other R_j for the values of $\alpha \in \left(\frac{2\theta}{q-2+2\theta}, 1\right)$.

Thus for $2 < q < 8/3$ we get the bound

$$C_{27} \frac{1}{n^{\alpha(q-2)/2}} I\left(\left(0 < \alpha \le \frac{2\theta}{q+2\theta}\right) \cup \left(\frac{\theta}{1+\theta} < \alpha \le \frac{2\theta}{q-2+2\theta}\right)\right) +$$
$$+ C_{28} \frac{b_n^2}{n^{\theta-\alpha(1+\theta)}} I\left(\frac{2\theta}{q+2\theta} < \alpha \le \frac{\theta}{1+\theta}\right) + C_{29} \frac{1}{n^{\theta(1-\alpha)}} I\left(\frac{2\theta}{q-2+2\theta} < \alpha < 1\right).$$

This can be simplified further. Since $n^{-\alpha(q-2)/2}$ decreases as α increases the best rate contributed by the first term is for the maximum value of α. So we compare for $\alpha = \frac{2\theta}{q+2\theta}$ and $\frac{2\theta}{q-2+2\theta}$ and get the best rate $n^{-\frac{\theta(q-2)}{q+2\theta}}$. On the other hand for the same value of q, the second term gives the rate $b_n^2\ n^{-\frac{\theta(q-2)}{q+2\theta}}$, which is dominated by the previously obtained rate because $b_n^2 \to 0$ while the third term gives the rate $n^{-\frac{\theta(q-2)}{q-2+2\theta}}$ which too is dominated by $n^{-\frac{\theta(q-2)}{q+2\theta}}$. Thus for $2 < q \le 8/3$ we get the best rate $n^{-\frac{\theta(q-2)}{q+2\theta}}$.

In the case $8/3 \le q \le 3$ the bound for the expression on the right side of (4.1) turns out to be

$$C_{28} \frac{b_n^2}{n^{\theta-\alpha(1+\theta)}} I\left(\frac{8\theta}{q+8+8\theta} < \alpha \le \frac{\theta}{1+\theta}\right) + C_{29} \frac{1}{n^{\theta(1-\alpha)}} I\left(\frac{8\theta}{q+8\theta} \le \alpha < 1\right) +$$
$$+ C_{30} \frac{1}{n^{q\alpha/8}} I\left(\left(0 < \alpha \le \frac{8\theta}{q+8+8\theta}\right) \cup \left(\frac{\theta}{1+\theta} < \alpha < \frac{8\theta}{q+8\theta}\right)\right).$$

Thus in the case $8/3 \le q \le 3$ the best rate is $n^{-\frac{q\theta}{8+q+8\theta}}$. In the case $q \ge 3$ the bound for the expression on the right side of (4.1) turns out to be

$$C_{32}\frac{1}{n^{3\alpha/8}}\, I\left(\left(0<\alpha\le\frac{8\theta}{11+8\theta}\right)\cup\left(\frac{\theta}{1+\theta}<\alpha\le\frac{8\theta}{3+8\theta}\right)\right)+$$
$$+\,C_{29}\frac{b_n^2}{n^{\theta-\alpha(1+\theta)}}\, I\left(\frac{8\theta}{11+8\theta}<\alpha\le\frac{\theta}{1+\theta}\right)+C_{30}\frac{1}{n^{(1-\alpha)\theta}}\left(\frac{8\theta}{3+8\theta}<\alpha<1\right).$$

Thus in the case $q \ge 3$ the best rate is $C_{33}\, n^{-\frac{3\theta}{11+8\theta}}$.

The best bound turns out to be

$$C_{22}\, n^{-\frac{3\theta}{11+8\theta}}.$$

This completes the proof of the Theorem 4.1. □

Remark 4.2. 1. In the border case of $q = 8/3$ both the bounds $n^{-\frac{\theta(q-2)}{q+2\theta}}$ and $n^{-\frac{q\theta}{q+8+8\theta}}$ coincide.

2. The bound in the Theorem 4.1 is independent of α. However in Çağin et al. [2] the bound depends on α.

3. For $q = 3$ the rate is $n^{-\frac{3\theta}{11+8\theta}}$, which as $\theta \to \infty$, goes to $n^{-3/8}$ and this is far better rate than the rate $n^{-1/5}$ given in Oliveira's book [8].

4. As is to be expected the rate of convergence in the CLT improves as q increases in the interval (2, 3). Further as in the case of independent and identically distributed rvs the rate remains the same with finiteness of the moments of order ≥ 3.

5. If μ_n is chosen as $e^{-\mu\alpha}$, $0 < \mu < 1/2$ instead of the above choice of μ_n with $\mu = 3/8$, the calculations become more complicated and we have to consider three cases; *viz.,* $2 < q < 1/\mu$, $(1/\mu) < q < (2\mu/(1-2\mu))$ and $(2\mu/(1-2\mu)) < q \le 3$ instead of $2 < q < 8/3$ and $8/3 \le q < 3$ when $q < 3$. Further, in each case the interval $(0, 1)$ to which α belongs has to be split into subintervals depending on the value of μ. The best rate turns out to be $n^{-\frac{\mu\theta}{\mu+1+\theta}}$ for any choice of $q \in [(2\mu/(1-2\mu)),\ 3]$. Interestingly the above interval collapses to the single point set consisting of 3 when $\mu = 3/8$.

5. A moderate deviation result

Çağin et al. [2] recently obtained a moderate deviation result for associated rvs under strong conditions. Before we state and prove the moderate deviation result, we shall recall a result of Frolov [4] and apply it to the coupling block rvs introduced earlier.

Theorem 5.1 (Theorem 1.1 in Frolov [4]). *Let* $\{Y_{k,n}, k = 1, 2, \ldots, k_n, n = 1, 2, \ldots\}$ *be an array of column-wise independent centered rvs with* $\mathsf{E}Y_{k,n}^2 = \sigma_{k,n}^2 < \infty$. *Denote* $T_n = \sum_{k=1}^{k_n} Y_{k,n}$ *and* $B_n = \sum_{k=1}^{k_n} \sigma_{k,n}^2$. *Assume for some* $q > 2$, $\mathsf{E}[Y_{k,n}^q I(Y_{k,n} > 0)] = = \beta_{k,n} < \infty$, $B_n \to \infty$ *and set*

$$M_n = \sum_{k=1}^{k_n} \beta_{k,n}, \quad L_n = \frac{M_n}{B_n^{q/2}},$$

$$\Lambda_n(t, s, \delta) = \frac{t}{B_n}\sum_{k=1}^{k_n} \mathsf{E}\Big(Y_{k,n}^2 I\big(-\infty < Y_{k,n} < -\delta\sqrt{B_n}/s\big)\Big).$$

Assume that $L_n \to 0$, *and that for each* $\delta > 0$, $\Lambda_n(x^4, x^5, \delta) \to 0$. *If* $x_n \to \infty$ *such that*

$$x_n^2 - 2\log(1/L_n) - (q-1)\log\log(1/L_n) \to -\infty, \tag{5.1}$$

then

$$\mathsf{P}(T_n \le x_n s_n) = (1 - \Phi(x_n))(1 + o(1)).$$

Let the assumption A_1 hold for the original rvs X_n. Recall that the block rvs $Y^*_{k,n}$, $k = 1, 2, \dots, m_n$ are independent and identically distributed for each n with $\mathsf{E}|Y_{k,n}|^q < \infty$ where $q > 2$. With $Y_{k,n} = Y^*_{k,n}$, $k_n = m_n$,

$$B_n = m_n\, s^2_{p_n} \sim n\sigma^2, \quad M_n \le m_n\, \mathsf{E}|Y^*_{k,n}|^q \sim n^{\alpha+(1-\alpha)q/2}, \quad L_n \sim n^{\alpha(2-q)/2} \to 0$$

as $n \to \infty$. Further

$$\Lambda_n(x^4, x^5, \delta) \le \frac{x^4}{\sigma^2\, n^{1-\alpha}}\, \mathsf{E}\big(Y^2_{k,n} I(D_n)\big), \tag{5.2}$$

where D_n is the event $|Y_{k.n}| > \delta\sqrt{n}\sigma/x^5$ since $Y^*_{k,n} \overset{D}{=} Y_{k,n}$. By the Hölder's inequality and finiteness of moment of order q for $Y_{k,n}$, we get

$$\begin{aligned}\mathsf{E}\big(Y^2_{k,n} I(D_n)\big) &\le \Big(\mathsf{E}\big(Y^2_{k,n} I(D_n)\big)^{q/2}\Big)^{2/q} \Big(\mathsf{E}(I(D_n))^{q/(q-2)}\Big)^{(q-2)/q} \le \\ &\le (\mathsf{E}|Y_{1,n}|^q)^{2/q} (\mathsf{P}(D_n))^{(q-2)/2} \le p_n \left(\frac{\mathsf{E}|Y_{1,n}|^q x^{5q}}{\delta^q n^{q/2} \sigma^q}\right)^{(q-2)/q},\end{aligned}$$

which results in the following bound from (5.2)

$$\begin{aligned}\Lambda_n(x^4, x^5, \delta) &\le x^4 \left(\frac{p_n^{q/2} x^{5q}}{\delta^q \sigma^q n^{q/2}}\right)^{(q-2)/q} \le \\ &\le C_{34} \frac{x^{5q-6}}{n^{\alpha(q-2)/2}}.\end{aligned}$$

If $x = x_n \sim (\log n)^\kappa$ and $\kappa > 0$ then $\Lambda_n(x_n^4, x_n^5, \delta) \to 0$ as $n \to \infty$, so that all the conditions of the Theorem 5.1 hold and we then get the following moderate deviation result for the coupling block rvs $Y^*_{k,n}$.

Theorem 5.2. *If $\{X_n\}$ is a sequence of centered associate rvs satisfying the assumption A_1 then for the coupling block rvs $Y^*_{j,n}$*

$$\mathsf{P}\left(\sum_{j=1}^{m_n} Y^*_{j,n} > x_n s_n\right) = (1 - \Phi(x_n))(1 + o(1))$$

whenever x_n satisfies

$$\limsup_{n\to\infty} \frac{x_n^2}{\log n} = \lambda < \alpha(q-2).$$

Remark 5.3. In the Theorem 4.2 of Çağin et al. [2] the Assumption (B2) states the condition differently but a close look at the proof reveals that they indeed use

$$\limsup_{n\to\infty} x_n^2\, (\log n)^{-1} < 1$$

which is similar to our assumption.

Corollary 5.4. *Recall $\mu_n = n^{-3\alpha/8}$. If x_n satisfies the relation (5.1) then so will $x_n \pm \mu_n$ and we have*

$$\mathsf{P}\left(\sum_{j=1}^{m_n} Y^*_{j,n} > (x_n \pm \mu_n) s_n\right) = (1 - \Phi(x_n))(1 + o(1))$$

because $\mu_n = o(1 - \Phi(x_n))$. Here we use the fact $|\Phi(x + \epsilon) - \Phi(x)| < \epsilon$.

Now we state and prove the moderate deviation result for S_n.

Theorem 5.5. *Let $\{X_n\}$ be a sequence of centered stationary associated rvs satisfying the assumptions A_1 and A_2. Assume further*
(i) $\limsup_{n\to\infty} \frac{x_n^2}{\log n} = \lambda < \frac{q-2}{2}$,
(ii) θ *in the assumption A_2 is such that*

$$\theta > 1 + \lambda. \tag{5.3}$$

Then

$$\mathsf{P}(S_n > x_n\, s_n) = (1 - \Phi(x_n))(1 + o(1)). \tag{5.4}$$

Proof. Choose α in the definition of p_n such that

$$\frac{1}{2} < \alpha < \frac{2\theta - \lambda}{2\theta + 2}. \tag{5.5}$$

This is possible because of the assumption at (5.3). Let $\epsilon_n = n^{-\epsilon}$ where

$$0 < \epsilon < \frac{q\alpha - \lambda}{2\, q}. \tag{5.6}$$

This is possible because $\lambda < (q-2)/2$ and $\alpha > 1/2$. The stated result follows from the Corollary 5.4 and the assumption (i) above if we prove

$$\text{(a)} \qquad \left| \mathsf{P}(S_n > x_n\, s_n) - \left(\sum_{j=1}^{m_n} Y_{j,n} > (x_n \pm \epsilon_n)\, s_n \right) \right| = o(1 - \Phi(x_n))$$

and

$$\text{(b)} \qquad \left| \left(\sum_{j=1}^{m_n} Y_{j,n} > (x_n \pm \epsilon_n)\, s_n \right) - \left(\sum_{j=1}^{m_n} Y_{j,n}^* > (x_n \pm \epsilon_n)\, s_n \right) \right| = o(1 - \Phi(x_n)).$$

To prove (a) recall from the Proposition 3.1

$$\left| \mathsf{P}(S_n > x_n\, s_n) - \mathsf{P}\left(\sum_{j=1}^{m_n} Y_{j,n} > (x_n \pm \epsilon_n)\, s_n \right) \right| \le \mathsf{P}(|Y_{m_n+1,n}| > \epsilon_n\, s_n) <$$
$$< C_{35}\, \frac{p_n^{q/2}}{\epsilon_n^q\, n^{q/2}} < C_{36}\, n^{-q\,(\alpha - 2\epsilon)/2}. \tag{5.7}$$

We get the result (a) if

$$\frac{\sqrt{\log n}}{n^{(q(\alpha-2\epsilon)-\lambda)/2}} \to 0$$

which follows from (5.5).
Next to prove (b) recall from the Proposition 3.3

$$\left| \left(\sum_{j=1}^{m_n} Y_{j,n} > (x_n \pm \epsilon_n)\, s_n \right) - \left(\sum_{j=1}^{m_n} Y_{j,n}^* > (x_n \pm \epsilon_n)\, s_n \right) \right| <$$
$$< C_3\, \frac{b_n^2}{n^{\theta - \alpha(1+\theta)}} I\left(\frac{2\theta}{3 + 2\theta} \le \alpha < \frac{\theta}{1+\theta} \right) + C_4\, \frac{1}{b_n n^{\alpha/2}} I\left(\alpha < \frac{2\theta}{3+2\theta} \right). \tag{5.8}$$

The first term on the right side above is $o(1-\Phi(x_n))$ because (5.5) implies $\theta - \alpha(1+\theta) > \frac{\lambda}{2}$. The second term on the right side of (5.8) is $o(1 - \Phi(x_n))$ because $\lambda < \alpha$. This completes the proof of the theorem. □

Remark 5.6. Çağin et al. [2] proved the Theorem 5.1 making complicated assumptions of the type A_2 as well as A_3 in their paper along with the conditions that $\theta > 4$ and $q > 3$. Further our proof does not require dealing with odd numbered and even numbered blocks separately nor does it need introduction of Gaussian centered variables similar to odd and even block sums.

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References

1. T. Birkel, *On the convergence rate in the central limit theorem for associated processes*, Ann. Probab, **16** (1988), 1689–1698.
2. T. Çağin, P. E. Oliveira, N. Torrado, *A moderate deviation for associated randon variables*, J. Korean Statist. Soc., **45** (2016), 285–294.
3. A. Dembo, A. Zeitouni, *Large deviation techniques and applications*, 2nd ed., Springer, New York, 1998.
4. A. N. Frolov, *On probabilities of moderate deviations of sums for independent random variables*, Jl. Math. Sci., **127** (2005), 1787–1796.
5. F. D. Hollander, *Large deviations*, Fields Inst. Monographs, Amer. Math. Soc., Providence, Rhodes Island, 2000.
6. M. Loève, *Probability Theory*, I, 4th Edn., Springer, New York, 1977.
7. C. M. Newman, *Normal fluctuations and FKG-inequality*, Commun. Math. Phys., **74** (1980), 119–128.
8. P. E. Oliveira, *Asymptotics for associated random variables*, Heidelberg, Springer, 2012.
9. A. N. Tikhomirov, *On the convergence rate in the central limit theorem for weakly dependent random variables*, Theory Probab. Appl., **25** (1980), 790–809.
10. S. R. S. Varadhan, *Large deviations and applications*, S. I. A. M., Philadelphia, 1984.
11. W. Wang, *Large deviations for sums of random vectors attracted to operator semi-stable laws*, Jl. Theor. Probab. (2015).
12. T. E. Wood, *A Berry–Esseen theorem for associated random variables*, Ann. Probab., **11** (1983), 1041–1047.

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ПОРЯДОК АПРОКСИМАЦІЇ В ЦЕНТРАЛЬНІЙ ГРАНИЧНІЙ ТЕОРЕМІ ДЛЯ АСОЦІЙОВАНИХ ВИПАДКОВИХ ВЕЛИЧИН ТА РЕЗУЛЬТАТ ПРО ПОМІРНІ ВІДХИЛЕННЯ

М. СРІГАРІ

Анотація. Одержано оцінку порядку апроксимації в центральній граничній теоремі для строго стаціонарних асоційованих випадкових величин зі скінченними моментами порядку $q > 2$. Також отримано результат про помірні відхилення. Уточнено останні результати із [2]. Одержаний порядок апроксимації є вдосконаленням відповідного результату із [12].

ПОРЯДОК АППРОКСИМАЦИИ В ЦЕНТРАЛЬНОЙ ПРЕДЕЛЬНОЙ ТЕОРЕМЕ ДЛЯ АССОЦИИРОВАННЫХ СЛУЧАЙНЫХ ВЕЛИЧИН И РЕЗУЛЬТАТ ОБ УМЕРЕННЫХ УКЛОНЕНИЯХ

М. СРИХАРИ

Аннотация. Получена оценка порядка аппроксимации в центральной предельной теореме для строго стационарных ассоциированных случайных величин с конечными моментами порядка $q > 2$. Также получен результат об умеренных уклонениях. Уточнены недавние результаты из [2]. Полученный порядок аппроксимации является улучшением соответствующего результата из [12].