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WAVELET ANALYSIS OF A MULTIFRACTIONAL PROCESS IN AN ARBITRARY WIENER CHAOS

A. AYACHE, Y. ESMILI

Abstract. The well-known multifractional Brownian motion (mBm) is the paradigmatic example of a continuous Gaussian process with non-stationary increments whose local regularity changes from point to point. In this article, using a wavelet approach, we construct a natural extension of mBm which belongs to a homogeneous Wiener chaos of an arbitrary order. Then, we study its global and local behavior.

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1. Introduction

A *fractional Brownian motion* (fBm) of an arbitrary Hurst parameter $H \in (0,1)$, denoted by $\{B_H(t) : t \in \mathbb{R}\}$, is defined, up to a multiplicative constant, as the unique (in distribution) Gaussian process with stationary increments which is globally self-similar of order H. Recall that a stochastic process $\{F(t) : t \in \mathbb{R}\}$ is said to have stationary increments if, for any fixed point $t_0 \in \mathbb{R}$, one has:

$$\big\{F(t_0+u) - F(t_0) : u \in \mathbb{R}\big\} \overset{\text{law}}{=} \big\{F(u) - F(0) : u \in \mathbb{R}\big\}\,; \tag{1.1}$$

and it is said to be globally self-similar of order H if, for each fixed positive real number ν, one has:

$$\big\{\nu^{-H} F(\nu t) : t \in \mathbb{R}\big\} \overset{\text{law}}{=} \big\{F(t) : t \in \mathbb{R}\big\}\,. \tag{1.2}$$

The representation of an fBm as a well-balanced moving average is given, for every $t \in \mathbb{R}$, by the Wiener integral over $\mathbb{R}$:

$$B_H(t) = \int_{\mathbb{R}} \Big[|t-s|^{H-1/2} - |s|^{H-1/2}\Big] dB(s), \tag{1.3}$$

with the convention that $|t-s|^0 - |s|^0 = \log|t-s| - \log|s|$. FBm was first introduced by Kolmogorov in 1940 as a way for generating Gaussian spirals in Hilbert spaces [14]. Later, in 1968, the well-known article [16] by Mandelbrot and Van Ness emphasised its importance as a model in several areas of application: hydrology, geology, finance, and so on. Since then many applied and theoretical aspects of this stochastic process have been extensively explored in the literature and, among many other things, its path behavior has been well understood. Despite its importance in modeling, fBm does not always succeed in giving a sufficiently reliable description of real-life signals. Indeed, fBm suffers from two main limitations:

(*a*) its Gaussian character,

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(b) local roughness of its paths remains everywhere the same; more precisely, their local and pointwise Hölder exponents are everywhere equal to the Hurst parameter H.

In order to overcome the limitation (b) of fBm, the so-called *multifractional Brownian motion* (mBm) was introduced, about twenty years ago, independently by Benassi, Jaffard and Roux [5] and by Lévy Véhel and Peltier [20]. The latter continuous Gaussian process with non-stationary increments can be obtained by substituting the constant Hurst parameter H in (1.3) by a deterministic continuous function $H(\cdot)$ depending on t and with values in the open interval $(0,1)$. Nowadays mBm has become a quite useful model in the fields of financial modeling and signal processing (see for instance [6–8]).

In order to overcome the limitations (a) and (b) together, the extensions of mBm whose Hurst parameter is a stochastic process or more generally a sequence of stochastic processes were introduced in [2, 3]. Other extensions of mBm to frames of heavy-tailed stable distributions were proposed in [10, 22, 23]. More recently, [21] constructed a multifractional generalized Rosenblatt process belonging to the second order homogeneous Wiener chaos. In our present article, we construct a multifractional process, denoted by $\{Z(t) : t \in \mathbb{R}\}$, which belongs to a homogeneous Wiener chaos of an arbitrary integer order $d \geq 2$. The latter multifractional process is not a generalization of the Rosenblatt process but of a process $\{Y_H(t) : t \in \mathbb{R}\}$ consisting in a very natural chaotic extension of the fBm in (1.3). Namely, it is defined, for all $t \in \mathbb{R}$, through the multiple Wiener integral on $\mathbb{R}^d$:

$$Y_H(t) = \int_{\mathbb{R}^d} \left[\|\mathbf{t}^* - \mathbf{x}\|_2^{H-\frac{d}{2}} - \|\mathbf{x}\|_2^{H-\frac{d}{2}} \right] dB_{x_1} \ldots dB_{x_d} \,, \tag{1.4}$$

where $\mathbf{t}^* = (t, \ldots, t) \in \mathbb{R}^d$ and $\|\cdot\|_2$ denotes the Euclidian norm over $\mathbb{R}^d$. A class of chaotic self-similar processes with stationary increments, which implicitly includes $\{Y_H(t) : t \in \mathbb{R}\}$, had been first introduced and investigated in [18]. Long time later, $\{Y_H(t) : t \in \mathbb{R}\}$ was explicitly introduced and studied in its own right in [1] through wavelet methods inspired by the ones in [2, 3].

Recall that a centred non-Gaussian square integrable real-valued random variable, on the underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$, belongs to the homogeneous Wiener chaos of an arbitrary integer order $d \geq 2$ when it can be represented by a multiple Wiener integral over $\mathbb{R}^d$. We always denote by $I_d(\cdot)$ this stochastic integral, and use the classical convention that, for every $f \in L^2(\mathbb{R}^d)$, one has $I_d(f) = I_d(\tilde{f})$; the function $\tilde{f}$ being the symmetrization of f, defined, for all $(t_1, \ldots, t_d) \in \mathbb{R}^d$, as

$$\tilde{f}(t_1, \ldots, t_d) = \frac{1}{d!} \sum_{\sigma \in \mathcal{S}_d} f\big(t_{\sigma(1)}, \ldots, t_{\sigma(d)}\big),$$

where $\mathcal{S}_d$ refers to the set of all permutations of $\{1, \ldots, d\}$ (observe that $d!$ is the cardinality of $\mathcal{S}_d$). A very important property of multiple Wiener integrals, which somehow can be viewed as an *isometry property*, is that, for all function $f \in L^2(\mathbb{R}^d)$, one has

$$\mathsf{E}\big(|I_d(f)|^2\big) = d! \, \|\tilde{f}\|^2_{L^2(\mathbb{R}^d)} \leq d! \, \|f\|^2_{L^2(\mathbb{R}^d)} \,. \tag{1.5}$$

Before ending these very short recalls on multiple Wiener integrals, it is worth mentioning that two well-known books on them and related topics are [13, 19].

Roughly speaking, we define $Z = \{Z(t) : t \in \mathbb{R}\}$, the multifractional generalization of $Y_H = \{Y_H(t) : t \in \mathbb{R}\}$, as $Z(t) = Y_{H(t)}(t)$, for all $t \in \mathbb{R}$, where $H(\cdot)$ is an arbitrary deterministic continuous function over $\mathbb{R}$ with values in the open interval $(0,1)$. More precisely, let us consider the chaotic stochastic field $X = \{X(u,v) : (u,v) \in \mathbb{R} \times (0,1)\}$, such that, for every $(u,v) \in \mathbb{R} \times (0,1)$, one has

$$X(u,v) = \int_{\mathbb{R}^d} \left[\|\mathbf{u}^* - \mathbf{x}\|_2^{v-\frac{d}{2}} - \|\mathbf{x}\|_2^{v-\frac{d}{2}} \right] dB_{x_1} \dots dB_{x_d}. \tag{1.6}$$

We mention in passing that, for each fixed $v \in (0,1)$, the stochastic processes $X(\cdot, v) = \{X(u,v) : u \in \mathbb{R}\}$ and $Y_v = \{Y_v(u) : u \in \mathbb{R}\}$ have the same law. The multifractional process $Z = \{Z(t) : t \in \mathbb{R}\}$ is defined, for all $t \in \mathbb{R}$, as

$$Z(t) = X(t, H(t)). \tag{1.7}$$

By expanding, for each fixed $(t,H) \in \mathbb{R} \times (0,1)$, the kernel function

$$\mathbf{x} \mapsto \|\mathbf{t}^* - \mathbf{x}\|_2^{H-\frac{d}{2}} - \|\mathbf{x}\|_2^{H-\frac{d}{2}}$$

in (1.4) into a Meyer wavelet basis of $L^2(\mathbb{R}^d)$ (see e.g. [9, 15, 17]), a random series representation for the chaotic fractional process $\{Y_H(t) : t \in \mathbb{R}\}$ has been constructed in [1], which also has shown that this series is almost surely absolutely convergent, for each fixed $(t,H) \in \mathbb{R} \times (0,1)$. The main goal of Section 2 of our article is to transpose these two results into the setting of the chaotic stochastic field $\{X(u,v) : (u,v) \in \mathbb{R} \times (0,1)\}$, and more importantly to show that the random series representation of this field is almost surely uniformly convergent in (u,v) on each compact subset of $\mathbb{R} \times (0,1)$. Then, thanks to this nice representation, global and local behavior of $\{X(u,v) : (u,v) \in \mathbb{R} \times (0,1)\}$ and $\{Z(t) : t \in \mathbb{R}\}$ are studied in Sections 3 and 4 of our article, by using a wavelet methodology which is, to a certain extent, inspired by the one introduced in [2, 3].

2. Uniformly convergent random series representation

First, we need to introduce some additional notations. We denote by $S(\mathbb{R}^d)$ the *Schwartz class*, that is the space of the infinitely differentiable complex-valued functions over $\mathbb{R}^d$ which, as well as all their partial derivatives of any order, vanish at infinity faster than any power function.

Let $E = \{0,1\}^d \setminus \{(0,\dots,0)\}$. A Meyer wavelet basis of $L^2(\mathbb{R}^d)$ is an orthonormal (or Hilbertian) basis of $L^2(\mathbb{R}^d)$ of the form:

$$\left\{ 2^{\frac{jd}{2}} \psi^{(\epsilon)}(2^j \mathbf{x} - \mathbf{k}) : j \in \mathbb{Z}, \mathbf{k} \in \mathbb{Z}^d, \epsilon \in E \right\}; \tag{2.1}$$

for the sake of convenience, one sets:

$$\psi^{(\epsilon)}_{j,\mathbf{k}}(\mathbf{x}) = 2^{\frac{jd}{2}} \psi^{(\epsilon)}(2^j \mathbf{x} - \mathbf{k}). \tag{2.2}$$

The $2^d - 1$ real-valued functions $\psi^{(\epsilon)}$, $\epsilon \in E$, which generate the basis are called *the d-variate Meyer mother wavelets.* They can be expressed as tensor products of ψ^0 and ψ^1 which respectively denote a 1-variate Meyer father and mother wavelets. More precisely, for each $\epsilon = (\epsilon_1, \dots, \epsilon_d) \in E$ and $\mathbf{x} = (x_1, \dots, x_d) \in \mathbb{R}^d$, one has that:

$$\psi^{(\epsilon)}(\mathbf{x}) = \prod_{l=1}^{d} \psi^{\epsilon_l}(x_l). \tag{2.3}$$

Let us emphasize that the 1-variate Meyer father and mother wavelets belong to $S(\mathbb{R})$. Moreover, their Fourier transforms $\mathcal{F}(\psi^0)$ and $\mathcal{F}(\psi^1)$ are infinitely differentiable compactly supported functions satisfying: $\operatorname{supp} \mathcal{F}(\psi^0) \subseteq [-\frac{4\pi}{3}, \frac{4\pi}{3}]$, $\operatorname{supp} \mathcal{F}(\psi^1) \subseteq \{\xi \in \mathbb{R} : \frac{2\pi}{3} \le |\xi| \le \frac{8\pi}{3}\}$, and $\mathcal{F}(\psi^0)(\xi) = 1$, for all $\xi \in [-\frac{2\pi}{3}, \frac{2\pi}{3}]$. Thus, in view of (2.3), the d-variate Meyer mother wavelets $\psi^{(\epsilon)}$, $\epsilon \in E$, belong to $S(\mathbb{R}^d)$ and have infinitely differentiable compactly supported Fourier transforms which vanish in a neighborhood of 0. Using these nice properties of the d-variate Meyer mother wavelets, for each $\epsilon \in E$, it can be shown (see [2, 3] for instance) that the real-valued function Ψ^ϵ defined, for all $(\mathbf{u}, v) \in \mathbb{R}^d \times [0,1]$, as

$$\Psi^\epsilon(\mathbf{u}, v) = \int_{\mathbb{R}^d} \|\mathbf{u}-\mathbf{s}\|_2^{v-d/2}\, \psi^{(\epsilon)}(\mathbf{s}) d\mathbf{s}\,,$$

is infinitely differentiable on $\mathbb{R}^d \times (0,1)$ and satisfies, as well as all its partial derivatives of any order, the following very useful localization property:

$$\forall\, (n, \mathbf{p}, q) \in \mathbb{Z}_+ \times \mathbb{Z}_+^d \times \mathbb{Z}_+\,,$$
$$\sup\big\{ \big(\alpha + \|\mathbf{u}\|_2\big)^n \, \big|(\partial_{\mathbf{u}}^{\mathbf{p}} \partial_v^q \Psi^\epsilon)(\mathbf{u}, v)\big| : (\mathbf{u}, v) \in \mathbb{R}^d \times (0,1)\big\} < +\infty\,, \quad (2.4)$$

where α is an arbitrary positive fixed real number.

Before giving the random series representation of the field X derived from the Meyer wavelet basis (2.1), let us state the following important lemma borrowed from [1].

Lemma 2.1. *For each $(j, \mathbf{k}, \epsilon) \in \mathbb{Z}\times\mathbb{Z}^d\times E$, let $I_d\Big(\psi_{j,\mathbf{k}}^{(\epsilon)}\Big)$ be the multiple Wiener integral over $\mathbb{R}^d$ of the wavelet function defined in (2.2). That is, one has*

$$I_d\Big(\psi_{j,\mathbf{k}}^{(\epsilon)}\Big) = \int_{\mathbb{R}^d} \psi_{j,\mathbf{k}}^{(\epsilon)}(\mathbf{x}) dB_{x_1} \dots dB_{x_d}. \quad (2.5)$$

Then, there exist an event Ω^* of probability 1 and a finite positive random variable C_d such that, for all $\omega \in \Omega^*$ and for each $(j, \mathbf{k}, \epsilon) \in \mathbb{Z} \times \mathbb{Z}^d \times E$, one has

$$\Big| I_d\Big(\psi_{j,\mathbf{k}}^{(\epsilon)}\Big)(\omega)\Big| \le C_d(\omega)\big(\log(e + |j| + \|\mathbf{k}\|_1)\big)^{\frac{d}{2}}, \quad (2.6)$$

where $\|\cdot\|_1$ denotes the 1-norm over $\mathbb{R}^d$; that is, $\|\mathbf{k}\|_1 = \sum_{l=1}^d |k_l|$, the k_l's being the coordinates of $\mathbf{k}$.

The following proposition provides the random series representation of the field X derived from the Meyer wavelet basis (2.1). The proposition has been obtained in [1] with Y_H in place of X (see (1.4) and (1.6)).

Proposition 2.2. *For each fixed $(u, v, \omega) \in \mathbb{R} \times (0,1) \times \Omega^*$, one has*

$$\sum_{j\in\mathbb{Z}} \sum_{\mathbf{k}\in\mathbb{Z}^d} \sum_{\epsilon\in E} 2^{-jv} \Big| \big(\Psi^\epsilon(2^j\mathbf{u}^* - \mathbf{k}, v) - \Psi^\epsilon(-\mathbf{k}, v)\big) I_d\Big(\psi_{j,\mathbf{k}}^{(\epsilon)}\Big)(\omega)\Big| < +\infty\,.$$

This means that the series of real numbers

$$\sum_{j\in\mathbb{Z}} \sum_{\mathbf{k}\in\mathbb{Z}^d} \sum_{\epsilon\in E} 2^{-jv} \big(\Psi^\epsilon(2^j\mathbf{u}^* - \mathbf{k}, v) - \Psi^\epsilon(-\mathbf{k}, v)\big) I_d\Big(\psi_{j,\mathbf{k}}^{(\epsilon)}\Big)(\omega)$$

is absolutely convergent, and consequently that it converges to a finite limit not depending on the way the terms of the series are ordered. Moreover, for all fixed $(u, v) \in \mathbb{R} \times (0,1)$, one has, almost surely,

$$X(u,v) = \sum_{j\in\mathbb{Z}} \sum_{\mathbf{k}\in\mathbb{Z}^d} \sum_{\epsilon\in E} 2^{-jv} \big(\Psi^\epsilon(2^j\mathbf{u}^* - \mathbf{k}, v) - \Psi^\epsilon(-\mathbf{k}, v)\big) I_d\Big(\psi_{j,\mathbf{k}}^{(\epsilon)}\Big). \quad (2.7)$$

Remark 2.3. From now on and till the end of our article, the chaotic stochastic field $X = \{X(u, v) : (u, v) \in \mathbb{R} \times (0,1)\}$ will be systematically identified with its modification

$$\left\{ \sum_{j\in\mathbb{Z}} \sum_{\mathbf{k}\in\mathbb{Z}^d} \sum_{\epsilon\in E} 2^{-jv} \big(\Psi^\epsilon(2^j\mathbf{u}^* - \mathbf{k}, v) - \Psi^\epsilon(-\mathbf{k}, v)\big) I_d\Big(\psi_{j,\mathbf{k}}^{(\epsilon)}\Big) : (u, v) \in \mathbb{R} \times (0,1) \right\},$$

which has just been introduced in Proposition 2.2. Also, we will always assume that X vanishes outside of Ω^*, the event of probability 1 introduced in Lemma 2.1. The low and high frequency parts of X are the two chaotic stochastic fields, denoted respectively by

$X^{lf} = \{X^{lf}(u,v) : (u,v) \in \mathbb{R}\times(0,1)\}$ and $X^{hf} = \{X^{hf}(u,v) : (u,v) \in \mathbb{R}\times(0,1)\}$, which vanish outside of Ω^*, and are defined, for every $(u,v,\omega) \in \mathbb{R}\times(0,1)\times\Omega^*$, as:

$$X^{lf}(u,v,\omega) = \sum_{j=-\infty}^{-1}\sum_{\mathbf{k}\in\mathbb{Z}^d}\sum_{\epsilon\in E} 2^{-jv}\big(\Psi^\epsilon(2^j\mathbf{u}^*-\mathbf{k},v)-\Psi^\epsilon(-\mathbf{k},v)\big)I_d\Big(\psi_{j,\mathbf{k}}^{(\epsilon)}\Big)(\omega) \quad (2.8)$$

and

$$X^{hf}(u,v,\omega) = \sum_{j=0}^{+\infty}\sum_{\mathbf{k}\in\mathbb{Z}^d}\sum_{\epsilon\in E} 2^{-jv}\big(\Psi^\epsilon(2^j\mathbf{u}^*-\mathbf{k},v)-\Psi^\epsilon(-\mathbf{k},v)\big)I_d\Big(\psi_{j,\mathbf{k}}^{(\epsilon)}\Big)(\omega). \quad (2.9)$$

One clearly has, for all $(u,v,\omega) \in \mathbb{R}\times(0,1)\times\Omega$, that

$$X(u,v,\omega) = X^{lf}(u,v,\omega) + X^{hf}(u,v,\omega). \quad (2.10)$$

Recall that Ω is the underlying probability space.

Let us now state the main result of the present section.

Theorem 2.4. *The random series in the right-hand side of (2.7) is, on the event Ω^* of probability 1, uniformly convergent in (u,v), on each compact subset of $\mathbb{R}\times(0,1)$.*

The following lemma will play a major role in the proof of Theorem 2.4 and in other important proofs in our article.

Lemma 2.5. *For all fixed $(\mathbf{p},q) \in \mathbb{Z}_+^d\times\mathbb{Z}_+$, there exists a positive finite random variable $C_{\mathbf{p},q}$ such that, for all $(j,\mathbf{u},\omega) \in \mathbb{Z}\times\mathbb{R}^d\times\Omega^*$, one has*

$$\sum_{\epsilon\in E}\sum_{\mathbf{k}\in\mathbb{Z}^d}\Big|I_d\Big(\psi_{j,\mathbf{k}}^{(\epsilon)}\Big)(\omega)\Big| \sup_{v\in(0,1)}\big|(\partial_{\mathbf{u}}^{\mathbf{p}}\partial_v^q\Psi^\epsilon)(\mathbf{u}-\mathbf{k},v)\big| \le C_{\mathbf{p},q}(\omega)\big(\log(e+|j|+\|\mathbf{u}\|_1)\big)^{\frac{d}{2}}. \quad (2.11)$$

As a straightforward consequence, for all $(j,\mathbf{u},v,\omega) \in \mathbb{Z}\times\mathbb{R}^d\times(0,1)\times\Omega^*$, the series

$$\Phi_j(\mathbf{u},v,\omega) = \sum_{\epsilon\in E}\sum_{\mathbf{k}\in\mathbb{Z}^d} I_d\Big(\psi_{j,\mathbf{k}}^{(\epsilon)}\Big)(\omega)\,\Psi^\epsilon(\mathbf{u}-\mathbf{k},v) \quad (2.12)$$

is absolutely convergent, and the real-valued function $\Phi_j(\cdot,\cdot,\omega) : (\mathbf{u},v) \mapsto \Phi_j(\mathbf{u},v,\omega)$ is well-defined and infinitely differentiable on $\mathbb{R}^d\times(0,1)$. Moreover, for each $(j,\mathbf{u},v,\omega) \in$ $\in \mathbb{Z}\times\mathbb{R}^d\times(0,1)\times\Omega^*$, one has

$$(\partial_{\mathbf{u}}^{\mathbf{p}}\partial_v^q\Phi_j)(\mathbf{u},v,\omega) = \sum_{\epsilon\in E}\sum_{\mathbf{k}\in\mathbb{Z}^d} I_d\Big(\psi_{j,\mathbf{k}}^{(\epsilon)}\Big)(\omega)(\partial_{\mathbf{u}}^{\mathbf{p}}\partial_v^q\Psi^\epsilon)(\mathbf{u}-\mathbf{k},v),$$

and the following inequality holds:

$$\sup_{v\in(0,1)}\big|(\partial_{\mathbf{u}}^{\mathbf{p}}\partial_v^q\Phi_j)(\mathbf{u},v,\omega)\big| \le C_{\mathbf{p},q}(\omega)\big(\log(e+|j|+\|\mathbf{u}\|_1)\big)^{\frac{d}{2}}. \quad (2.13)$$

Proof of Lemma 2.5. It easily follows from (2.4) and from the finiteness of the set E that, for all $(\mathbf{p},q)$ fixed in $\mathbb{Z}_+^d\times\mathbb{Z}_+$, there is a positive finite deterministic constant $c_{\mathbf{p},q}$ for which the following inequality holds for every $(\mathbf{u},\mathbf{k},\epsilon) \in \mathbb{R}^d\times\mathbb{Z}^d\times E$:

$$\sup_{v\in(0,1)}\big|(\partial_{\mathbf{u}}^{p}\partial_v^q\Psi^\epsilon)(\mathbf{u}-\mathbf{k},v)\big| \le \frac{c_{\mathbf{p},q}}{(\sqrt{d}+1+\|\mathbf{u}-\mathbf{k}\|_2)^{2d}}.$$

In the sequel, one sets $\lfloor\mathbf{u}\rfloor = (\lfloor u_1\rfloor,\dots,\lfloor u_d\rfloor)$, where $\lfloor u_l\rfloor$ denotes the integer part of the l-th coordinate u_l of the vector $\mathbf{u}$. Then, using Lemma 2.1, the change of variable $\mathbf{m} = \mathbf{k}+\lfloor\mathbf{u}\rfloor$, the triangle inequality and the inequality

$$\forall\,(x,y)\in\mathbb{R}_+^2, \quad \big(\log(e+x+y)\big)^{\frac{d}{2}} \le \big(\log(e+x)\big)^{\frac{d}{2}}\big(\log(e+y)\big)^{\frac{d}{2}}, \quad (2.14)$$

one gets, for some positive finite constant $C'_{\mathbf{p},q}(\omega)$ and for all $(j,\mathbf{u},\omega)\in\mathbb{Z}\times\mathbb{R}^d\times\Omega^*$, that:

$$\begin{aligned}
&\sum_{\epsilon\in E}\sum_{\mathbf{k}\in\mathbb{Z}^d}\Big|I_d\Big(\psi_{j,\mathbf{k}}^{(\epsilon)}\Big)(\omega)\Big|\sup_{v\in(0,1)}\big|(\partial_{\mathbf{u}}^{\mathbf{p}}\partial_v^q\Psi^\epsilon)(\mathbf{u}-\mathbf{k},v)\big|\le\\
&\quad\le C'_{\mathbf{p},q}(\omega)\sum_{\mathbf{k}\in\mathbb{Z}^d}\frac{(\log(e+|j|+\|\mathbf{k}\|_1))^{\frac{d}{2}}}{(\sqrt{d}+1+\|\mathbf{u}-\mathbf{k}\|_2)^{2d}}\le\\
&\quad\le C'_{\mathbf{p},q}(\omega)\sum_{\mathbf{m}\in\mathbb{Z}^d}\frac{(\log(e+|j|+\|\mathbf{m}+\lfloor\mathbf{u}\rfloor\|_1))^{\frac{d}{2}}}{(\sqrt{d}+1+\|\mathbf{u}-\lfloor\mathbf{u}\rfloor-\mathbf{m}\|_2)^{2d}}\le\\
&\quad\le C'_{\mathbf{p},q}(\omega)\sum_{\mathbf{m}\in\mathbb{Z}^d}\frac{(\log(e+|j|+\|\mathbf{m}\|_1+\|\lfloor\mathbf{u}\rfloor\|_1))^{\frac{d}{2}}}{(\sqrt{d}+1-\|\mathbf{u}-\lfloor\mathbf{u}\rfloor\|_2+\|\mathbf{m}\|_2)^{2d}}\le\\
&\quad\le C'_{\mathbf{p},q}(\omega)\big(\log(e+|j|+\|\lfloor\mathbf{u}\rfloor\|_1)\big)^{\frac{d}{2}}\sum_{\mathbf{m}\in\mathbb{Z}^d}\frac{(\log(e+\|\mathbf{m}\|_1))^{\frac{d}{2}}}{(1+\|\mathbf{m}\|_2)^{2d}}.
\end{aligned}$$

Thus, in view of the fact that

$$\sum_{\mathbf{m}\in\mathbb{Z}^d}\frac{(\log(e+\|\mathbf{m}\|_1))^{\frac{d}{2}}}{(1+\|\mathbf{m}\|_2)^{2d}}<+\infty\,,$$

it turns out that (2.11) is satisfied. □

Proof of Theorem 2.4. First, observe that, in view of (2.7) and (2.12), $X(u,v,\omega)$ can be expressed, for all $(u,v,\omega)\in\mathbb{R}\times(0,1)\times\Omega^*$, as:

$$X(u,v,\omega)=\sum_{j\in\mathbb{Z}}A_j(u,v,\omega)\,, \tag{2.15}$$

where, for each $j\in\mathbb{Z}$, $A_j(\cdot,\cdot,\omega)$ is the infinitely differentiable function on $\mathbb{R}\times(0,1)$ defined as:

$$\forall\,(u,v)\in\mathbb{R}\times(0,1),\quad A_j(u,v,\omega)=2^{-jv}\big(\Phi_j(2^j\mathbf{u}^*,v,\omega)-\Phi_j(\mathbf{0},v,\omega)\big)\,. \tag{2.16}$$

Thus, in order to prove the theorem, one has to show that the convergence of the series in (2.15) holds uniformly in (u,v) on each compact subset of $\mathbb{R}\times(0,1)$. To this end, it is enough to prove that, for all fixed positive real numbers ν and $a<b<1$, one has

$$\sum_{j\in\mathbb{Z}}\sup_{(u,v)\in[-\nu,\nu]\times[a,b]}|A_j(u,v,\omega)|<+\infty. \tag{2.17}$$

Let us set

$$T_1(\omega)=\sum_{j=0}^{+\infty}\sup_{(u,v)\in[-\nu,\nu]\times[a,b]}|A_j(u,v,\omega)|$$

and

$$T_2(\omega)=\sum_{j=-\infty}^{-1}\sup_{(u,v)\in[-\nu,\nu]\times[a,b]}|A_j(u,v,\omega)|.$$

One can derive from (2.16) and (2.13) that

$$T_1(\omega)\le 2C_{\mathbf{0},0}(\omega)\sum_{j=0}^{+\infty}2^{-ja}\big(\log(e+j+2^jd\nu)\big)^{\frac{d}{2}}<+\infty\,. \tag{2.18}$$

On the other hand, using the mean value theorem and (2.13), one obtains, for every $(j,u,v,\omega)\in\mathbb{Z}_-\times[-\nu,\nu]\times[a,b]\times\Omega^*$, that

$$|A_j(u,v,\omega)|\le 2^{-ja}|\Phi_j(2^j\mathbf{u}^*,v,\omega)-\Phi_j(\mathbf{0},v,\omega)|\le$$

$$\le 2^{j(1-a)}|u|\sum_{i=1}^{d}\sup_{x\in[0\wedge 2^j u,0\vee 2^j u]}|\partial_{u_i}\Phi_j(\mathbf{x}^*,v,\omega)|\le$$
$$\le d\nu C_{\mathbf{1},0}(\omega)2^{j(1-a)}\big(\log(e+|j|+2^j d\nu)\big)^{\frac{d}{2}}.$$

Thus, one gets that

$$T_2(\omega)\le d\nu C_{\mathbf{1},0}(\omega)\sum_{j=1}^{\infty}2^{-j(1-a)}\big(\log(e+j+2^{-j}d\nu)\big)^{\frac{d}{2}}<+\infty. \tag{2.19}$$

Finally, combining (2.18) and (2.19), we see that (2.17) is satisfied. □

3. Global behavior

First, we state the main results of the section and then we give their proofs.

Theorem 3.1. *Let X^{lf} and X^{hf} be the low and high frequency parts of the field X which were introduced in Remark 2.3. The following two results hold, for all $\omega\in\Omega^*$.*

(i) The function $X^{lf}(\cdot,\cdot,\omega):(u,v)\mapsto X^{lf}(u,v,\omega)$ is infinitely many times differentiable on $\mathbb{R}\times(0,1)$.

(ii) For all fixed $u\in\mathbb{R}$, the function $X^{hf}(u,\cdot,\omega):v\mapsto X^{hf}(u,v,\omega)$ is infinitely many times differentiable on $(0,1)$. Moreover, for each fixed $q\in\mathbb{Z}_+$, the function $(\partial_v^q X)(\cdot,\cdot,\omega):(u,v)\mapsto(\partial_v^q X)(u,v,\omega)$ is continuous on $\mathbb{R}\times(0,1)$.

Corollary 3.2. *For each $(\omega,q)\in\Omega^*\times\mathbb{Z}_+$, and for all non-degenerate compact intervals $\mathcal{J}\subset\mathbb{R}$ and $\mathcal{H}\subset(0,1)$, one has:*

$$\sup_{(u,v_1,v_2)\in\mathcal{J}\times\mathcal{H}^2}\left\{\frac{|(\partial_v^q X)(u,v_1,\omega)-(\partial_v^q X)(u,v_2,\omega)|}{|v_1-v_2|}\right\}<+\infty. \tag{3.1}$$

Theorem 3.3. *For each $(\omega,q)\in\Omega^*\times\mathbb{Z}_+$, and for all non-degenerate compact intervals $\mathcal{J}\subset\mathbb{R}$ and $\mathcal{H}\subset(0,1)$, one has:*

$$\sup_{(u_1,u_2,v_1,v_2)\in\mathcal{J}^2\times\mathcal{H}^2}\left\{\frac{|(\partial_v^q X)(u_1,v_1,\omega)-(\partial_v^q X)(u_2,v_2,\omega)|}{|u_1-u_2|^{v_1\vee v_2}\big(1+\big|\log|u_1-u_2|\big|\big)^{q+\frac{d}{2}}+|v_1-v_2|}\right\}<+\infty. \tag{3.2}$$

Corollary 3.4. *Let $H(\cdot)$ be the continuous functional parameter of the chaotic multifractional process $\{Z(t):t\in\mathbb{R}\}$ (see (1.7)). Let $\mathcal{L}\subset\mathbb{R}$ be an arbitrary non-degenerate compact interval. One sets*

$$\underline{H}(\mathcal{L}):=\min\{H(t):t\in\mathcal{L}\}\quad\text{and}\quad\overline{H}(\mathcal{L}):=\max\{H(t):t\in\mathcal{L}\}. \tag{3.3}$$

Assuming that

$$H(\cdot)\in C^{\gamma_{\mathcal{L}}}(\mathcal{L})\quad\text{for some}\quad\gamma_{\mathcal{L}}\in[\underline{H}(\mathcal{L}),1), \tag{3.4}$$

where $C^{\gamma_{\mathcal{L}}}(\mathcal{L})$ denotes the global space of Hölder on $\mathcal{L}$ of order $\gamma_{\mathcal{L}}$. Then, for all $\omega\in\Omega^*$, one has:

$$\sup_{(t_1,t_2)\in\mathcal{L}^2}\left\{\frac{|Z(t_1,\omega)-Z(t_2,\omega)|}{|t_1-t_2|^{\underline{H}(\mathcal{L})}\big(1+\big|\log|t_1-t_2|\big|\big)^{\frac{d}{2}}}\right\}<+\infty\,. \tag{3.5}$$

Proof of Theorem 3.1. First, we point out that one knows from the proof of Theorem 2.4 that, for all $\omega\in\Omega^*$, one has

$$X^{lf}(u,v,\omega)=\sum_{j=-\infty}^{-1}A_j(u,v,\omega) \tag{3.6}$$

and

$$X^{hf}(u,v,\omega)=\sum_{j=0}^{+\infty}A_j(u,v,\omega)\,, \tag{3.7}$$

where the series in (3.6) and (3.7) are uniformly convergent in (u,v) on each compact subset of $\mathbb{R}\times(0,1)$. Moreover, one knows that, for each $j\in\mathbb{Z}$, the function $A_j(\cdot,\cdot,\omega)$ is infinitely differentiable on $\mathbb{R}\times(0,1)$. Thus, in order to prove the theorem, it is enough to show that, for all $(m,q,\omega)\in\mathbb{Z}_+\times\mathbb{Z}_+\times\Omega^*$, and for each positive real numbers ν and $a<b$, one has

$$\sum_{j=-\infty}^{-1}\sup_{(u,v)\in[-\nu,\nu]\times[a,b]}\big|(\partial_u^m\partial_v^q A_j)(u,v,\omega)\big|<+\infty$$

and

$$\sum_{j=0}^{+\infty}\sup_{(u,v)\in[-\nu,\nu]\times[a,b]}\big|(\partial_v^q A_j)(u,v,\omega)\big|<+\infty.$$

This can be done by following the main lines of the proof of (2.18) and (2.19). □

Proof of Corollary 3.2. It follows from Theorem 3.1 that, for all fixed $(u,\omega)\in\mathbb{R}\times\Omega^*$, the function $X(u,\cdot,\omega)$ is infinitely differentiable on $(0,1)$; and for all $q\in\mathbb{Z}_+$, the function $(\partial_v^q X)(\cdot,\cdot,\omega)$ is continuous. That is enough to prove that (3.1) holds. □

Corollary 3.2 and the following lemma are the two main ingredients of the proof of Theorem 3.3.

Lemma 3.5. *For each $(\omega,q)\in\Omega^*\times\mathbb{Z}_+$, and for all non-degenerate compact intervals $\mathcal{J}\subset\mathbb{R}$ and $\mathcal{H}\subset(0,1)$, one has*

$$\sup_{(u_1,u_2,v)\in\mathcal{J}^2\times\mathcal{H}}\left\{\frac{|(\partial_v^q X)(u_1,v,\omega)-(\partial_v^q X)(u_2,v,\omega)|}{|u_1-u_2|^v\big(1+\big|\log|u_1-u_2|\big|\big)^{q+\frac{d}{2}}}\right\}<+\infty. \tag{3.8}$$

Proof of Lemma 3.5. First, notice that Theorem 3.1 entails that the lemma holds when X in (3.8) is replaced by X^{lf}. Thus, one only has to prove that the lemma is true when X in (3.8) is replaced by X^{hf}. Using the continuity property of the function $(\partial_v^q X)(\cdot,\cdot,\omega)$ (see Theorem 3.1), one has that

$$\sup_{(u_1,u_2,v)\in\mathcal{K}'}\left\{\frac{\big|\big(\partial_v^q X^{hf}\big)(u_1,v,\omega)-\big(\partial_v^q X^{hf}\big)(u_2,v,\omega)\big|}{|u_1-u_2|^v\big(1+\big|\log|u_1-u_2|\big|\big)^{q+\frac{d}{2}}}\right\}<+\infty, \tag{3.9}$$

where $\mathcal{K}'$ is a compact subset of $\mathbb{R}^2\times(0,1)$ defined as

$$\mathcal{K}'=\big\{(u_1,u_2,v)\in\mathcal{J}^2\times\mathcal{H}:\quad |u_1-u_2|\geq 2^{-1}\big\}.$$

Thus, in order to derive the lemma, it is enough to prove that:

$$\sup_{(u_1,u_2,v)\in\mathcal{K}}\left\{\frac{\big|\big(\partial_v^q X^{hf}\big)(u_1,v,\omega)-\big(\partial_v^q X^{hf}\big)(u_2,v,\omega)\big|}{|u_1-u_2|^v\big(1+\big|\log|u_1-u_2|\big|\big)^{q+\frac{d}{2}}}\right\}<+\infty, \tag{3.10}$$

where $\mathcal{K}$ is a compact subset of $\mathbb{R}^2\times(0,1)$ defined as

$$\mathcal{K}=\big\{(u_1,u_2,v)\in\mathcal{J}^2\times\mathcal{H}:\quad |u_1-u_2|\leq 2^{-1}\big\}.$$

We will show (3.10) for $q=0$; the proof can be done in a rather similar way in the general case where q is an arbitrary nonnegative integer. There is no restriction to assume that $\mathcal{J}=[-\nu,\nu]$ and $\mathcal{H}=[a,b]\subset(0,1)$, where ν and $a<b$ are fixed positive real numbers. Let $(u_1,u_2,v)\in\mathcal{K}$ be arbitrary; there is no restriction to assume that $u_1\neq u_2$ since

(3.10) is clearly satisfied when $u_1 = u_2$. Then, denote by j_0 the biggest nonnegative integer satisfying $|u_1 - u_2| \le 2^{-j_0}$. Observe that $j_0 \ge 1$ and that one has:

$$2^{-(j_0+1)} < |u_1 - u_2| \le 2^{-j_0}, \tag{3.11}$$

which means that

$$j_0 = \left\lfloor \frac{\log\left(|u_1 - u_2|^{-1}\right)}{\log 2} \right\rfloor. \tag{3.12}$$

Notice that one knows from Lemma 2.5 and (2.16) that the function $A_0(\cdot,\cdot,\omega)$ is infinitely many times differentiable on $\mathbb{R}\times(0,1)$, which implies that it satisfies (3.10). This allows to assume that the sum over j in (3.7) starts from $j=1$ instead of $j=0$. Thus, one has that

$$\left|X^{hf}(u_1,v,\omega) - X^{hf}(u_2,v,\omega)\right| \le S_1(u_1,u_2,v,\omega) + S_2(u_1,u_2,v,\omega), \tag{3.13}$$

where

$$S_1(u_1,u_2,v,\omega) = \sum_{j=1}^{j_0} |A_j(u_1,v,\omega) - A_j(u_2,v,\omega)| \tag{3.14}$$

and

$$S_2(u_1,u_2,v,\omega) = \sum_{j=j_0+1}^{+\infty} |A_j(u_1,v,\omega) - A_j(u_2,v,\omega)|. \tag{3.15}$$

In order to derive appropriate upper bounds for $S_1(u_1,u_2,v,\omega)$ and $S_2(u_1,u_2,v,\omega)$, notice that there exists a deterministic positive finite constant c such that:

$$\forall\, x \ge 1, \quad \log(e + x + 2^x d\nu) \le cx. \tag{3.16}$$

Using (3.15), (2.16), the triangle inequality, Lemma 2.5, (3.16), the inequality

$$\forall\,(x,y)\in\mathbb{R}_+^2, \quad (1+x+y)^{\frac{d}{2}} \le (1+x)^{\frac{d}{2}}(1+y)^{\frac{d}{2}},$$

(3.11) and (3.12), one gets:

$$\begin{aligned}
S_2(u_1,u_2,v,\omega) &\le 2\sum_{j=j_0+1}^{+\infty} 2^{-jv} \sup_{(u,v)\in[-\nu,\nu]\times[a,b]} |\Phi_j(2^j\mathbf{u}^*,v,\omega)| \le \\
&\le C_2(\omega)\sum_{j=j_0+1}^{+\infty} 2^{-jv}\big(\log(e+j+2^j d\nu)\big)^{\frac{d}{2}} \le \\
&\le C_2'(\omega)\sum_{j=j_0+1}^{+\infty} 2^{-jv} j^{\frac{d}{2}} \le \\
&\le C_2'(\omega) 2^{-(j_0+1)v}(1+j_0)^{\frac{d}{2}} \sum_{j=0}^{+\infty} 2^{-ja}(1+j)^{\frac{d}{2}} \le \\
&\le C_2''(\omega) 2^{-(j_0+1)v}(1+j_0)^{\frac{d}{2}} \le \\
&\le C_2'''(\omega)|u_1-u_2|^v\big(1+\big|\log|u_1-u_2|\big|\big)^{\frac{d}{2}},
\end{aligned} \tag{3.17}$$

where C_2 is a positive finite random variable not depending on (u_1,u_2,v) derived from Lemma 2.5, $C_2' = C_2 c^{d/2}$, $C_2'' = C_2' \sum_{j=0}^{+\infty} 2^{-ja}(1+j)^{\frac{d}{2}} < +\infty$ and $C_2''' = (\log 2)^{-\frac{d}{2}} C_2''$.

On the other hand, using (3.14), (2.16), the mean value theorem, the triangle inequality, Lemma 2.5, (3.16), (3.11) and (3.12), one gets:

$$S_1(u_1,u_2,v,\omega) \le \sum_{j=1}^{j_0} 2^{j(1-v)}|u_1-u_2| \sum_{i=1}^{d} \sup_{(u,v)\in[-\nu,\nu]\times[a,b]} |\partial_{u_i}\Phi_j(2^j\mathbf{u}^*,v,\omega)| \le$$

$$
\begin{aligned}
&\le C_1(\omega)|u_1-u_2|\sum_{j=1}^{j_0} 2^{j(1-v)}\big(\log(e+j+2^j d\nu)\big)^{\frac{d}{2}} \le\\
&\le C_1'(\omega)|u_1-u_2|\sum_{j=1}^{j_0} 2^{j(1-v)}(j+1)^{\frac{d}{2}} \le\\
&\le C_1''(\omega)|u_1-u_2|(1+j_0)^{\frac{d}{2}}2^{j_0(1-v)} \le\\
&\le C_1'''(\omega)|u_1-u_2|^{v}\big(1+\big|\log|u_1-u_2|\big|\big)^{\frac{d}{2}},
\end{aligned} \tag{3.18}
$$

where C_1 is a positive finite random variable not depending on (u_1,u_2,v) derived from Lemma 2.5, $C_1' = C_1 c^{d/2}$, $C_1'' = 2^{1-a}(2^{1-b}-1)^{-1}C_1'$ and $C_1''' = (\log 2)^{-\frac{d}{2}}C_1''$.

Finally, puting together (3.13), (3.17) and (3.18), one obtains (3.10). □

Proof of Theorem 3.3. For all $(u_1,u_2,v_1,v_2)\in\mathcal{J}^2\times\mathcal{H}^2$, one sets:

$$
f(u_1,u_2,v_1,v_2) = \frac{|(\partial_v^q X)(u_1,v_1,\omega)-(\partial_v^q X)(u_2,v_2,\omega)|}{|u_1-u_2|^{v_1\vee v_2}\big(1+\big|\log|u_1-u_2|\big|\big)^{q+\frac{d}{2}}+|v_1-v_2|},
$$

with the convention that $\frac{0}{0}=0$. Observe that one has:

$$
f(u_1,u_2,v_1,v_2) = f(u_2,u_1,v_2,v_1).
$$

Thus, one gets that:

$$
\sup_{(u_1,u_2,v_1,v_2)\in\mathcal{J}^2\times\mathcal{H}^2} f(u_1,u_2,v_1,v_2) = \sup_{(u_1,u_2,v_1,v_2)\in\mathcal{J}^2\times\mathcal{H}^2} f(u_1,u_2,v_1\vee v_2,v_1\wedge v_2). \tag{3.19}
$$

Moreover, using the triangle inequality, one obtains that:

$$
\begin{aligned}
&\sup_{(u_1,u_2,v_1,v_2)\in\mathcal{J}^2\times\mathcal{H}^2} f(u_1,u_2,v_1\vee v_2,v_1\wedge v_2) \le\\
&\quad\le \sup_{(u_1,u_2,v_1,v_2)\in\mathcal{J}^2\times\mathcal{H}^2}\left\{\frac{|(\partial_v^q X)(u_1,v_1\vee v_2,\omega)-(\partial_v^q X)(u_2,v_1\vee v_2,\omega)|}{|u_1-u_2|^{v_1\vee v_2}\big(1+\big|\log|u_1-u_2|\big|\big)^{q+\frac{d}{2}}+|v_1-v_2|}\right\}+\\
&\quad\quad+\sup_{(u_1,u_2,v_1,v_2)\in\mathcal{J}^2\times\mathcal{H}^2}\left\{\frac{|(\partial_v^q X)(u_2,v_1\vee v_2,\omega)-(\partial_v^q X)(u_2,v_1\wedge v_2,\omega)|}{|u_1-u_2|^{v_1\vee v_2}\big(1+\big|\log|u_1-u_2|\big|\big)^{q+\frac{d}{2}}+|v_1-v_2|}\right\}\le\\
&\quad\le \sup_{(u_1,u_2,v_1,v_2)\in\mathcal{J}^2\times\mathcal{H}^2}\left\{\frac{|(\partial_v^q X)(u_1,v_1\vee v_2,\omega)-(\partial_v^q X)(u_2,v_1\vee v_2,\omega)|}{|u_1-u_2|^{v_1\vee v_2}\big(1+\big|\log|u_1-u_2|\big|\big)^{q+\frac{d}{2}}}\right\}+\\
&\quad\quad+\sup_{(u_1,u_2,v_1,v_2)\in\mathcal{J}^2\times\mathcal{H}^2}\left\{\frac{|(\partial_v^q X)(u_2,v_1\vee v_2,\omega)-(\partial_v^q X)(u_2,v_1\wedge v_2,\omega)|}{|v_1-v_2|}\right\}\le\\
&\quad\le \sup_{(u_1,u_2,v)\in\mathcal{J}^2\times\mathcal{H}}\left\{\frac{|(\partial_v^q X)(u_1,v,\omega)-(\partial_v^q X)(u_2,v,\omega)|}{|u_1-u_2|^{v}\big(1+\big|\log|u_1-u_2|\big|\big)^{q+\frac{d}{2}}}\right\}+\\
&\quad\quad+\sup_{(u,v_1,v_2)\in\mathcal{J}\times\mathcal{H}^2}\left\{\frac{|(\partial_v^q X)(u,v_1,\omega)-(\partial_v^q X)(u,v_2,\omega)|}{|v_1-v_2|}\right\}.
\end{aligned} \tag{3.20}
$$

Finally, putting together (3.19), (3.20), Corollary 3.2 and Lemma 3.5, one gets that

$$
\sup_{(u_1,u_2,v_1,v_2)\in\mathcal{J}^2\times\mathcal{H}^2} f(u_1,u_2,v_1,v_2) < +\infty,
$$

which shows that (3.2) holds. □

Proof of Corollary 3.4. Using (1.7) and Theorem 3.3, in the case where $q = 0$, $\mathcal{J} = \mathcal{L}$ and $\mathcal{H} = \big[\underline{H}(\mathcal{L}), \overline{H}(\mathcal{L})\big]$ (see (3.3)), one obtains, for all $\omega \in \Omega^*$, that:

$$\sup_{(t_1,t_2)\in\mathcal{L}^2} \left\{ \frac{|Z(t_1,\omega) - Z(t_2,\omega)|}{|t_1-t_2|^{H(t_1)\vee H(t_2)}\big(1+\big|\log|t_1-t_2|\big|\big)^{\frac{d}{2}} + |H(t_1)-H(t_2)|} \right\} < +\infty. \quad (3.21)$$

Then, (3.4) and (3.21) imply that (3.5) holds. □

4. Local behavior

First, we state the main results of the section and then we give their proofs.

Theorem 4.1. *Let $u_0 \in \mathbb{R}$ be an arbitrary fixed point. Then, one has, almost surely, for every $q \in \mathbb{Z}_+$ and a non-degenerate compact interval $\mathcal{H} \subset (0,1)$, that:*

$$\sup_{(u,v)\in[u_0-1,u_0+1]\times\mathcal{H}} \left\{ \frac{|(\partial_v^q X)(u,v) - (\partial_v^q X)(u_0,v)|}{|u-u_0|^v\big(1+\big|\log|u-u_0|\big|\big)^q\Big(\log\big(e+\big|\log|u-u_0|\big|\big)\Big)^{\frac{d}{2}}} \right\} < +\infty. \quad (4.1)$$

Corollary 4.2. *Let $t_0 \in \mathbb{R}$ be an arbitrary fixed point. Assume that there exists a constant $\gamma_{t_0} \in (H(t_0),1)$ such that the continuous function $H(\cdot)$ satisfies*

$$\sup_{t\in\mathbb{R}} \left\{ \frac{|H(t)-H(t_0)|}{|t-t_0|^{\gamma_{t_0}}} \right\} < +\infty. \quad (4.2)$$

Then, one has, almost surely:

$$\sup_{t\in[t_0-1,t_0+1]} \left\{ \frac{|Z(t)-Z(t_0)|}{|t-t_0|^{H(t_0)}\Big(\log\big(e+\big|\log|t-t_0|\big|\big)\Big)^{\frac{d}{2}}} \right\} < +\infty. \quad (4.3)$$

The following theorem shows that the chaotic multifractional process $\{Z(t) : t \in \mathbb{R}\}$ has *a local asymptotic self-similarity* property rather similar to the one satisfied by the classical Gaussian multifractional Brownian motion (see [4, 11, 12]).

Theorem 4.3. *Let $t_0 \in \mathbb{R}$ be an arbitrary fixed point such that the condition (4.2) holds. Then, the stochastic process $\{Z(t) : t \in \mathbb{R}\}$ is at t_0, strongly locally asymptotically self-similar of order $H(t_0)$ and the tangent process is $\big\{X(s,H(t_0)) : s \in \mathbb{R}\big\}$. More precisely, let $(\nu_n)_{n\in\mathbb{N}}$ be an arbitrary sequence of positive real numbers which converges to 0. For each $n \in \mathbb{N}$, let $T_{t_0,\nu_n}Z = \big\{(T_{t_0,\nu_n}Z)(s) : s \in \mathbb{R}\big\}$ be a stochastic process with continuous paths, defined, for all $s \in \mathbb{R}$, as*

$$(T_{t_0,\nu_n}Z)(s) = \frac{Z(t_0+\nu_n s) - Z(t_0)}{\nu_n^{H(t_0)}}. \quad (4.4)$$

Then, when n goes to $+\infty$, the probability measure induced on $\mathcal{C}(\mathcal{J})$ by $\big\{(T_{t_0,\nu_n}Z)(s) : s \in \mathbb{R}\big\}$ converges to the one induced on $\mathcal{C}(\mathcal{J})$ by $\big\{X(s,H(t_0)) : s \in \mathbb{R}\big\}$, where $\mathcal{C}(\mathcal{J})$ denotes the usual Banach space of the real-valued continuous functions over an arbitrary non-degenerate compact interval $\mathcal{J}$ of the real line equipped with the uniform norm.

Remark 4.4. One can derive from Theorem 4.3 and the zero-one law that, for any fixed arbitrarily small positive real number η, one has, almost surely,

$$\sup_{t\in[t_0-1,t_0+1]} \left\{ \frac{|Z(t)-Z(t_0)|}{|t-t_0|^{H(t_0)+\eta}} \right\} = +\infty,$$

which means that the exponent $H(t_0)$ in (4.3) is optimal. Moreover, when $\gamma_{\mathcal{L}}$ in (3.3) belongs to $\big[\overline{H}(\mathcal{L}),1\big)$, then, using similar arguments, it can be shown that the exponent $\underline{H}(\mathcal{L})$ in (3.5) is optimal: one has, almost surely,

$$\sup_{(t_1,t_2)\in\mathcal{L}^2}\left\{\frac{|Z(t_1)-Z(t_2)|}{|t_1-t_2|^{\underline{H}(\mathcal{L})+\eta}}\right\}=+\infty\,.$$

Theorem 4.5. *Let $\delta\in(0,+\infty)$ be arbitrary and fixed. One sets $\mathbb{R}_\delta^+=\{u\in\mathbb{R};\ \ |u|\ge\delta\}$. Then, for all $\omega\in\Omega^*$ and non-degenerate compact interval $\mathcal{H}\subset(0,1)$, one has:*

$$\sup_{(u,v)\in\mathbb{R}_\delta^+\times\mathcal{H}}\left\{\frac{\big|X^{lf}(u,v,\omega)\big|}{|u|^v\Big(\log\big(e+\big|\log|u|\big|\big)\Big)^{\frac{d}{2}}}\right\}<+\infty \tag{4.5}$$

and

$$\sup_{(u,v)\in\mathbb{R}_\delta^+\times\mathcal{H}}\left\{\frac{\big|X^{hf}(u,v,\omega)\big|}{\big(\log(e+|u|)\big)^{\frac{d}{2}}}\right\}<+\infty. \tag{4.6}$$

Notice that a straightforward consequence of (4.5), (4.6) and (2.10) is that:

$$\sup_{(u,v)\in\mathbb{R}_\delta^+\times\mathcal{H}}\left\{\frac{|X(u,v,\omega)|}{|u|^v\Big(\log\big(e+\big|\log|u|\big|\big)\Big)^{\frac{d}{2}}}\right\}<+\infty.$$

Corollary 4.6. *Assume that the continuous function $H(\cdot)$ is with values in a compact interval included in $(0,1)$ (this means that $\inf_{t\in\mathbb{R}}H(t)>0$ and $\sup_{t\in\mathbb{R}}H(t)<1$). Then, for each fixed $\omega\in\Omega^*$ and $\delta>0$, one has:*

$$\sup_{|t|\ge\delta}\left\{\frac{|Z(t,\omega)|}{|t|^{H(t)}\Big(\log\big(e+\big|\log|t|\big|\big)\Big)^{\frac{d}{2}}}\right\}<+\infty. \tag{4.7}$$

The following lemma will play a crucial role in the proof of Theorem 4.1.

Lemma 4.7. *For a fixed integer $j\ge1$ and $(\mathbf{u},\theta)\in\mathbb{R}^d\times[1,+\infty)$, let $D_j(\mathbf{u},\theta)$ be a finite non-empty set defined as:*

$$D_j(\mathbf{u},\theta)=\big\{(\epsilon,\mathbf{k})\in E\times\mathbb{Z}^d:\ \|\mathbf{u}-2^{-j}\mathbf{k}\|_1\le d\,j^\theta2^{-j}\big\}. \tag{4.8}$$

Then, for each fixed $(\mathbf{u},\theta)\in\mathbb{R}^d\times[1,+\infty)$ there is a deterministic positive finite constant c_*, only depending on (u,θ,d), such that one has, almost surely:

$$\limsup_{j\to+\infty}\left\{\frac{\max_{(\epsilon,\mathbf{k})\in D_j(\mathbf{u},\theta)}\Big|I_d\Big(\psi_{j,\mathbf{k}}^{(\epsilon)}\Big)\Big|}{\big(\log(2+j)\big)^{\frac{d}{2}}}\right\}\le c_*\,. \tag{4.9}$$

Proof of Lemma 4.7. The lemma can be derived from the Borel–Cantelli Lemma by showing that for some fixed well-chosen deterministic positive finite constant $a\ge2$, one has

$$\sum_{j=1}^{+\infty}\mathsf{P}\bigg(\max_{(\epsilon,\mathbf{k})\in D_j(\mathbf{u},\theta)}\Big|I_d\Big(\psi_{j,\mathbf{k}}^{(\epsilon)}\Big)\Big|>a\sqrt{d!}\big(\log(2+j)\big)^{\frac{d}{2}}\bigg)<+\infty. \tag{4.10}$$

It can easily be seen that, for all $j\ge1$, the following inequality holds:

$$\begin{aligned}\mathsf{P}\bigg(&\max_{(\epsilon,\mathbf{k})\in D_j(\mathbf{u},\theta)}\Big|I_d\Big(\psi_{j,\mathbf{k}}^{(\epsilon)}\Big)\Big|>a\sqrt{d!}\big(\log(2+j)\big)^{\frac{d}{2}}\bigg)\le\\&\le\sum_{(\epsilon,\mathbf{k})\in D_j(\mathbf{u},\theta)}\mathsf{P}\Big(\Big|I_d\Big(\psi_{j,\mathbf{k}}^{(\epsilon)}\Big)\Big|>a\sqrt{d!}\big(\log(2+j)\big)^{\frac{d}{2}}\Big).\end{aligned} \tag{4.11}$$

Let us conveniently bound from above the probabilities in the right-hand side of (4.11). Observe that (1.5) and the equality $\left\|\psi_{j,\mathbf{k}}^{(\epsilon)}\right\|_{L^2(\mathbb{R}^d)} = 1$ imply, for all $(j,\mathbf{k},\epsilon) \in \mathbb{Z}\times\mathbb{Z}^d\times E$, that

$$\mathsf{E}\Big(\Big|I_d\Big(\psi_{j,\mathbf{k}}^{(\epsilon)}\Big)\Big|^2\Big) = d!\Big\|\tilde{\psi}_{j,\mathbf{k}}^{(\epsilon)}\Big\|^2_{L^2(\mathbb{R}^d)} \le d!\Big\|\psi_{j,\mathbf{k}}^{(\epsilon)}\Big\|^2_{L^2(\mathbb{R}^d)} = d!\,. \tag{4.12}$$

Then (4.12) and Theorem 6.7 in [13] entail that, for all real number $\alpha \ge 2$, one has

$$\mathsf{P}\Big(\Big|I_d\Big(\psi_{j,\mathbf{k}}^{(\epsilon)}\Big)\Big| > \alpha\sqrt{d!}\Big) \le \mathsf{P}\Big(|I_d(\psi_{j,\mathbf{k}}^{(\epsilon)})| > \alpha\big\|I_d(\psi_{j,\mathbf{k}}^{(\epsilon)})\big\|_{L^2(\Omega)}\Big) \le \exp\big(-\kappa_d\alpha^{\frac{2}{d}}\big)\,, \tag{4.13}$$

where κ_d is a deterministic positive finite constant only depending on d. Thus, setting in (4.13) $\alpha = a\big(\log(2+j)\big)^{\frac{d}{2}}$, one gets that

$$\mathsf{P}\Big(\Big|I_d\Big(\psi_{j,\mathbf{k}}^{(\epsilon)}\Big)\Big| > a\sqrt{d!}\big(\log(2+j)\big)^{\frac{d}{2}}\Big) \le \exp\Big(-\kappa_d\, a^{\frac{2}{d}}\log(2+j)\Big) = (2+j)^{-\kappa_d a^{\frac{2}{d}}}\,. \tag{4.14}$$

On the other hand, one can easily derive from (4.8) that there exists a deterministic positive finite constant c such that, for all $(\mathbf{u},\theta) \in \mathbb{R}^d \times [1,+\infty)$, one has

$$\mathrm{Card}\,\big(D_j(\mathbf{u},\theta)\big) \le c\, j^{d\theta}. \tag{4.15}$$

Putting together (4.11), (4.14) and (4.15), one obtains, for all $j \ge 1$, that

$$\mathsf{P}\Bigg(\max_{(\epsilon,\mathbf{k})\in D_j(\mathbf{u},\theta)}\Big|I_d\Big(\psi_{j,\mathbf{k}}^{(\epsilon)}\Big)\Big| > a\sqrt{d!}\big(\log(2+j)\big)^{\frac{d}{2}}\Bigg) \le c\, j^{d\theta-\kappa_d a^{2/d}}\,. \tag{4.16}$$

Thus, assuming that the constant a has been chosen big enough so that $d\theta - \kappa_d a^{\frac{2}{d}} < -1$, then it follows from (4.16) that (4.10) holds. □

Proof of Theorem 4.1. Using the same arguments as in the proof of Lemma 3.5, it turns out that in order to derive the statement of the theorem it is enough to show that (4.1) holds when X in it is repalced by X^{hf}, and one can assume that the sums over j in (3.7) and in (2.9) start from $j=1$ instead of $j=0$. Also, for the sake of simplicity one focuses on the case where $q=0$. The proof can be done in a rather similar way in the general case where q is an arbitrary nonnegative integer.

Let us express the compact interval $\mathcal{H}$ as $\mathcal{H} = [a,b]$, where the real numbers a and b are such that $0 < a < b < 1$. Let then $(u,v) \in [u_0-1,u_0+1]\times[a,b] \subset \mathbb{R}\times(0,1)$ be arbitrary and fixed. There is no restriction to assume that $0 < |u-u_0| \le 2^{-15}$, since sample paths of X^{hf} are almost surely continuous functions. One denotes by j_1 the biggest nonnegative integer which satisfies $|u-u_0| \le 2^{-(j_1-1)}$. Then, one has:

$$2^{-j_1} < |u-u_0| \le 2^{-(j_1-1)}\,, \tag{4.17}$$

which means that

$$j_1 = 1 + \left\lfloor \frac{\log\big(|u-u_0|^{-1}\big)}{\log 2}\right\rfloor. \tag{4.18}$$

One sets

$$j_2 = j_1 + \left\lfloor \frac{d\log j_1}{2a\log 2}\right\rfloor. \tag{4.19}$$

Observe that one has $j_1 \ge 16$ and $j_2 \ge j_1+2$. Moreover, for any $j \in \{j_1+1,\dots,j_2\}$, the following inequality holds:

$$j^{\frac{d}{a}} \ge 2^{j-j_1+2}. \tag{4.20}$$

Next, for all integer $j \ge 1$, let $\mathcal{D}_j(u_0)$ be a finite non-empty set defined as $\mathcal{D}_j(u_0) = D_j(\mathbf{u}_0^*, \frac{d}{a})$, where, as usual, $\mathbf{u}_0^*$ denotes the vector of $\mathbb{R}^d$ whose coordinates are all equal to the real number u_0, and $D_j(\mathbf{u}_0^*, \frac{d}{a})$ is defined through (4.8) with $\mathbf{u} = \mathbf{u}_0^*$ and $\theta = \frac{d}{a}$. That is:

$$\mathcal{D}_j(u_0) = \Big\{(\epsilon, \mathbf{k}) \in E \times \mathbb{Z}^d : \|\mathbf{u}_0^* - 2^{-j}\mathbf{k}\|_1 \le d\, j^{\frac{d}{a}} 2^{-j}\Big\}. \tag{4.21}$$

Then, Lemma 4.7 entails that one has almost surely, for all integer $j \ge 1$,

$$\max_{(\epsilon,\mathbf{k})\in\mathcal{D}_j(u_0)} \Big| I_d\Big(\psi_{j,\mathbf{k}}^{(\epsilon)}\Big)\Big| \le C\big(\log(2+j)\big)^{\frac{d}{2}}, \tag{4.22}$$

where C is a positive almost surely finite random variable not depending on j. Next, one denotes by $\mathcal{D}_j^{co}(u_0)$ the complement of $\mathcal{D}_j(u_0)$ in $E \times \mathbb{Z}^d$, that is:

$$\mathcal{D}_j^{co}(u_0) = \Big\{(\epsilon, \mathbf{k}) \in E \times \mathbb{Z}^d : \|\mathbf{u}_0^* - 2^{-j}\mathbf{k}\|_1 > d\, j^{\frac{d}{a}} 2^{-j}\Big\}. \tag{4.23}$$

Let us mention in passing that

$$\mathcal{D}_j^{co}(u_0) \subset \bigcup_{l=1}^{d} \Big\{(\epsilon, \mathbf{k}) \in E \times \mathbb{Z}^d : |u_0 - 2^{-j}k_l| > j^{\frac{d}{a}} 2^{-j}\Big\}, \tag{4.24}$$

where k_l is the l-th coordinate of $\mathbf{k}$. One can derive from (2.9) (where the sum over j is assumed to start from $j = 1$ instead of $j = 0$), (4.21), (4.23) and the triangle inequality that

$$\big|X^{hf}(u,v) - X^{hf}(u_0,v)\big| \le R_1(u,u_0,v) + R_2(u,u_0,v) + R_3(u,u_0,v) + R_4(u,u_0,v), \tag{4.25}$$

where

$$R_1(u,u_0,v) = \sum_{j=1}^{j_1} \sum_{(\epsilon,\mathbf{k})\in\mathcal{D}_j(u_0)} 2^{-jv}\Big|I_d\Big(\psi_{j,\mathbf{k}}^{(\epsilon)}\Big)\Big|\Big|\Psi^\epsilon\big(2^j\mathbf{u}^* - \mathbf{k}, v\big) - \Psi^\epsilon\big(2^j\mathbf{u}_0^* - \mathbf{k}, v\big)\Big|, \tag{4.26}$$

$$R_2(u,u_0,v) = \sum_{j=j_1+1}^{+\infty} \sum_{(\epsilon,\mathbf{k})\in\mathcal{D}_j(u_0)} 2^{-jv}\Big|I_d\Big(\psi_{j,\mathbf{k}}^{(\epsilon)}\Big)\Big|\Big|\Psi^\epsilon\big(2^j\mathbf{u}^* - \mathbf{k}, v\big) - \Psi^\epsilon\big(2^j\mathbf{u}_0^* - \mathbf{k}, v\big)\Big|, \tag{4.27}$$

$$R_3(u,u_0,v) = \sum_{j=1}^{j_2} \sum_{(\epsilon,\mathbf{k})\in\mathcal{D}_j^{co}(u_0)} 2^{-jv}\Big|I_d\Big(\psi_{j,\mathbf{k}}^{(\epsilon)}\Big)\Big|\Big|\Psi^\epsilon\big(2^j\mathbf{u}^* - \mathbf{k}, v\big) - \Psi^\epsilon\big(2^j\mathbf{u}_0^* - \mathbf{k}, v\big)\Big|, \tag{4.28}$$

and

$$R_4(u,u_0,v) = \sum_{j=j_2+1}^{+\infty} \sum_{(\epsilon,\mathbf{k})\in\mathcal{D}_j^{co}(u_0)} 2^{-jv}\Big|I_d\Big(\psi_{j,\mathbf{k}}^{(\epsilon)}\Big)\Big|\Big|\Psi^\epsilon\big(2^j\mathbf{u}^* - \mathbf{k}, v\big) - \Psi^\epsilon\big(2^j\mathbf{u}_0^* - \mathbf{k}, v\big)\Big|. \tag{4.29}$$

From now on, our goal is to derive an appropriate upper bound for each term in the right-hand side of (4.25). In all the sequel, one assumes that L is an arbitrary large fixed positive integer. Therefore, one has

$$c := \sup_{y\in\mathbb{R}} \left\{ \sum_{\mathbf{k}\in\mathbb{Z}^d} \frac{\big(\log(e + \|\mathbf{y}^* - \mathbf{k}\|_2)\big)^{\frac{d}{2}}}{\big(1 + \|\mathbf{y}^* - \mathbf{k}\|_2\big)^L} \right\} < +\infty. \tag{4.30}$$

Putting together (4.27), (4.22), (2.4), (4.30), (2.14), (4.17) and (4.18), one obtains that almost surely:

$$
\begin{aligned}
R_2(u,u_0,v) &\le K_2' \sum_{j=j_1+1}^{+\infty} 2^{-jv}\big(\log(2+j)\big)^{\frac{d}{2}} \le \\
&\le K_2' \sum_{j=0}^{+\infty} 2^{-(j+j_1+1)v}\big(\log(2+j+j_1+1)\big)^{\frac{d}{2}} \le \\
&\le K_2' 2^{-(j_1+1)v}\big(\log(e+j_1+1)\big)^{\frac{d}{2}} \sum_{j=0}^{+\infty} 2^{-ja}\big(\log(e+j)\big)^{\frac{d}{2}} \le \\
&\le K_2|u-u_0|^v\Big(\log\big(e+\big|\log|u-u_0|\big|\big)\Big)^{\frac{d}{2}}, \qquad (4.31)
\end{aligned}
$$

where K_2' and K_2 are two positive almost surely finite random variables not depending on (u,v).

Next, using (4.29), (2.11), the fact that $|u|\le|u_0|+1$, the inequality $2^j>j$ for all $j\in\mathbb{Z}_+$, (2.14), (4.17), (4.18) and (4.19), one gets, on the event of probability 1 Ω^*, that:

$$
\begin{aligned}
R_4(u,u_0,v) &\le K_4' \sum_{j=j_2+1}^{+\infty} 2^{-jv}\Big(\log\big(e+(|u_0|+d+1)2^j\big)\Big)^{\frac{d}{2}} \le \\
&\le K_4'' 2^{-(j_2+1)v}(j_2+1)^{\frac{d}{2}} \sum_{j=0}^{+\infty} 2^{-ja}\Big(\log\big(e+(|u_0|+d+1)2^j\big)\Big)^{\frac{d}{2}} \le \\
&\le K_4''' 2^{-(j_2+1)v}(j_2+1)^{\frac{d}{2}} = \\
&= K_4''' 2^{-j_1 v}\exp\left(-(v\log 2)\left(1+\left\lfloor\frac{d\log j_1}{2a\log 2}\right\rfloor\right)+\frac{d\log(j_2+1)}{2}\right) \le \\
&\le K_4'''|u-u_0|^v \exp\left(\frac{d}{2}\log\left(\frac{j_2+1}{j_1}\right)\right) \le \\
&\le K_4|u-u_0|^v, \qquad (4.32)
\end{aligned}
$$

where K_4', K_4'' and K_4''' are three positive finite random variables not depending on (u,v), and where $K_4 = K_4'''\big(2+\frac{d}{2a\log 2}\big)^{\frac{d}{2}}$.

Next, observe that using the mean value theorem, it can be shown that, for all fixed $(\epsilon,j,\mathbf{k})\in E\times\mathbb{N}\times\mathbb{Z}^d$, there exists a real number $\lambda_{j,\mathbf{k}}^{\epsilon}(u,u_0)\in(0,1)$ such that:

$$
\Psi^{\epsilon}(2^j\mathbf{u}^*-\mathbf{k},v)-\Psi^{\epsilon}(2^j\mathbf{u}_0^*-\mathbf{k},v) = 2^j(u-u_0)\sum_{n=1}^{d}(\partial_{y_n}\Psi^{\epsilon})\big(2^j\mathbf{u}_0^*+\lambda_{j,\mathbf{k}}^{\epsilon}(u,u_0)2^j(\mathbf{u}^*-\mathbf{u}_0^*)\big). \qquad (4.33)
$$

Then, combining (4.26), (4.33), (4.22), (2.4), (4.30), (4.17) and (4.18), one gets almost surely that

$$
\begin{aligned}
R_1(u,u_0,v) &\le K_1'|u-u_0| \sum_{j=1}^{j_1} 2^{j(1-v)}\big(\log(2+j)\big)^{\frac{d}{2}} \le \\
&\le K_1'|u-u_0|\big(\log(2+j_1)\big)^{\frac{d}{2}} \sum_{j=1}^{j_1} 2^{(j_1-j+1)(1-v)} \le \\
&\le K_1'|u-u_0|2^{(j_1+1)(1-v)}\big(\log(2+j_1)\big)^{\frac{d}{2}} \sum_{j=1}^{+\infty} 2^{-j(1-b)} \le
\end{aligned}
$$

$$\leq K_1|u-u_0|^v\Big(\log\big(2+\big|\log|u-u_0|\big|\big)\Big)^{\frac{d}{2}}\,, \tag{4.34}$$

where K_1' and K_1 are two positive almost surely finite random variables not depending on (u,v).

It only remains to obtain a convenient upper bound for $R_3(u,u_0,v)$. Notice that, using the equivalence of all norms on $\mathbb{R}^d$, one deduces from (2.4) that:

$$\forall\,(n,\mathbf{p},q)\in\mathbb{N}\times\mathbb{Z}^d\times\mathbb{Z},\quad \sup\big\{\big(\alpha+\|\mathbf{u}\|_1\big)^n\,|\partial_{\mathbf{u}}^{\mathbf{p}}\partial_v^q\Psi^{\epsilon}(\mathbf{u},v)|:\mathbf{u}\in\mathbb{R}^d,v\in[a,b]\big\}<+\infty\,, \tag{4.35}$$

where α is an abritrary positive real number. Next, combining (4.28), (4.33) and (4.35), one obtains that

$$R_3(u,u_0,v)\leq \kappa_3|u-u_0|\times$$
$$\times\sum_{j=1}^{j_2}\sum_{(\epsilon,\mathbf{k})\in\mathcal{D}_j^{co}(u_0)}2^{j(1-v)}\frac{\Big|I_d\Big(\psi_{j,\mathbf{k}}^{(\epsilon)}\Big)\Big|}{\big(2d+1+\|2^j\mathbf{u}_0^*-\mathbf{k}+\lambda_{j,\mathbf{k}}^{\epsilon}(u,u_0)2^j(\mathbf{u}^*-\mathbf{u}_0^*)\|_1\big)^L}, \tag{4.36}$$

where κ_3 denotes a positive finite and deterministic constant. Observe that using the triangle inequality, (4.17) and the fact that $\lambda_{j,\mathbf{k}}^{\epsilon}(u,u_0)\in(0,1)$ one has, for all $j\in\{1,\ldots,j_1\}$ and $(\epsilon,\mathbf{k})\in\mathcal{D}_j^{co}(u_0)$, that

$$\begin{aligned}2d+1+\|2^j\mathbf{u}_0^*-\mathbf{k}+\lambda_{j,\mathbf{k}}^{\epsilon}(u,u_0)2^j(\mathbf{u}^*-\mathbf{u}_0^*)\|_1\geq\\ \geq\|2^j\mathbf{u}_0^*-\mathbf{k}\|_1-\lambda_{j,\mathbf{k}}^{\epsilon}(u,u_0)\,d\,2^j|u-u_0|+2d+1\geq\\ \geq 1+\|2^j\mathbf{u}_0^*-\mathbf{k}\|_1.\end{aligned} \tag{4.37}$$

Also observe that using the triangle inequality, the fact that $\lambda_{j,\mathbf{k}}^{\epsilon}(u,u_0)\in(0,1)$, (4.17), (4.23) and (4.20), one obtains, for all $j\in\{j_1+1,\ldots,j_2\}$ and all $(\epsilon,\mathbf{k})\in\mathcal{D}_j^{co}(u_0)$, that

$$\begin{aligned}2d+1+\|2^j\mathbf{u}_0^*-\mathbf{k}+\lambda_{j,\mathbf{k}}^{\epsilon}(u,u_0)2^j(\mathbf{u}^*-\mathbf{u}_0^*)\|_1\geq\\ \geq\|2^j\mathbf{u}_0^*-\mathbf{k}\|_1-\lambda_{j,\mathbf{k}}^{\epsilon}(u,u_0)\,d\,2^j|u-u_0|+2d+1\geq\\ \geq\frac{1}{2}\|2^j\mathbf{u}_0^*-\mathbf{k}\|_1+\frac{1}{2}\|2^j\mathbf{u}_0^*-\mathbf{k}\|_1-d\,2^{j-j_1+1}+2d+1\geq\\ \geq\frac{1}{2}\|2^j\mathbf{u}_0^*-\mathbf{k}\|_1+\frac{d}{2}j^{\frac{d}{a}}-d\,2^{j-j_1+1}+2d+1\geq\\ \geq\frac{1}{2}(1+\|2^j\mathbf{u}_0^*-\mathbf{k}\|_1).\end{aligned} \tag{4.38}$$

Thus, combining (4.37), (4.38) and (4.36), one gets that

$$R_3(u,u_0,v)\leq 2\kappa_3|u-u_0|\sum_{j=1}^{j_2}\sum_{(\epsilon,\mathbf{k})\in\mathcal{D}_j^{co}(u_0)}2^{j(1-v)}\frac{\Big|I_d\Big(\psi_{j,\mathbf{k}}^{(\epsilon)}\Big)\Big|}{\big(1+\|2^j\mathbf{u}_0^*-\mathbf{k}\|_1\big)^L}\,. \tag{4.39}$$

Next, using (4.39), Lemma 2.1, the inequalities

$$\begin{aligned}\log\big(e+j+\|\mathbf{k}\|_1\big)\leq\log\big(e+j+\|2^j\mathbf{u}_0^*\|_1+\|2^j\mathbf{u}_0^*-\mathbf{k}\|_1\big)\leq\\ \leq\log\big(e+j+\|2^j\mathbf{u}_0^*\|_1\big)\prod_{l=1}^{d}\log\big(e+|2^ju_0-k_l|\big)\,,\end{aligned}$$

$$\big(1+\|2^j\mathbf{u}_0^*-\mathbf{k}\|_1\big)^L\geq\prod_{l=1}^{d}\big(1+|2^ju_0-k_l|\big)^{\frac{L}{d}}\,,$$

(4.24), and (4.30) with $\mathbb{Z}$ in place of $\mathbb{Z}^d$, it turns out that, on the event of probability 1 Ω^*, one has

$$R_3(u,u_0,v) \le K_3'|u-u_0|\sum_{j=1}^{j_2} j^{\frac{d}{2}}\, 2^{j(1-v)} \sum_{k\in\mathcal{D}_{1,j}^{co}(u_0)} \frac{\big(\log(e+|2^j u_0-k|)\big)^{\frac{d}{2}}}{\big(1+|2^j u_0-k|\big)^{\frac{L}{d}}}\,, \tag{4.40}$$

where K_3' is a positive finite random variable not depending on (u,v), and where

$$\mathcal{D}_{1,j}^{co}(u_0) = \Big\{k\in\mathbb{Z} : |2^j u_0 - k| > j^{\frac{d}{a}}\Big\}\,. \tag{4.41}$$

Next, let us assume that η is an arbitrarily small fixed positive real number. Using (4.41), and the fact that $x\mapsto \log(e+x)$ and $x\mapsto x$ are increasing functions over $\mathbb{R}_+$, one gets that

$$\sum_{k\in\mathcal{D}_{1,j}^{co}(u_0)} \frac{\big(\log(e+|2^j u_0-k|)\big)^{\frac{d}{2}}}{\big(1+|2^j u_0-k|\big)^{\frac{L}{d}}} \le 2\int_{j^{\frac{d}{a}}}^{+\infty} \frac{\big(\log(e+1+x)\big)^{\frac{d}{2}}}{x^{\frac{L}{d}}}\,dx \le \kappa_3'\, j^{-(\frac{L-d}{a}-\eta)}\,, \tag{4.42}$$

where κ_3' is a positive finite deterministic constant not depending on j. Moreover the assumption that L is an arbitrarily large integer allows to assume that

$$\frac{L-d}{a} - \eta - \frac{d}{2} > \frac{d}{2a} > 0\,. \tag{4.43}$$

Thus, using the fact that $v\in[a,b]\subset(0,1)$, one obtains that

$$\begin{aligned}\sum_{j=1}^{j_2} j^{-(\frac{L-d}{a}-\eta-\frac{d}{2})}\, 2^{j(1-v)} &\le \sum_{j=1}^{\lfloor j_2/2\rfloor} 2^{j(1-v)} + \big(j_2/2\big)^{-(\frac{L-d}{a}-\eta-\frac{d}{2})} \sum_{\lfloor j_2/2\rfloor+1}^{j_2} 2^{j(1-v)} \le \\ &\le 4\big(2^{1-b}-1\big)^{-1}\Big(2^{j_2(1-v)/2} + \big(j_2/2\big)^{-(\frac{L-d}{a}-\eta-\frac{d}{2})}\, 2^{j_2(1-v)}\Big) \le \\ &\le \kappa_3''\, j_2^{-(\frac{L-d}{a}-\eta-\frac{d}{2})}\, 2^{j_2(1-v)}\,,\end{aligned} \tag{4.44}$$

where the finite deterministic constant

$$\kappa_3'' = 4\big(2^{1-b}-1\big)^{-1}\Big(2^{(\frac{L-d}{a}-\eta-\frac{d}{2})} + \sup_{n\in\mathbb{N}}\Big\{2^{-n(1-b)/2}\, n^{(\frac{L-d}{a}-\eta-\frac{d}{2})}\Big\}\Big)\,.$$

Moreover, one can derive from (4.18), (4.19) and (4.43) that

$$\begin{aligned} j_2^{-(\frac{L-d}{a}-\eta-\frac{d}{2})}\, 2^{j_2(1-v)} &\le 4|u-u_0|^{v-1}\, j_2^{-(\frac{L-d}{a}-\eta-\frac{d}{2})}\, 2^{d\log(j_1)/(2a\log 2)} \le \\ &\le 4|u-u_0|^{v-1}\, j_2^{-(\frac{L-d}{a}-\eta-\frac{d}{2}-\frac{d}{2a})} \le 4|u-u_0|^{v-1}\,. \end{aligned} \tag{4.45}$$

Next, putting together (4.40), (4.42), (4.44) and (4.45), it turns out that, on the event of probability 1 Ω^*, one has

$$R_3(u,u_0,v) \le K_3|u-u_0|^v\,, \tag{4.46}$$

where K_3 is a positive finite random variable not depending on (u,v).

Finally, combining (4.25), (4.31), (4.32), (4.34) and (4.46), one obtains the statement of the theorem. □

Proof of Corollary 4.2. Using (1.7) and the triangle inequality, one gets that

$$\sup_{t\in[t_0-1,t_0+1]} \left\{ \frac{|Z(t)-Z(t_0)|}{|t-t_0|^{H(t_0)}\big(\log\big(e+\big|\log|t-t_0|\big|\big)\big)^{\frac{d}{2}}} \right\} \le U_1(t_0)+U_2(t_0), \tag{4.47}$$

where

$$U_1(t_0) = \sup_{t\in[t_0-1,t_0+1]} \left\{ \frac{|X(t,H(t)) - X(t,H(t_0))|}{|t-t_0|^{H(t_0)} \big(\log\big(e + \big|\log|t-t_0|\big|\big)\big)^{\frac{d}{2}}} \right\} \tag{4.48}$$

and

$$U_2(t_0) = \sup_{t\in[t_0-1,t_0+1]} \left\{ \frac{|X(t,H(t_0)) - X(t_0,H(t_0))|}{|t-t_0|^{H(t_0)} \big(\log\big(e + \big|\log|t-t_0|\big|\big)\big)^{\frac{d}{2}}} \right\}. \tag{4.49}$$

Next, observe that it follows from the assumption (4.2) that

$$\sup_{t\in[t_0-1,t_0+1]} \left\{ \frac{|H(t) - H(t_0)|}{|t-t_0|^{H(t_0)} \big(\log\big(e + \big|\log|t-t_0|\big|\big)\big)^{\frac{d}{2}}} \right\} < +\infty\,. \tag{4.50}$$

On the other hand, denoting by $\mathcal{H}$ the compact interval included in $(0,1)$ defined as

$$\mathcal{H} = H([t_0-1,t_0+1]) = \big\{H(t) : t\in[t_0-1,t_0+1]\big\}\,,$$

one clearly has that

$$\sup_{t\in[t_0-1,t_0+1]} \left\{ \frac{|X(t,H(t)) - X(t,H(t_0))|}{|H(t)-H(t_0)|} \right\} \le$$

$$\le \sup_{(u,v_1,v_2)\in[t_0-1,t_0+1]\times\mathcal{H}^2} \left\{ \frac{|X(u,v_1) - X(u,v_2)|}{|v_1-v_2|} \right\}. \tag{4.51}$$

Thus, combining (4.48), (4.50) and (4.51) with Corollary 3.2, one almost surely gets that:

$$U_1(t_0) < +\infty. \tag{4.52}$$

On the other hand, it easily follows from (4.49) and Theorem 4.1 that:

$$U_2(t_0) \le \sup_{(u,v)\in[t_0-1,t_0+1]\times\mathcal{H}} \left\{ \frac{|X(u,v) - X(u_0,v)|}{|u-u_0|^{v} \big(\log\big(e + \big|\log|u-u_0|\big|\big)\big)^{\frac{d}{2}}} \right\} < +\infty. \tag{4.53}$$

Finally, putting together (4.52), (4.53) and (4.47), one obtains (4.3). □

Proof of Theorem 4.3. It easily follows from (4.4) and (1.7) that, for every $n\in\mathbb{N}$, the stochastic process $T_{t_0,\nu_n}Z = \big\{(T_{t_0,\nu_n}Z)(s) : s\in\mathbb{R}\big\}$ can be expressed as the sum of two stochastic processes $T^1_{t_0,\nu_n}X = \big\{(T^1_{t_0,\nu_n}X)(s) : s\in\mathbb{R}\big\}$ and $T^2_{t_0,\nu_n}X = \big\{(T^2_{t_0,\nu_n}X)(s) : s\in\mathbb{R}\big\}$, defined, for all $s\in\mathbb{R}$, as:

$$\big(T^1_{t_0,\nu_n}X\big)(s) = \frac{X\big(t_0+\nu_n s, H(t_0)\big) - X\big(t_0, H(t_0)\big)}{\nu_n^{H(t_0)}} \tag{4.54}$$

and

$$\big(T^2_{t_0,\nu_n}X\big)(s) = \frac{X\big(t_0+\nu_n s, H(t_0+\nu_n s)\big) - X\big(t_0+\nu_n s, H(t_0)\big)}{\nu_n^{H(t_0)}}\,. \tag{4.55}$$

Next, using (4.54), the stationary increments property of the stochastic process

$$X(\cdot, H(t_0)) = \big\{X(u,H(t_0)) : u\in\mathbb{R}\big\}$$

(see (1.1)), its global self-similar property of order $H(t_0)$ (see (1.2)), and the equality $X(0,H(t_0)) \overset{\text{a.s.}}{=} 0$, one gets that

$$\big\{(T^1_{t_0,\nu_n}X)(s) : s\in\mathbb{R}\big\} \overset{\text{law}}{=} \big\{X(s,H(t_0)) : s\in\mathbb{R}\big\}\,.$$

This equatility in the sense of finite-dimensional distributions and the fact that the processes $\big\{(T^1_{t_0,\nu_n}X)(s) : s\in\mathbb{R}\big\}$ and $\big\{X(s,H(t_0)) : s\in\mathbb{R}\big\}$ have continuous paths imply that these two processes induce the same probability distribution on the space of continuous functions $\mathcal{C}(\mathcal{J})$. Thus, in order to derive the statement of the theorem, it is enough to show that $T^2_{t_0,\nu_n}X$, viewed as a random variable with values in the space $\mathcal{C}(\mathcal{J})$, converges to 0 in this space, when n goes to $+\infty$. That is,

$$\lim_{n\to+\infty}\sup_{s\in\mathcal{J}}\big|(T^2_{t_0,\nu_n}X)(s)\big| \overset{\text{a.s.}}{=} 0\,. \tag{4.56}$$

There is no rectriction to assume that $\mathcal{J}=[-M,M]$ for some fixed positive real number M, and that $\nu_n\in(0,1]$, for every $n\in\mathbb{N}$. Let then $\mathcal{I}$ and $\mathcal{H}$ be compact intervals defined as $\mathcal{I}=[t_0-M,t_0+M]$ and $\mathcal{H}=H(\mathcal{I})=\big\{H(t):t\in\mathcal{I}\big\}$. It follows from Corollary 3.2 that the positive random variable A defined as

$$A=\sup_{(u,v_1,v_2)\in\mathcal{I}\times\mathcal{H}^2}\left\{\frac{|X(u,v_1)-X(u,v_2)|}{|v_1-v_2|}\right\} \tag{4.57}$$

is finite on the event of probability 1 Ω^*. Moreover, one can derive from (4.55) and (4.57) that on Ω^*, for all $n\in\mathbb{N}$, one has

$$\sup_{s\in\mathcal{J}}\big|(T^2_{t_0,\nu_n}X)(s)\big|\le\nu_n^{-H(t_0)}A\sup_{s\in\mathcal{J}}\big\{\big|H(t_0+\nu_n s)-H(t_0)\big|\big\}\,. \tag{4.58}$$

Finally, combining (4.2) and (4.58) one obtains (4.56). □

Proof of Theorem 4.5. In view of the fact that on the event of probability 1 Ω^* the fields X^{lf} and X^{hf} are with continuous paths, one can assume without any restriction that $\delta=2$. Let $\mathcal{H}=[a,b]\subset(0,1)$ be an arbitrary compact interval and let u be an arbitrary real number such that $|u|\ge 2$. One denotes by j_3 the biggest positive integer satisfying: $|u|\ge 2^{j_3}$. Then, one gets that:

$$2^{j_3}\le|u|<2^{j_3+1}\,, \tag{4.59}$$

which means that

$$j_3=\left\lfloor\frac{\log|u|}{\log 2}\right\rfloor. \tag{4.60}$$

First, one shows that (4.5) holds. Recall that, for all $(v,\omega)\in(0,1)\times\Omega^*$, one has:

$$X^{lf}(u,v,\omega)=\sum_{j=1}^{+\infty}2^{jv}\big(\Phi_{-j}(2^{-j}\mathbf{u}^*,v,\omega)-\Phi_{-j}(\mathbf{0},v,\omega)\big)\,, \tag{4.61}$$

where $\Phi_{-j}(\cdot,\cdot,\omega)$ is the infinitely differentiable function on $\mathbb{R}^d\times(0,1)$, introduced in (2.12). Also, recall that the series in (4.61) is uniformly convergent in (u,v) on each compact subset of $\mathbb{R}\times(0,1)$. Using the mean value theorem, one gets, for all $(j,v)\in\mathbb{N}\times\mathcal{H}$, that:

$$|\Phi_{-j}(2^{-j}\mathbf{u}^*,v,\omega)-\Phi_{-j}(\mathbf{0},v,\omega)|\le 2^{-j}|u|\sum_{n=1}^{d}\sup_{(y,v)\in[0\wedge 2^{-j}u,0\vee 2^{-j}u]\times\mathcal{H}}|\partial_{y_n}\Phi_{-j}(\mathbf{y}^*,v,\omega)|.$$

Then, Lemma 2.5 entails that:

$$|\Phi_{-j}(2^{-j}\mathbf{u}^*,v,\omega)-\Phi_{-j}(\mathbf{0},v,\omega)|\le C_d(\omega)2^{-j}|u|\big(\log(e+j+2^{-j}|u|)\big)^{\frac{d}{2}}\,, \tag{4.62}$$

where C_d is a positive finite random variable not depending on u. Thus, one can derive from (4.62), (2.14), (4.59) and (4.60) that:

$$\sum_{j=j_3+1}^{+\infty}2^{jv}|\Phi_{-j}(2^{-j}\mathbf{u}^*,v,\omega)-\Phi_{-j}(\mathbf{0},v,\omega)|\le$$

$$\le C_d(\omega)|u|\sum_{j=j_3+1}^{+\infty}2^{-j(1-v)}\big(\log(e+1+j)\big)^{\frac{d}{2}}\le$$

$$\le C_d(\omega)|u|2^{-(j_3+1)(1-v)}\sum_{l=0}^{+\infty}2^{-l(1-v)}\big(\log(e+2+l+j_3)\big)^{\frac{d}{2}}\le$$
$$\le C'_d(\omega)2^{j_3 v}\big(\log(e+j_3)\big)^{\frac{d}{2}}\le$$
$$\le C''_d(\omega)|u|^v\big(\log\big(e+\big|\log|u|\big|\big)\big)^{\frac{d}{2}}, \tag{4.63}$$

where

$$C'_d=2^b\,C_d\sum_{l=0}^{+\infty}2^{-l(1-b)}\big(\log(e+2+l)\big)^{\frac{d}{2}}<+\infty,$$

and $C''_d=C'_d(\log(e+\frac{1}{\log 2}))^{\frac{d}{2}}$. On the other hand, Lemma 2.5, (4.59), (2.14), and (4.60) imply that:

$$\sum_{j=1}^{j_3}2^{jv}|\Phi_{-j}(2^{-j}\mathbf{u}^*,v,\omega)-\Phi_{-j}(\mathbf{0},v,\omega)|\le$$
$$\le\widetilde{C}_d(\omega)\sum_{j=1}^{j_3}2^{jv}\big(\log(e+j+2^{-j}|u|)\big)^{\frac{d}{2}}=$$
$$=\widetilde{C}_d(\omega)2^{(j_3+1)v}\sum_{j=1}^{j_3}2^{-jv}\Big(\log\Big(e+(j_3+1-j)+2^{-(j_3+1-j)}|u|\Big)\Big)^{\frac{d}{2}}\le$$
$$\le\widetilde{C}'_d(\omega)|u|^v\sum_{j=1}^{j_3}2^{-jv}\big(\log(e+j_3+2^j)\big)^{\frac{d}{2}}\le$$
$$\le\widetilde{C}''_d(\omega)|u|^v\big(\log(e+j_3)\big)^{\frac{d}{2}}\le$$
$$\le\widetilde{C}'''_d(\omega)|u|^v\big(\log\big(e+\big|\log|u|\big|\big)\big)^{\frac{d}{2}}, \tag{4.64}$$

where $\widetilde{C}_d$ is a positive finite random variable not depending on u, $\widetilde{C}'_d=2^b\widetilde{C}_d$,

$$\widetilde{C}''_d=\widetilde{C}'_d\sum_{l=1}^{+\infty}2^{-la}\big(\log\big(e+2^l\big)\big)^{\frac{d}{2}},$$

and $\widetilde{C}'''_d=(\log(e+\frac{1}{\log 2}))^{\frac{d}{2}}\widetilde{C}''_d$. Finally, (4.63) and (4.64) entail that (4.5) holds.

Now, let us show that (4.6) is satisfied. Recall that, for all $(v,\omega)\in(0,1)\times\Omega^*$, one has:

$$X^{hf}(u,v,\omega)=\sum_{j=0}^{+\infty}2^{-jv}\big(\Phi_j(2^j\mathbf{u}^*,v,\omega)-\Phi_j(\mathbf{0},v,\omega)\big), \tag{4.65}$$

where $\Phi_j(\cdot,\cdot,\omega)$ is the infinitely differentiable function on $\mathbb{R}^d\times(0,1)$, introduced in (2.12). Also, recall that the series in (4.65) is uniformly convergent in (u,v) on each compact subset of $\mathbb{R}\times(0,1)$. Next, let us mention that thanks to the convexity property of the function $z\mapsto z^{\frac{d}{2}}$, one has the following inequaty:

$$\forall\,(x,y)\in\mathbb{R}^2_+,\quad (x+y)^{\frac{d}{2}}\le 2^{\frac{d}{2}-1}\Big(x^{\frac{d}{2}}+y^{\frac{d}{2}}\Big). \tag{4.66}$$

Also, one mentions that the inequality

$$\forall\,(x,y,z)\in\mathbb{R}^3_+,\quad \log(e+x+yz)\le\log(e+x+y)+\log(e+z) \tag{4.67}$$

holds, since $(e+x+yz) \leq (e+x+y)(e+z)$ and the logarithm is an increasing function. Using (4.65), the triangle inequality, Lemma 2.5, (4.67) and (4.66), one obtains:

$$\begin{aligned}|X^{hf}(u,v,\omega)| &\leq \widehat{C}_d(\omega)\sum_{j=0}^{+\infty} 2^{-ja}\big[\log(e+j+2^j|u|)\big]^{\frac{d}{2}} \leq \\ &\leq \widehat{C}_d(\omega)\sum_{j=0}^{+\infty} 2^{-ja}\Big[\log(e+|u|)+\log(e+j+2^j)\Big]^{\frac{d}{2}} \leq \\ &\leq \widehat{C}'_d(\omega)\sum_{j=0}^{+\infty} 2^{-ja}\Big[\big(\log(e+|u|)\big)^{\frac{d}{2}}+\big(\log(e+j+2^j)\big)^{\frac{d}{2}}\Big] \leq \\ &\leq \widehat{C}''_d(\omega)\big(\log(e+|u|)\big)^{\frac{d}{2}}+\widehat{C}'''_d(\omega),\end{aligned} \tag{4.68}$$

where $\widehat{C}_d$ is a positive finite random variable not depending on u,

$$\widehat{C}'_d = 2^{\frac{d}{2}-1}\,\widehat{C}_d, \quad \widehat{C}''_d = \frac{2^a}{2^a-1}\widehat{C}'_d, \quad \text{and} \quad \widehat{C}'''_d = \widehat{C}'_d\sum_{j=0}^{+\infty} 2^{-ja}\big(\log(e+j+2^j)\big)^{\frac{d}{2}} < +\infty\,.$$

It easily results from (4.68) that (4.6) holds. □

Proof of Corollary 4.6. The corollary is a straightforward consequence of Theorem 4.5 and (1.7). □

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UMR CNRS 8524, Laboratoire Paul Painlevé, Université de Lille, Cité Scientifique, Bâtiment M2, 59655 Villeneuve d'Ascq, France
E-mail address: `Antoine.Ayache@univ-lille.fr`

UMR CNRS 8524, Laboratoire Paul Painlevé, Université de Lille, Cité Scientifique, Bâtiment M2, 59655 Villeneuve d'Ascq, France
E-mail address: `Yassine.Esmili@univ-lille.fr`

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ВЕЙВЛЕТ-АНАЛІЗ МУЛЬТИДРОБОВОГО ПРОЦЕСУ У ДОВІЛЬНОМУ ВІНЕРІВСЬКОМУ ХАОСІ

А. АЙЯШ, Я. ЕСМІЛІ

Анотація. Відомий процес мультидробового броунівського руху (мбр) є зразковим прикладом неперервного гауссівського процесу з нестаціонарними приростами, локальна регулярність якого змінюється від точки до точки. У статті за допомогою вейвлет-підходу побудовано природне узагальнення мбр, що належить до однорідного вінерівського хаосу довільного порядку. Вивчається його глобальна та локальна поведінка.

ВЕЙВЛЕТ-АНАЛИЗ МУЛЬТИДРОБНОГО ПРОЦЕССА В ПРОИЗВОЛЬНОМ ВИНЕРОВСКОМ ХАОСЕ

А. АЙЯШ, Я. ЭСМИЛИ

Аннотация. Известный процесс мультидробного броуновского движения (мбд) является образцовым примером непрерывного гауссовского процесса с нестационарными приращениями, локальная регулярность которого изменяется от точки к точке. В статье с помощью вейвлет-подхода построено естественное обобщение мбд, которое принадлежит однородному винеровскому хаосу произвольного порядка. Изучается его глобальное и локальное поведение.