Теорія ймовірностей
та математична статистика
Вип. 2(99)/2018, с. 37–50

Teoriya Ĭmovirnosteĭ
ta Matematychna Statystyka
No. 2(99)/2018, pp. 37–50

UDC 519.21

ON THE PRODUCT OF A SINGULAR WISHART MATRIX AND A SINGULAR GAUSSIAN VECTOR IN HIGH DIMENSION

T. BODNAR, S. MAZUR, S. MUHINYUZA, N. PAROLYA

Abstract. In this paper we consider the product of a singular Wishart random matrix and a singular normal random vector. A very useful stochastic representation of this product is derived, using which its characteristic function and asymptotic distribution under the double asymptotic regime are established. We further document a good finite sample performance of the obtained high-dimensional asymptotic distribution via an extensive Monte Carlo study.

Key words and phrases. Singular Wishart distribution, singular normal distribution, stochastic representation, high-dimensional asymptotics.

2010 *Mathematics Subject Classification.* Primary: 60E05, 60E10, 60F05, 62H10, 62E20.

1. Introduction

The multivariate normal distribution is one of the basic distributions in probability theory and a building block in multivariate statistical analysis. It is also used as a standard assumption in many applications where the normal distribution is usually accompanied by the Wishart distribution. For instance, when we consider a sample of size n from a k-dimensional normal distribution, then the unbiased estimators for the mean vector and covariance matrix have a k-dimensional normal distribution and a k-dimensional Wishart distribution, respectively. Moreover, they are independent (see e. g. [16, Chapter 3]).

A number of papers deal either with the properties of the sample mean vector or with the properties of the sample covariance matrix, although these two random objects often appear together in the expressions of statistics. Consequently, a question arises how the distributions of functions involving both a Wishart matrix and a normal vector can be characterised. Recently, this topic has attracted a lot of attention in the literature from both the theoretical perspectives (cf. [3, 6]) and the applications (see e. g. [2, 12, 13]). While [6, 15] derived the exact distribution and the approximative distribution of the product of an inverse Wishart matrix and a normal vector, [3] presented similar results for the product of a Wishart matrix and a normal vector. The product of an inverse Wishart matrix and a normal vector has direct applications in discriminant analysis (cf. [19]) and in portfolio theory (see e. g. [7]), whereas the product of a Wishart matrix and a normal vector arises in Bayesian statistics when the aim is to infer the coefficients of the discriminant function or the optimal portfolio weights by employing the inverse Wishart - normal prior which is a conjugate prior for the mean vector and the covariance matrix under normality (see e. g. [1]).

Singular covariance matrix is present in practical applications as well, especially when data generating process is large-dimensional. For example, the construction of an optimal portfolio with a singular covariance matrix has become an important topic in finance (see e. g. [4, 17]). While the normal distribution with the singular covariance matrix is known as the singular normal distribution in statistical literature, there is no unique definition in the case of the Wishart distribution. The singular Wishart distribution introduced by [14] and [20] deals with the case when the number of degrees of freedom is smaller than the process dimension. Its practical relevance was discussed in [22], while some theoretical findings were derived in [5, 21]. Another type of the singular Wishart distributions, the so-called pseudo-Wishart distribution, was defined in [8] where a model with a singular covariance matrix was proposed. The latter stochastic model is considered in the present paper.

We contribute to the existent literature by deriving a stochastic representation for the product of a singular Wishart matrix and a normal vector, which provides an elegant way of characterising the finite sample distribution of the product. Also, it appears to be very useful in the derivation of the asymptotic distribution under the high-dimensional asymptotic regime, i.e. when both the sample size and the process dimension become very large.

The rest of the paper is structured as follows. Section 2 contains several distributional properties of the singular Wishart distribution which are used as a tool to prove the main results of the paper presented in Section 3. Here, the distribution of the product of a singular Wishart matrix and a singular normal random vector is derived in terms of a stochastic representation from which we also obtain the characteristic function of the product. Furthermore, we prove the asymptotic normality of the product under the high-dimensional asymptotic regime. The finite sample performance of the obtained asymptotic results is discussed in Section 4, while Section 5 presents the summary.

2. Preliminary results

We start this section with the formal definition of the singular normal distribution and singular Wishart distribution.

Definition 1. A random vector $\mathbf{z}$ is said to have a singular normal distribution with mean vector μ and covariance matrix $\mathbf{\Sigma}$ if its characteristic function is given by

$$\varphi_{\mathbf{z}}(\mathbf{u}) = \exp\left(i\mu^T\mathbf{u} - \frac{1}{2}\mathbf{u}^T\mathbf{\Sigma}\mathbf{u}\right),$$

where $\mathbf{\Sigma}$ is a positive semi-definite matrix with $\operatorname{rank}(\mathbf{\Sigma}) = r < k$. We denote this distribution by $\mathbf{z} \sim \mathcal{N}_k(\mu, \mathbf{\Sigma})$.

Definition 2. Let $\mathbf{z}_1, ..., \mathbf{z}_n$ be independent and identically distributed where $\mathbf{z}_i$ is singular normal with zero mean vector and covariance matrix $\mathbf{\Sigma}$, $\operatorname{rank}(\mathbf{\Sigma}) = r < k$, and let $\mathbf{Z} = [\mathbf{z}_1, ..., \mathbf{z}_n]$. Then the random matrix $\mathbf{A} = \mathbf{Z}\mathbf{Z}^T$ has a singular Wishart distribution with n degrees of freedom and covariance matrix $\mathbf{\Sigma}$. We denote this distribution by $\mathbf{A} \sim \mathcal{W}_k(n, \mathbf{\Sigma})$.

Throughout the paper, no assumption is made about the relationship between the degrees of freedom n and the dimension k. The results are valid in both cases $n \geq k$ (the Wishart distribution with positive semi-definite covariance matrix $\mathbf{\Sigma}$) and $k < n$ (the singular Wishart distribution with positive semi-definite covariance matrix $\mathbf{\Sigma}$). Also, we use the symbol $\mathbf{I}_k$ to denote the $k \times k$ identity matrix, $\otimes$ is the Kronecker product, and the symbol $\stackrel{d}{=}$ stands for the equality in distribution.

Next, we present several distributional properties of the singular Wishart distribution which are used in proving the main results of the paper. In Proposition 1, we derive the distribution of a linear symmetric transformation of the singular Wishart random matrix.

Proposition 1. *Let* $\mathbf{A} \sim \mathcal{W}_k(n, \mathbf{\Sigma})$ *with* $\operatorname{rank}(\mathbf{\Sigma}) = r < k$ *and let* $\mathbf{M} : p \times k$ *be a matrix of constants with* $\operatorname{rank}(\mathbf{M}) = p$ *such that* $\mathbf{M}\mathbf{\Sigma} \neq \mathbf{0}$*. Then*

$$\mathbf{M}\mathbf{A}\mathbf{M}^T \sim \mathcal{W}_p(n, \mathbf{M}\mathbf{\Sigma}\mathbf{M}^T).$$

Moreover, if $\operatorname{rank}(\mathbf{M}\mathbf{\Sigma}) = p \leq r$*, then* $\mathbf{M}\mathbf{A}\mathbf{M}^T$ *and* $\mathbf{M}\mathbf{\Sigma}\mathbf{M}^T$ *are of the full rank* p*.*

Proof. From Theorem 5.2 of [21], we have that the stochastic representation of $\mathbf{A}$ is given by

$$\mathbf{A} \stackrel{d}{=} \mathbf{X}\mathbf{X}^T \quad \text{with} \quad \mathbf{X} \sim \mathcal{N}_{k,n}(\mathbf{0}, \mathbf{\Sigma} \otimes \mathbf{I}_n).$$

Then, using Theorem 2.4.2 of [10], we get

$$\mathbf{M}\mathbf{A}\mathbf{M}^T \stackrel{d}{=} \mathbf{M}\mathbf{X}\mathbf{X}^T\mathbf{M}^T \stackrel{d}{=} \mathbf{Y}\mathbf{Y}^T,$$

where $\mathbf{Y} \sim \mathcal{N}_{p,n}(\mathbf{0}, (\mathbf{M}\mathbf{\Sigma}\mathbf{M}^T) \otimes \mathbf{I}_n)$. This completes the proof of the proposition. □

An application of Proposition 1 leads to the following result summarized in Proposition 2.

Proposition 2. *Let* $\mathbf{A} \sim \mathcal{W}_k(n, \mathbf{\Sigma})$ *with* $\operatorname{rank}(\mathbf{\Sigma}) = r < k$ *and let* $\mathbf{W} : p \times k$ *be a random matrix which is independent of* $\mathbf{A}$ *such that* $\operatorname{rank}(\mathbf{W}\mathbf{\Sigma}) = p \le r \le n$ *with probability one. Then*

$$(\mathbf{W}\mathbf{\Sigma}\mathbf{W}^T)^{-1/2}(\mathbf{W}\mathbf{A}\mathbf{W}^T)(\mathbf{W}\mathbf{\Sigma}\mathbf{W}^T)^{-1/2} \sim \mathcal{W}_p(n, \mathbf{I}_p)$$

and is independent of $\mathbf{W}$.

Proof. Using the fact that $\mathbf{W}$ and $\mathbf{A}$ are independently distributed, we obtain that the conditional distribution of $\mathbf{W}\mathbf{A}\mathbf{W}^T|(\mathbf{W} = \mathbf{W}_0)$ is equal to the distribution of $\mathbf{W}_0\mathbf{A}\mathbf{W}_0^T$. Then, applying Proposition 1, we obtain

$$(\mathbf{W}_0\mathbf{\Sigma}\mathbf{W}_0^T)^{-1/2}(\mathbf{W}_0\mathbf{A}\mathbf{W}_0^T)(\mathbf{W}_0\mathbf{\Sigma}\mathbf{W}_0^T)^{-1/2} \sim \mathcal{W}_p(n, \mathbf{I}_p).$$

Since this distribution does not depend on $\mathbf{W}$, it is also the unconditional distribution of $(\mathbf{W}\mathbf{\Sigma}\mathbf{W}^T)^{-1/2}(\mathbf{W}\mathbf{A}\mathbf{W}^T)(\mathbf{W}\mathbf{\Sigma}\mathbf{W}^T)^{-1/2}$. The proposition is proved. □

In the next corollary, we consider a special case of Proposition 2 with $p = 1$.

Corollary 1. *Let* $\mathbf{A} \sim \mathcal{W}_k(n, \mathbf{\Sigma})$ *with* $\operatorname{rank}(\mathbf{\Sigma}) = r \le k$ *and let* $\mathbf{w}$ *be a* k*-dimensional vector which is independent of* $\mathbf{A}$ *with* $P(\mathbf{w}^T\mathbf{\Sigma} = \mathbf{0}) = 0$. *Then*

$$\frac{\mathbf{w}^T\mathbf{A}\mathbf{w}}{\mathbf{w}^T\mathbf{\Sigma}\mathbf{w}} \sim \chi^2_n,$$

and is independent of $\mathbf{w}$.

3. Main results

In this section, we present the main results of the paper which are complementary to the ones obtained in [3] to the case of high-dimensional data and singular covariance matrix.

3.1. Finite sample results.

Let $\mathbf{z}$ be a k-dimensional singular normally distributed random vector with mean vector μ and covariance matrix $\kappa\mathbf{\Sigma}$, $\kappa > 0$, such that $\operatorname{rank}(\mathbf{\Sigma}) = r < k$, i.e. $\mathbf{z} \sim \mathcal{N}_k(\mu, \kappa\mathbf{\Sigma})$. Also, let $\mathbf{M}$ be a $p \times k$ matrix of constants with $\operatorname{rank}(\mathbf{M}) = p \le r \le \min\{n, k\}$ such that $\mathbf{M}\mathbf{\Sigma} \neq \mathbf{0}$. We are interested in the distribution of $\mathbf{M}\mathbf{A}\mathbf{z}$, when $\mathbf{A}$ and $\mathbf{z}$ are independently distributed where $\mathbf{A}$ has a singular Wishart distribution as defined in Section 2.

In Theorem 1, we derive a stochastic representation for $\mathbf{M}\mathbf{A}\mathbf{z}$. The stochastic representation is a tool in the theory of multivariate statistics and it is frequently used in Monte Carlo simulations (cf. [9]). Its importance in the theory of elliptically contoured distributions is well described by [11].

Theorem 1. *Let* $\mathbf{A} \sim \mathcal{W}_k(n, \mathbf{\Sigma})$ *with* $\operatorname{rank}(\mathbf{\Sigma}) = r < k$ *and let* $\mathbf{z} \sim \mathcal{N}_k(\mu, \kappa\mathbf{\Sigma})$*,* $\kappa > 0$*. We assume that* $\mathbf{A}$ *and* $\mathbf{z}$ *are independently distributed. Also, let* $\mathbf{M} : p \times k$ *be a matrix of constants of rank* $p < r \leq n$ *and denote* $\mathbf{Q} = \mathbf{P}^T\mathbf{P}$ *with* $\mathbf{P} = (\mathbf{M\Sigma M}^T)^{-1/2}\mathbf{M\Sigma}^{1/2}$*. Then the stochastic representation of* $\mathbf{MAz}$ *is given by*

$$\mathbf{MAz} \stackrel{d}{=} \zeta\mathbf{M\Sigma}^{1/2}\mathbf{t} + \sqrt{\zeta}(\mathbf{M\Sigma M}^T)^{1/2}\left[\sqrt{\mathbf{t}^T\mathbf{t}}\mathbf{I}_p - \frac{\sqrt{\mathbf{t}^T\mathbf{t}} - \sqrt{\mathbf{t}^T(\mathbf{I}_k - \mathbf{Q})\mathbf{t}}}{\mathbf{t}^T\mathbf{Q}\mathbf{t}}\mathbf{P}\mathbf{t}\mathbf{t}^T\mathbf{P}^T\right]\mathbf{z}_0,$$

where $\zeta \sim \chi^2_n$*,* $\mathbf{t} \sim \mathcal{N}_k(\mathbf{\Sigma}^{1/2}\mu, \kappa\mathbf{\Sigma}^2)$*, and* $\mathbf{z}_0 \sim \mathcal{N}_p(\mathbf{0}, \mathbf{I}_p)$*;* ζ*,* $\mathbf{t}$*, and* $\mathbf{z}_0$ *are mutually independent.*

Proof. Since $\mathbf{A}$ and $\mathbf{z}$ are independently distributed, it holds that the conditional distribution of $\mathbf{MAz}|(\mathbf{z} = \mathbf{z}^*)$ is equal to the distribution of $\mathbf{MAz}^*$.

Let $\widetilde{\mathbf{M}}$ be the matrix which is obtained from $\mathbf{M}$ by adding a row vector $\mathbf{z}^*$, i.e. $\widetilde{\mathbf{M}} = (\mathbf{M}^T, \mathbf{z}^*)^T$. Consider the following two partitioned matrices

$$\widetilde{\mathbf{A}} = \widetilde{\mathbf{M}}\mathbf{A}\widetilde{\mathbf{M}}^T = \begin{pmatrix} \mathbf{MAM}^T & \mathbf{MAz}^* \\ \mathbf{z}^{*T}\mathbf{AM}^T & \mathbf{z}^{*T}\mathbf{Az}^* \end{pmatrix} = \begin{pmatrix} \widetilde{\mathbf{A}}_{11} & \widetilde{\mathbf{A}}_{12} \\ \widetilde{\mathbf{A}}_{21} & \widetilde{A}_{22} \end{pmatrix}$$

and

$$\widetilde{\mathbf{\Sigma}} = \widetilde{\mathbf{M}}\mathbf{\Sigma}\widetilde{\mathbf{M}}^T = \begin{pmatrix} \mathbf{M\Sigma M}^T & \mathbf{M\Sigma z}^* \\ \mathbf{z}^{*T}\mathbf{\Sigma M}^T & \mathbf{z}^{*T}\mathbf{\Sigma z}^* \end{pmatrix} = \begin{pmatrix} \widetilde{\mathbf{\Sigma}}_{11} & \widetilde{\mathbf{\Sigma}}_{12} \\ \widetilde{\mathbf{\Sigma}}_{21} & \widetilde{\Sigma}_{22} \end{pmatrix}.$$

Since $\mathbf{A} \sim \mathcal{W}_k(n, \mathbf{\Sigma})$ and $\operatorname{rank}(\widetilde{\mathbf{M}}) = p + 1 \leq r$, following Proposition 1, it holds that $\widetilde{\mathbf{A}} \sim \mathcal{W}_{p+1}(n, \widetilde{\mathbf{\Sigma}})$. Using Theorem 3.2.10 of [16], we get the conditional distribution of $\widetilde{\mathbf{A}}_{12} = \mathbf{MAz}^*$ given $\widetilde{A}_{22}$ can be expressed as

$$\widetilde{\mathbf{A}}_{12}|\widetilde{A}_{22} \sim \mathcal{N}_p\left(\widetilde{\mathbf{\Sigma}}_{12}\widetilde{\Sigma}_{22}^{-1}\widetilde{A}_{22}, \widetilde{\mathbf{\Sigma}}_{11\cdot 2}\widetilde{A}_{22}\right)$$

with $\widetilde{\mathbf{\Sigma}}_{11\cdot 2} = \widetilde{\mathbf{\Sigma}}_{11} - \widetilde{\mathbf{\Sigma}}_{12}\widetilde{\Sigma}_{22}^{-1}\widetilde{\mathbf{\Sigma}}_{21}$.

Let $\zeta = \widetilde{A}_{22}\widetilde{\Sigma}_{22}^{-1}$. Then, from Corollary 1, we get that $\zeta \sim \chi^2_n$, and it is independent of $\mathbf{z}$. Hence,

$$\mathbf{MAz}|\zeta, \mathbf{z} \sim \mathcal{N}_p\Big(\zeta\mathbf{M\Sigma z}, \zeta(\mathbf{z}^T\mathbf{\Sigma z M\Sigma M}^T - \mathbf{M\Sigma z z}^T\mathbf{\Sigma M}^T)\Big),$$

which leads to the stochastic representation of $\mathbf{MAz}$ given by

$$\mathbf{MAz} \stackrel{d}{=} \zeta\mathbf{M\Sigma z} + \sqrt{\zeta}(\mathbf{z}^T\mathbf{\Sigma z M\Sigma M}^T - \mathbf{M\Sigma z z}^T\mathbf{\Sigma M}^T)^{1/2}\mathbf{z}_0, \tag{1}$$

where $\zeta \sim \chi^2_n$, $\mathbf{z} \sim \mathcal{N}_k(\mu, \kappa\mathbf{\Sigma})$, and $\mathbf{z}_0 \sim \mathcal{N}_p(\mathbf{0}, \mathbf{I}_p)$. Moreover, ζ, $\mathbf{z}$, and $\mathbf{z}_0$ are mutually independent.

Next, we calculate the square root of $(\mathbf{z}^T\mathbf{\Sigma z M\Sigma M}^T - \mathbf{M\Sigma z z}^T\mathbf{\Sigma M}^T)$ using the following equality

$$(\mathbf{D} - \mathbf{b}\mathbf{b}^T)^{1/2} = \mathbf{D}^{1/2}(\mathbf{I}_p - c\mathbf{D}^{-1/2}\mathbf{b}\mathbf{b}^T\mathbf{D}^{-1/2})$$

with $c = \frac{1-\sqrt{1-\mathbf{b}^T\mathbf{D}^{-1}\mathbf{b}}}{\mathbf{b}^T\mathbf{D}^{-1}\mathbf{b}}$, $\mathbf{b} = \mathbf{M\Sigma z}$, and $\mathbf{D} = \mathbf{z}^T\mathbf{\Sigma z M\Sigma M}^T$ that leads to

$$\mathbf{MAz} \stackrel{d}{=} \zeta\mathbf{M\Sigma z} + \sqrt{\zeta}(\mathbf{M\Sigma M}^T)^{1/2} \times$$
$$\times \left[\sqrt{\mathbf{z}^T\mathbf{\Sigma z}}\mathbf{I}_p - \frac{\sqrt{\mathbf{z}^T\mathbf{\Sigma z}} - \sqrt{\mathbf{z}^T(\mathbf{\Sigma} - \mathbf{\Sigma}^{1/2}\mathbf{Q}\mathbf{\Sigma}^{1/2})\mathbf{z}}}{\mathbf{z}^T\mathbf{\Sigma}^{1/2}\mathbf{Q}\mathbf{\Sigma}^{1/2}\mathbf{z}}\mathbf{P}\mathbf{\Sigma}^{1/2}\mathbf{z}\mathbf{z}^T\mathbf{\Sigma}^{1/2}\mathbf{P}^T\right]\mathbf{z}_0,$$

where $\mathbf{P} = (\mathbf{M\Sigma M}^T)^{-1/2}\mathbf{M\Sigma}^{1/2}$ and $\mathbf{Q} = \mathbf{P}^T\mathbf{P}$.

Finally, making the transformation $\mathbf{t} = \mathbf{\Sigma}^{1/2}\mathbf{z} \sim \mathcal{N}_k(\mathbf{\Sigma}^{1/2}\mu, \kappa\mathbf{\Sigma}^2)$, we obtain the statement of the theorem. $\square$

Next, we consider the special case of Theorem 1 when $p = 1$ and $\mathbf{M} = \mathbf{m}^T$.

Corollary 2. *Let* $\mathbf{A} \sim \mathcal{W}_k(n, \mathbf{\Sigma})$ *with* $\operatorname{rank}(\mathbf{\Sigma}) = r < k$ *and let* $\mathbf{z} \sim \mathcal{N}_k(\mu, \kappa\mathbf{\Sigma})$, $\kappa > 0$. *We assume that* $\mathbf{A}$ *and* $\mathbf{z}$ *are independently distributed. Let* $\mathbf{m}$ *be a* k*-dimensional vector of constants such that* $\mathbf{m}^T\mathbf{\Sigma}\mathbf{m} > 0$. *Then the stochastic representation of* $\mathbf{m}^T\mathbf{A}\mathbf{z}$ *is given by*

$$\mathbf{m}^T\mathbf{A}\mathbf{z} \stackrel{d}{=} \zeta\mathbf{m}^T\mathbf{\Sigma}\mathbf{z} + \sqrt{\zeta}\big[\mathbf{z}^T\mathbf{\Sigma}\mathbf{z}\cdot\mathbf{m}^T\mathbf{\Sigma}\mathbf{m} - (\mathbf{m}^T\mathbf{\Sigma}\mathbf{z})^2\big]^{1/2} z_0, \tag{2}$$

where $\zeta \sim \chi^2_n$ *and* $z_0 \sim \mathcal{N}(0,1)$; ζ, z_0, *and* $\mathbf{z}$ *are mutually independent.*

The proof of Corollary 2 follows directly from (1). The result of the corollary is very useful from the viewpoint of computational statistics. Namely, in order to get a realization of $\mathbf{m}^T\mathbf{A}\mathbf{z}$ it is sufficient to simulate two random variables from the standard univariate distributions together with a random vector which has a singular multivariate normal distribution. There is no need to generate a large-dimensional object $\mathbf{A}$ and, as a result, the application of (2) speeds up the simulations where the product of $\mathbf{A}$ and $\mathbf{z}$ is present.

Another application of Corollary 2 leads to the expression of the characteristic function of $\mathbf{A}\mathbf{z}$ presented in the following theorem.

Theorem 2. *Let* $\mathbf{A} \sim \mathcal{W}_k(n, \mathbf{\Sigma})$ *with* $\operatorname{rank}(\mathbf{\Sigma}) = r < k$ *and let* $\mathbf{z} \sim \mathcal{N}_k(\mu, \kappa\mathbf{\Sigma})$. *We assume that* $\mathbf{A}$ *and* $\mathbf{z}$ *are independently distributed. Then the characteristic function of* $\mathbf{A}\mathbf{z}$ *is given by*

$$\varphi_{\mathbf{A}\mathbf{z}}(\mathbf{u}) = \frac{\exp\left(-\frac{\kappa^{-1}}{2}\mu^T\mathbf{R}\mathbf{\Lambda}^{-1}\mathbf{R}^T\mu\right)}{\kappa^{r/2}|\mathbf{\Lambda}|^{1/2}} \int_0^\infty |\mathbf{\Omega}(\zeta)|^{-1/2} f_{\chi^2_n}(\zeta) \times$$
$$\times \exp\left(i\zeta\nu(\zeta)^T\mathbf{\Lambda}\mathbf{R}^T\mathbf{u} - \frac{\zeta^2}{2}\mathbf{u}^T\mathbf{R}\mathbf{\Lambda}\mathbf{\Omega}(\zeta)^{-1}\mathbf{\Lambda}\mathbf{R}^T\mathbf{u} + \frac{1}{2}\nu(\zeta)^T\mathbf{\Omega}(\zeta)\nu(\zeta)\right) d\zeta,$$

where $\nu(\zeta) = \kappa^{-1}\mathbf{\Omega}(\zeta)^{-1}\mathbf{\Lambda}^{-1}\mathbf{R}^T\mu$,

$$\mathbf{\Omega}(\zeta) = \kappa^{-1}\mathbf{\Lambda}^{-1} + \zeta\big[\mathbf{\Lambda}\cdot\mathbf{u}^T\mathbf{\Sigma}\mathbf{u} - \mathbf{\Lambda}\mathbf{R}^T\mathbf{u}\mathbf{u}^T\mathbf{R}\mathbf{\Lambda}\big],$$

and $\mathbf{\Sigma} = \mathbf{R}\mathbf{\Lambda}\mathbf{R}^T$ *is the singular value decomposition of* $\mathbf{\Sigma}$ *with diagonal matrix* $\mathbf{\Lambda}$ *consisting of all* r *non-zero eigenvalues of* $\mathbf{\Sigma}$ *and the* $k \times r$ *matrix* $\mathbf{R}$ *of the corresponding eigenvectors;* $f_{\chi^2_n}$ *denotes the density function of the* χ^2 *distribution with* n *degrees of freedom.*

Proof. From the stochastic representation derived in Corollary 2, we get that

$$\begin{aligned}
\varphi_{\mathbf{A}\mathbf{z}}(\mathbf{u}) &= \mathbb{E}\big(\exp(i\mathbf{u}^T\mathbf{A}\mathbf{z})\big) = \\
&= \mathbb{E}\Big(\exp\Big(i\zeta\mathbf{u}^T\mathbf{\Sigma}\mathbf{z} + i\sqrt{\zeta}\big[\mathbf{z}^T\mathbf{\Sigma}\mathbf{z}\cdot\mathbf{u}^T\mathbf{\Sigma}\mathbf{u} - (\mathbf{u}^T\mathbf{\Sigma}\mathbf{z})^2\big]^{1/2} z_0\Big)\Big) = \\
&= \mathbb{E}\Big(\exp(i\zeta\mathbf{u}^T\mathbf{\Sigma}\mathbf{z})\mathbb{E}\Big(\exp\Big(i\sqrt{\zeta}\big[\mathbf{z}^T\mathbf{\Sigma}\mathbf{z}\cdot\mathbf{u}^T\mathbf{\Sigma}\mathbf{u} - (\mathbf{u}^T\mathbf{\Sigma}\mathbf{z})^2\big]^{1/2} z_0\Big)|\zeta,\mathbf{z}\Big)\Big) = \\
&= \mathbb{E}\left(\exp(i\zeta\mathbf{u}^T\mathbf{\Sigma}\mathbf{z})\exp\left(-\frac{1}{2}\zeta\big[\mathbf{z}^T\mathbf{\Sigma}\mathbf{z}\cdot\mathbf{u}^T\mathbf{\Sigma}\mathbf{u} - (\mathbf{u}^T\mathbf{\Sigma}\mathbf{z})^2\big]\right)\right) = \\
&= \mathbb{E}\left(\mathbb{E}\left(\exp(i\zeta\mathbf{u}^T\mathbf{\Sigma}\mathbf{z})\exp\left(-\frac{1}{2}\zeta\big[\mathbf{z}^T\mathbf{\Sigma}\mathbf{z}\cdot\mathbf{u}^T\mathbf{\Sigma}\mathbf{u} - (\mathbf{u}^T\mathbf{\Sigma}\mathbf{z})^2\big]\right)|\zeta\right)\right) = \\
&= \mathbb{E}\left(\mathbb{E}\left(\exp(i\zeta\mathbf{v}^T\mathbf{\Lambda}\mathbf{y})\exp\left(-\frac{1}{2}\zeta\big[\mathbf{y}^T\mathbf{\Lambda}\mathbf{y}\cdot\mathbf{v}^T\mathbf{\Lambda}\mathbf{v} - (\mathbf{v}^T\mathbf{\Lambda}\mathbf{y})^2\big]\right)|\zeta\right)\right),
\end{aligned}$$

where $\mathbf{v} = \mathbf{R}^T\mathbf{u}$; $\mathbf{\Sigma} = \mathbf{R}\mathbf{\Lambda}\mathbf{R}^T$ is the singular value decomposition of $\mathbf{\Sigma}$; $\mathbf{y} = \mathbf{R}^T\mathbf{z} \sim$ $\sim \mathcal{N}_r(\mathbf{R}^T\mu, \kappa\mathbf{\Lambda})$ has a non-singular multivariate normal distribution.

Hence,

$$\mathbb{E}\left(\exp(i\zeta\mathbf{v}^T\mathbf{\Lambda}\mathbf{y})\exp\left(-\frac{1}{2}\zeta\big[\mathbf{y}^T\mathbf{\Lambda}\mathbf{y}\cdot\mathbf{v}^T\mathbf{\Lambda}\mathbf{v} - (\mathbf{v}^T\mathbf{\Lambda}\mathbf{y})^2\big]\right)|\zeta\right) =$$

$$= \frac{1}{(2\pi\kappa)^{r/2}|\mathbf{\Lambda}|^{1/2}} \int_{\mathbb{R}^r} \exp\big(i\zeta \mathbf{v}^T\mathbf{\Lambda}\mathbf{y}\big) \exp\bigg(-\frac{1}{2}\zeta\big[\mathbf{y}^T\mathbf{\Lambda}\mathbf{y}\cdot\mathbf{v}^T\mathbf{\Lambda}\mathbf{v} - (\mathbf{v}^T\mathbf{\Lambda}\mathbf{y})^2\big]\bigg) \times$$
$$\times \exp\bigg(-\frac{\kappa^{-1}}{2}(\mathbf{y}-\mathbf{R}^T\mu)^T\mathbf{\Lambda}^{-1}(\mathbf{y}-\mathbf{R}^T\mu)\bigg)\mathrm{d}\mathbf{y}$$

where

$$\kappa^{-1}(\mathbf{y}-\mathbf{R}^T\mu)^T\mathbf{\Lambda}^{-1}(\mathbf{y}-\mathbf{R}^T\mu) + \zeta\big[\mathbf{y}^T\mathbf{\Lambda}\mathbf{y}\cdot\mathbf{v}^T\mathbf{\Lambda}\mathbf{v} - (\mathbf{v}^T\mathbf{\Lambda}\mathbf{y})^2\big] =$$
$$= (\mathbf{y}-\nu(\zeta))^T\mathbf{\Omega}(\zeta)(\mathbf{y}-\nu(\zeta)) + d$$

with

$$\mathbf{\Omega}(\zeta) = \kappa^{-1}\mathbf{\Lambda}^{-1} + \zeta\big[\mathbf{\Lambda}\cdot\mathbf{v}^T\mathbf{\Lambda}\mathbf{v} - \mathbf{\Lambda}\mathbf{v}\mathbf{v}^T\mathbf{\Lambda}\big],$$
$$\nu(\zeta) = \kappa^{-1}\mathbf{\Omega}(\zeta)^{-1}\mathbf{\Lambda}^{-1}\mathbf{R}^T\mu,$$
$$d = \kappa^{-1}\mu^T\mathbf{R}\mathbf{\Lambda}^{-1}\mathbf{R}^T\mu - \nu(\zeta)^T\mathbf{\Omega}(\zeta)\nu(\zeta) = \kappa^{-1}\mu^T\mathbf{\Sigma}^+\mu - \nu(\zeta)^T\mathbf{\Omega}(\zeta)\nu(\zeta),$$

and $\mathbf{\Sigma}^+$ the Moore–Penrose inverse.

As a result, we get

$$\varphi_{\mathbf{Az}}(\mathbf{u}) = \frac{\exp\Big(-\frac{\kappa^{-1}}{2}\mu^T\mathbf{\Sigma}^+\mu\Big)}{\kappa^{r/2}|\mathbf{\Lambda}|^{1/2}} \int_0^\infty |\mathbf{\Omega}(\zeta)|^{-1/2} f_{\chi^2_n}(\zeta) \times$$
$$\times \exp\bigg(i\zeta\nu(\zeta)^T\mathbf{\Lambda}\mathbf{v} - \frac{\zeta^2}{2}\mathbf{v}^T\mathbf{\Lambda}\mathbf{\Omega}(\zeta)^{-1}\mathbf{\Lambda}\mathbf{v} + \frac{1}{2}\nu(\zeta)^T\mathbf{\Omega}(\zeta)\nu(\zeta)\bigg)\mathrm{d}\zeta.$$

This completes the proof of the theorem. □

3.2. **Asymptotic distribution under double asymptotic regime.** In this section we derive the asymptotic distribution of $\mathbf{MAz}$ under double asymptotic regime, i.e. when both r and n tend to infinity such that $r/n \to c \in [0, +\infty)$. In the derivation of the asymptotic distribution we rely on the results of Corollary 2.

The following conditions are needed to ensure the validity of the asymptotic results presented in this section.

(A1) Let $(\lambda_i, \boldsymbol{u}_i)$ denote the set of non-zero eigenvalues and eigenvectors of $\mathbf{\Sigma}$. We assume that there exist l_1 and L_1 such that

$$0 < l_1 \le \lambda_1 \le \lambda_2 \le \ldots \le \lambda_r \le L_1 < \infty$$

uniformly on k.

(A2) There exists L_2 such that

$$|\boldsymbol{u}_i^T\mu| \le L_2 \text{ for all } i = 1, \ldots, r \text{ uniformly on } k.$$

It is noted that Assumptions (A1) and (A2) are valid uniformly on k, that is both constants L_1 and L_2 should not depend on k. Later on we also assume that κ increases with r. This condition is needed in order to ensure that the random vector $\mathbf{z}$ is well concentrated around its mean vector in large dimension. For example, fulfilled in the case, when $\mathbf{z}$ is the sample mean computed from the independent normal sample.

Theorem 3. *Let* $\mathbf{A} \sim \mathcal{W}_k(n, \mathbf{\Sigma})$ *with* $\mathrm{rank}(\mathbf{\Sigma}) = r < k$ *and let* $\mathbf{z} \sim \mathcal{N}_k(\mu, \kappa\mathbf{\Sigma}), \kappa > 0$. *Assume* $\frac{r}{n} = c + o(n^{-1/2}), c \in [0, +\infty)$ *and* $\kappa r = O(1)$ *as* $n \to \infty$. *Also, let* $\mathbf{m}$ *be a k-dimensional vector of constants such that* $\mathbf{m}^T\mathbf{\Sigma}\mathbf{m} > 0$ *and* $|\boldsymbol{u}_i^T\mathbf{m}| \le L_2$ *for all* $i = 1, \ldots, r$ *uniformly on k. Assume that* $\mathbf{A}$ *and* $\mathbf{z}$ *are independently distributed. Then, under (A1) and (A2), it holds that the asymptotic distribution of* $\mathbf{m}^T\mathbf{Az}$ *is given by*

$$\sqrt{n}\sigma^{-1}\bigg(\frac{1}{n}\mathbf{m}^T\mathbf{Az} - \mathbf{m}^T\mathbf{\Sigma}\mu\bigg) \xrightarrow{d} \mathcal{N}(0,1) \quad \textit{for } r/n \to c \textit{ as } n \to \infty,$$

where
$$\sigma^2 = \left(\mathbf{m}^T\mathbf{\Sigma}\mu\right)^2 + \mathbf{m}^T\mathbf{\Sigma}\mathbf{m}\left[\kappa tr(\mathbf{\Sigma}^2) + \mu^T\mathbf{\Sigma}\mu\right] + \frac{\kappa}{c}\mathbf{m}^T\mathbf{\Sigma}^3\mathbf{m}.$$

Proof. From Corollary 2, the stochastic representation of $\mathbf{m}^T\mathbf{A}\mathbf{z}$ is given by
$$\mathbf{m}^T\mathbf{A}\mathbf{z} \overset{d}{=} \zeta\mathbf{m}^T\mathbf{\Sigma}\mathbf{z} + \sqrt{\zeta}\left[\mathbf{z}^T\mathbf{\Sigma}\mathbf{z}\cdot\mathbf{m}^T\mathbf{\Sigma}\mathbf{m} - (\mathbf{m}^T\mathbf{\Sigma}\mathbf{z})^2\right]^{1/2} z_0,$$
with $\zeta \sim \chi^2_n$, $z_0 \sim \mathcal{N}(0,1)$ and $\mathbf{z} \sim \mathcal{N}_k(\mu, \kappa\mathbf{\Sigma}), \kappa > 0$; ζ, z_0, and $\mathbf{z}$ are mutually independent.

From the property of χ^2-distribution, we immediately obtain the asymptotic distribution of ζ given by
$$\sqrt{n}\left(\frac{\zeta}{n} - 1\right) \xrightarrow{d} \mathcal{N}(0,2) \text{ as } n \to \infty. \tag{3}$$
Further, it holds that $\sqrt{n}(z_0/\sqrt{n}) \sim \mathcal{N}(0,1)$ for all n, consequently it is its asymptotic distribution.

We next show that $\mathbf{m}^T\mathbf{\Sigma}\mathbf{z}$ and $\mathbf{z}^T\mathbf{\Sigma}\mathbf{z}$ are jointly asymptotically normally distributed under the high-dimensional asymptotic regime. For any $a_1 \in \mathbb{R}$ and $a_2 \in \mathbb{R}$, we consider
$$a_1\mathbf{z}^T\mathbf{\Sigma}\mathbf{z} + 2a_2\mathbf{m}^T\mathbf{\Sigma}\mathbf{z} = a_1\left(\mathbf{z} + \frac{a_2}{a_1}\mathbf{m}\right)^T\mathbf{\Sigma}\left(\mathbf{z} + \frac{a_2}{a_1}\mathbf{m}\right) - \frac{a_2^2}{a_1}\mathbf{m}^T\mathbf{\Sigma}\mathbf{m} =$$
$$= a_1\tilde{\mathbf{z}}^T\mathbf{\Sigma}\tilde{\mathbf{z}} - \frac{a_2^2}{a_1}\mathbf{m}^T\mathbf{\Sigma}\mathbf{m},$$
where $\tilde{\mathbf{z}} \sim \mathcal{N}_k(\mu_a, \kappa\mathbf{\Sigma})$ with $\mu_a = \mu + \frac{a_2}{a_1}\mathbf{m}$. By [18] the random variable $\tilde{\mathbf{z}}^T\mathbf{\Sigma}\tilde{\mathbf{z}}$ can be expressed as
$$\tilde{\mathbf{z}}^T\mathbf{\Sigma}\tilde{\mathbf{z}} \overset{d}{=} \kappa\sum_{i=1}^r \lambda_i^2\zeta_i \quad \text{with} \quad \zeta_i \overset{d}{\sim} \chi^2_1(\delta_i^2),\ \delta_i^2 = \kappa^{-1}\lambda_i^{-1}\left(\boldsymbol{u}_i^T\mu_a\right)^2,$$
where the symbol $\chi^2_d(\delta)$ denotes the non-central chi-squared distribution with d degrees of freedom and non-centrality parameter δ.

Next, we apply the Lindeberg central limit theorem to the i.i.d. random variables $V_i = \kappa\lambda_i^2\zeta_i$. Let $\sigma_i^2 = \mathbb{V}(V_i)$ and $s_n^2 = \mathbb{V}(\sum_{i=1}^r V_i)$. It holds that
$$s_n^2 = \mathbb{V}\left(\sum_{i=1}^r V_i\right) = \kappa^2\sum_{i=1}^r \lambda_i^4\mathbb{V}(\zeta_i) = \kappa^2\sum_{i=1}^r \lambda_i^4 2(1+2\delta_i^2) =$$
$$= \kappa^2\sum_{i=1}^r\left(2\lambda_i^4 + 4\kappa^{-1}\lambda_i^3(\boldsymbol{u}_i^T\mu_a)^2\right) = \kappa^2\left[2tr(\mathbf{\Sigma}^4) + 4\kappa^{-1}\mu_a^T\mathbf{\Sigma}^3\mu_a\right].$$

In order to verify the Lindeberg condition, we need to check if for any small $\epsilon > 0$ it holds that
$$\lim_{r\to\infty}\frac{1}{s_n^2}\sum_{i=1}^r \mathbb{E}\left[(V_i - \mathbb{E}(V_i))^2\mathbb{1}_{\{|V_i - \mathbb{E}(V_i)| > \epsilon s_n\}}\right] = 0, \tag{4}$$
where
$$\sum_{i=1}^r \mathbb{E}\left[(V_i - \mathbb{E}(V_i))^2\mathbb{1}_{\{|V_i - \mathbb{E}(V_i)| > \epsilon s_n\}}\right] \overset{\text{Cauchy–Schwarz}}{\leq}$$
$$\overset{\text{Cauchy–Schwarz}}{\leq} \sum_{i=1}^r \sqrt{\mathbb{E}[(V_i - \mathbb{E}(V_i))^4]}\sqrt{\mathbb{E}\left[\mathbb{1}_{\{|V_i - \mathbb{E}(V_i)| > \epsilon s_n\}}\right]} =$$
$$= \sum_{i=1}^r \sqrt{\mathbb{E}[(V_i - \mathbb{E}(V_i))^4]}\sqrt{\mathbb{P}[|V_i - \mathbb{E}(V_i)| > \epsilon s_n]} \overset{\text{Chebychev}}{\leq}$$

$$\overset{\text{Chebychev}}{\leq} \sum_{i=1}^{r}\sqrt{\mathbb{E}[(V_i-\mathbb{E}(V_i))^4]}\frac{\sigma_i}{\epsilon s_n} =$$
$$= 2\sqrt{3}\frac{\kappa^2}{\epsilon}\sum_{i=1}^{r}\lambda_i^4\sqrt{(1+2\delta_i^2)^2+4(1+4\delta_i^2)}\frac{\sigma_i}{s_n}.$$

By using

$$(1+2\delta_i^2)^2+4(1+4\delta_i^2)=(5+2\delta_i^2)^2-20\leq(5+2\delta_i^2)^2,$$

for $\sigma_{\max}=\sup_i\sigma_i$, we get the following inequality

$$\frac{1}{s_n^2}\sum_{i=1}^{r}\mathbb{E}\left[(V_i-\mathbb{E}(V_i))^2\mathbb{1}_{\{|V_i-\mathbb{E}(V_i)|>\epsilon s_n\}}\right]\leq 2\sqrt{3}\frac{\kappa^2}{\epsilon}\frac{\sigma_{\max}}{s_n}\frac{1}{s_n^2}\sum_{i=1}^{r}\lambda_i^4(5+2\delta_i^2)=$$
$$=\frac{\sqrt{3}}{\epsilon}\frac{\sigma_{\max}}{s_n}\frac{5tr(\mathbf{\Sigma}^4)+2\kappa^{-1}\mu_a^T\mathbf{\Sigma}^3\mu_a}{tr(\mathbf{\Sigma}^4)+2\kappa^{-1}\mu_a^T\mathbf{\Sigma}^3\mu_a}\leq\frac{5\sqrt{3}}{\epsilon}\frac{\sigma_{\max}}{s_n}.$$

Using

$$(\boldsymbol{u}_i^T\mu_a)^2=\left(\boldsymbol{u}_i^T\mu+\frac{a_2}{a_1}\boldsymbol{u}_i^T\boldsymbol{m}\right)^2=2(\boldsymbol{u}_i^T\mu)^2+2\left(\frac{a_2}{a_1}\boldsymbol{u}_i^T\boldsymbol{m}\right)^2=$$
$$=2L_2^2\left(1+\left(\frac{a_2}{a_1}\right)^2\right)<\infty$$

and Assumptions (A1) and (A2), we get

$$\frac{\sigma_{\max}^2}{s_n^2}=\frac{\sup_i(\lambda_i^4(1+2\delta_i^2))}{tr(\mathbf{\Sigma}^4)+2\kappa^{-1}\mu_a^T\mathbf{\Sigma}^3\mu_a}=\frac{\sup_i(\lambda_i^4+2\kappa^{-1}\lambda_i^3(\boldsymbol{u}_i^T\mu_a)^2)}{tr(\mathbf{\Sigma}^4)+2\kappa^{-1}\mu_a^T\mathbf{\Sigma}^3\mu_a}\to 0,$$

which verifies the Lindeberg condition.

Since

$$\sum_{i=1}^{r}\mathbb{E}(V_i)=\kappa\sum_{i=1}^{r}\lambda_i^2\mathbb{E}(\zeta_i)=\kappa\sum_{i=1}^{r}\lambda_i^2\left(1+\delta_i^2\right)=\kappa tr(\mathbf{\Sigma}^2)+\mu_a^T\mathbf{\Sigma}\mu_a$$

we obtain by using the Lindeberg central limit theorem that

$$\sqrt{\frac{1}{\kappa}}\frac{\tilde{\mathbf{z}}^T\mathbf{\Sigma}\tilde{\mathbf{z}}-\kappa tr(\mathbf{\Sigma}^2)-\mu_a^T\mathbf{\Sigma}\mu_a}{\sqrt{2\kappa tr(\mathbf{\Sigma}^4)+4\mu_a^T\mathbf{\Sigma}^3\mu_a}}\xrightarrow{d}\mathcal{N}(0,1).$$

Let $\mathbf{a}=(a_1,2a_2)^T$. Then the last identity leads to

$$\sqrt{n}\left[\mathbf{a}^T\begin{pmatrix}\mathbf{z}^T\mathbf{\Sigma}\mathbf{z}\\ \mathbf{m}^T\mathbf{\Sigma}\mathbf{z}\end{pmatrix}-\mathbf{a}^T\begin{pmatrix}\kappa tr(\mathbf{\Sigma}^2)+\mu^T\mathbf{\Sigma}\mu\\ \mathbf{m}^T\mathbf{\Sigma}\mu\end{pmatrix}\right]\xrightarrow{d}$$
$$\xrightarrow{d}\mathcal{N}\left(0,\mathbf{a}^T\frac{\kappa}{c}\begin{pmatrix}2\kappa tr(\mathbf{\Sigma}^4)+4\mu\mathbf{\Sigma}^3\mu & 2\mathbf{m}^T\mathbf{\Sigma}^3\mu\\ 2\mathbf{m}^T\mathbf{\Sigma}^3\mu & \mathbf{m}^T\mathbf{\Sigma}^3\mathbf{m}\end{pmatrix}\mathbf{a}\right), \tag{5}$$

which implies that the vector $\left(\mathbf{z}^T\mathbf{\Sigma}\mathbf{z},\mathbf{m}^T\mathbf{\Sigma}\mathbf{z}\right)^T$ is asymptotically multivariate normally distributed because $\mathbf{a}$ is an arbitrary fixed vector.

Taking into account (3),(5) and the fact that ζ, z_0, and $\mathbf{z}$ are mutually independent, we get the following asymptotic result

$$\sqrt{n}\left[\begin{pmatrix}\frac{\zeta}{n}\\ \mathbf{z}^T\mathbf{\Sigma}\mathbf{z}\\ \mathbf{m}^T\mathbf{\Sigma}\mathbf{z}\\ \frac{z_0}{\sqrt{n}}\end{pmatrix}-\begin{pmatrix}1\\ \kappa tr(\mathbf{\Sigma}^2)+\mu^T\mathbf{\Sigma}\mu\\ \mathbf{m}^T\mathbf{\Sigma}\mu\\ 0\end{pmatrix}\right]\xrightarrow{d}$$

$$\xrightarrow{d} \mathcal{N}\left(0, \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2\frac{\kappa^2}{c}tr(\mathbf{\Sigma}^4) + 4\frac{\kappa}{c}\mu^T\mathbf{\Sigma}^3\mu & 2\frac{\kappa}{c}\mathbf{m}^T\mathbf{\Sigma}^3\mu & 0 \\ 0 & 2\frac{\kappa}{c}\mathbf{m}^T\mathbf{\Sigma}^3\mu & \frac{\kappa}{c}\mathbf{m}^T\mathbf{\Sigma}^3\mathbf{m} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\right).$$

Finally, the application of the delta method leads to

$$\sqrt{n}\sigma^{-1}\left(\frac{1}{n}\mathbf{m}^T\mathbf{A}\mathbf{z} - \mathbf{m}^T\mathbf{\Sigma}\mu\right) \xrightarrow{d} \mathcal{N}(0,1),$$

where

$$\sigma^2 = \begin{pmatrix} \mathbf{m}^T\mathbf{\Sigma}\mu & 0 & 1 & \left[\left[\kappa tr(\mathbf{\Sigma}^2) + \mu^T\mathbf{\Sigma}\mu\right]\mathbf{m}^T\mathbf{\Sigma}\mathbf{m} - \left(\mathbf{m}^T\mathbf{\Sigma}\mu\right)^2\right]^{\frac{1}{2}} \end{pmatrix} \times$$

$$\times \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2\frac{\kappa^2}{c}tr(\mathbf{\Sigma}^4) + 4\frac{\kappa}{c}\mu^T\mathbf{\Sigma}^3\mu & 2\frac{\kappa}{c}\mathbf{m}^T\mathbf{\Sigma}^3\mu & 0 \\ 0 & 2\frac{\kappa}{c}\mathbf{m}^T\mathbf{\Sigma}^3\mu & \frac{\kappa}{c}\mathbf{m}^T\mathbf{\Sigma}^3\mathbf{m} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \times$$

$$\times \begin{pmatrix} \mathbf{m}^T\mathbf{\Sigma}\mu \\ 0 \\ 1 \\ \left[\left[\kappa tr(\mathbf{\Sigma}^2) + \mu^T\mathbf{\Sigma}\mu\right]\mathbf{m}^T\mathbf{\Sigma}\mathbf{m} - \left(\mathbf{m}^T\mathbf{\Sigma}\mu\right)^2\right]^{\frac{1}{2}} \end{pmatrix} =$$

$$= \left(\mathbf{m}^T\mathbf{\Sigma}\mu\right)^2 + \mathbf{m}^T\mathbf{\Sigma}\mathbf{m}\left[\kappa tr(\mathbf{\Sigma}^2) + \mu^T\mathbf{\Sigma}\mu\right] + \frac{\kappa}{c}\mathbf{m}^T\mathbf{\Sigma}^3\mathbf{m}. \qquad \square$$

Finally, we extend the results of Theorem 3 to the case of finite number of linear combinations of the elements of $\mathbf{Az}$. The results are summarised in the following theorem.

Theorem 4. *Let* $\mathbf{A} \sim \mathcal{W}_k(n, \mathbf{\Sigma})$ *with* $\text{rank}(\mathbf{\Sigma}) = r < k$ *and let* $\mathbf{z} \sim \mathcal{N}_k(\mu, \kappa\mathbf{\Sigma})$*,* $\kappa > 0$*. Assume* $\frac{r}{n} = c + o(n^{-1/2}), c \in [0, +\infty)$ *and* $\kappa r = O(1)$ *as* $n \to \infty$*. Let* $\mathbf{M} = (\mathbf{m}_1, \ldots, \mathbf{m}_p)^T : p \times k$ *be a matrix of constants of rank* $p < r \le n$ *with probability one and let* $|\boldsymbol{u}_i^T\mathbf{m}_j| \le L_2$ *for all* $i = 1, \ldots, r$ *and* $j = 1, \ldots, p$ *uniformly on* k*. Assume that* $\mathbf{A}$ *and* $\mathbf{z}$ *are independently distributed. Then under (A1) and (A2) the asymptotic distribution of* $\mathbf{MAz}$ *under the double asymptotic regime is given by*

$$\sqrt{n}\mathbf{\Omega}^{-1/2}\left(\frac{1}{n}\mathbf{MAz} - \mathbf{M\Sigma z}\right) \xrightarrow{d} \mathcal{N}_p(\mathbf{0}, \mathbf{I}_p) \quad \text{for } r/n \to c \text{ as } n \to \infty,$$

where

$$\mathbf{\Omega} = \mathbf{M\Sigma}\mu\mu^T\mathbf{\Sigma M}^T + \mathbf{M\Sigma M}^T\left[\kappa tr(\mathbf{\Sigma}^2) + \mu^T\mathbf{\Sigma}\mu\right] + \frac{\kappa}{c}\mathbf{M\Sigma}^3\mathbf{M}^T.$$

Proof. For all $\mathbf{l} \in \mathbb{R}^p$-fixed, we consider $\mathbf{l}^T\mathbf{MAz}$. The rest of the proof follows from Theorem 3 with $\mathbf{m} = \mathbf{M}^T\mathbf{l}$ and the fact that $\mathbf{l}$ is an arbitrary vector. $\square$

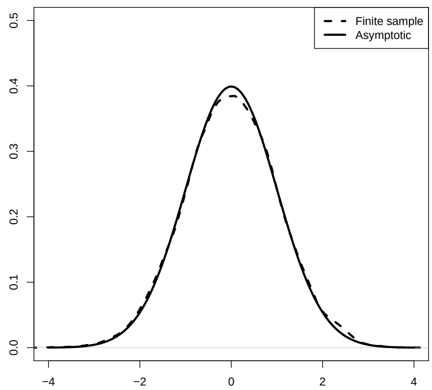
4. Finite sample performance

In this section, we present the results of a Monte Carlo simulation study where the performance of the obtained asymptotic distribution for the product of a singular Wishart matrix and a singular Gaussian vector is investigated.

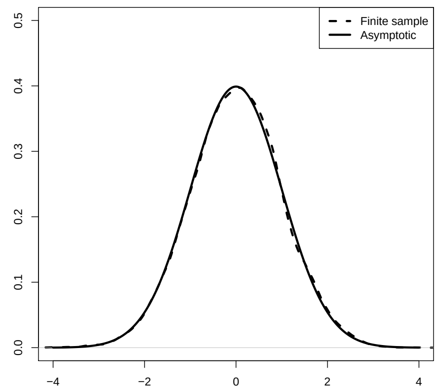
In our simulation, we fix $\mathbf{m} = \mathbf{1}/k$ where $\mathbf{1}$ denotes the k-dimensional vector of ones and generated each element of μ from the uniform distribution on $[-1, 1]$. The population covariance matrix was drawn in the following way:

- r non-zero eigenvalues of $\mathbf{\Sigma}$ were generated from the uniform distribution on $(0, 1)$ and the rest were set to be zero;
- the eigenvectors were generated from the Haar distribution by simulating a Wishart matrix with identity covariance matrix and calculating its eigenvectors.

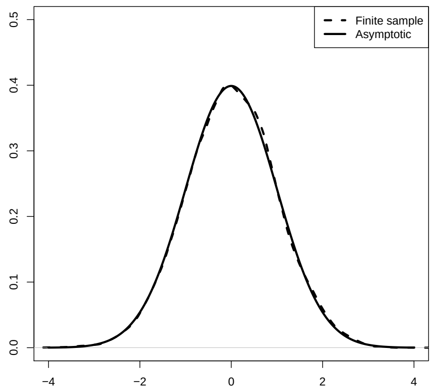
Both the mean vector and the population covariance matrix obtained by such setting satisfy the assumptions (A1) and (A2).



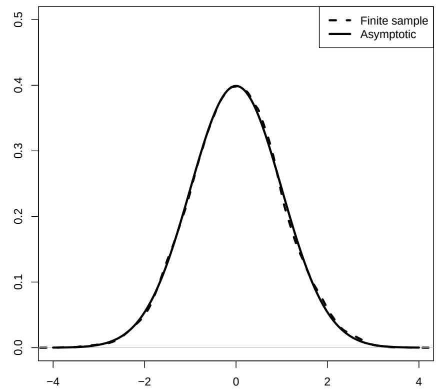
(a) $n = 500, c = 0.1, k = 750$



(b) $n = 500, c = 0.5, k = 750$



(c) $n = 500, c = 0.8, k = 750$



(d) $n = 500, c = 0.95, k = 750$

FIGURE 1. Asymptotic distribution and the kernel density estimator of the finite sample distribution calculated for the product of a singular Wishart matrix and a singular normal vector ($n = 500$)

We compare the asymptotic density of the standardized random variable $\mathbf{m}^T\mathbf{A}\mathbf{z}$ with its finite sample one which is obtained by applying the stochastic representation of Corollary 2. More precisely, we draw $N = 10^4$ independent realizations of the standardized random variable $\mathbf{m}^T\mathbf{A}\mathbf{z}$ by using the following algorithm.

a) Generate $\mathbf{m}^T\mathbf{A}\mathbf{z}$ by using stochastic representation (2) of Corollary 2 expressed as

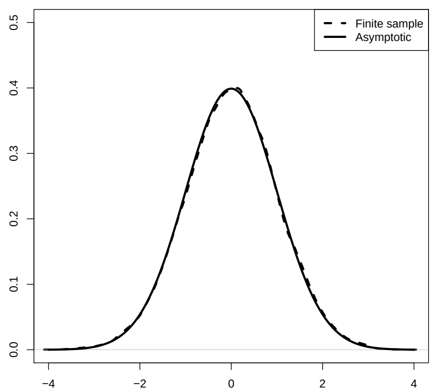
$$\mathbf{m}^T\mathbf{A}\mathbf{z} \stackrel{d}{=} \zeta\mathbf{m}^T\mathbf{\Sigma}\mathbf{z} + \sqrt{\zeta}\left[\mathbf{z}^T\mathbf{\Sigma}\mathbf{z}\cdot\mathbf{m}^T\mathbf{\Sigma}\mathbf{m} - (\mathbf{m}^T\mathbf{\Sigma}\mathbf{z})^2\right]^{1/2} z_0,$$

where $\zeta \sim \chi^2_n$, $z_0 \sim \mathcal{N}(0,1)$, $\mathbf{z} \sim \mathcal{N}_k(\mu, \kappa\mathbf{\Sigma})$; ζ, z_0, and $\mathbf{z}$ are mutually independent.

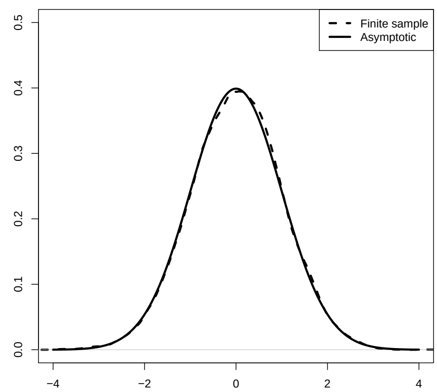
b) Compute
$$\sqrt{n}\sigma^{-1}\left(\frac{1}{n}\mathbf{m}^T\mathbf{A}\mathbf{z} - \mathbf{m}^T\boldsymbol{\Sigma}\mu\right),$$
where
$$\sigma^2 = \left(\mathbf{m}^T\boldsymbol{\Sigma}\mu\right)^2 + \mathbf{m}^T\boldsymbol{\Sigma}\mathbf{m}\left[\kappa tr(\boldsymbol{\Sigma}^2) + \mu^T\boldsymbol{\Sigma}\mu\right] + \frac{\kappa}{c}\mathbf{m}^T\boldsymbol{\Sigma}^3\mathbf{m}.$$

c) Repeat a)–b) N times.

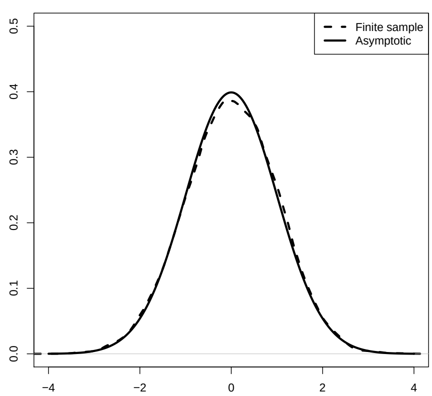
Then, the elements of the generated sample are used to construct a kernel density estimator which is compared to the asymptotic distribution, i.e. to the density of the standard normal distribution. As a kernel, we make use of the Epanechnikov kernel with the bandwidth chosen by applying Silverman's rule of thumb.



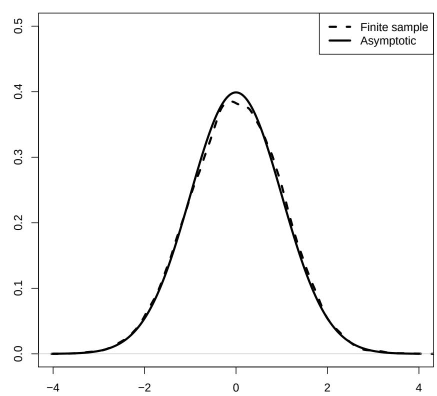
(a) $n = 1000, c = 0.1, k = 750$



(b) $n = 1000, c = 0.5, k = 750$

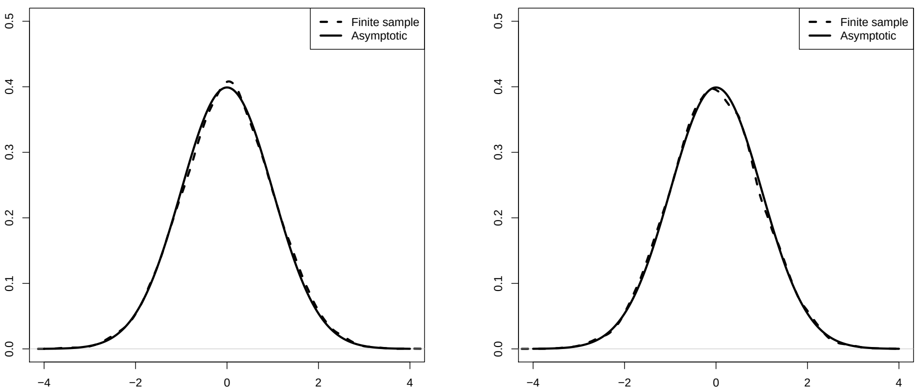


(c) $n = 1000, c = 0.8, k = 990$



(d) $n = 1000, c = 0.95, k = 990$

FIGURE 2. Asymptotic distribution and the kernel density estimator of the finite sample distribution calculated for the product of a singular Wishart matrix and a singular normal vector ($n = 1000$)



(a) $n = 500, c = 2, k = 1200$ (b) $n = 1000, c = 2, k = 2100$

FIGURE 3. Asymptotic distribution and the kernel density estimator of the finite sample distribution calculated for the product of a singular Wishart matrix and a singular normal vector ($n \in \{500, 1000\}$)

The results of the simulation study are summarized in Figure 1 for $n = 500$, in Figure 2 for $n = 1000$, and for $n \in \{500, 100\}$ with $c = 2$ in Figure 3. In all cases we set $\kappa = 1/n$. Finally, $k = 750$ is chosen for $n = 500$ and $k \in \{750, 990\}$ for $n = 1000$. For $c = 2$, $k = 1200$ is chosen for $n = 500$ and $k = 2100$ for $n = 1000$. Furthermore, several values of r are considered such that $c = \{0.1, 0.5, 0.8, 0.95\}$ in Figures 1 and 2, while $c = 2$ in Figure 3. The finite sample distributions are shown as dashed lines, while the asymptotic distributions are solid lines. All obtained results show a good performance of the asymptotic approximation which is almost indistinguishable from the corresponding finite sample density. This result remains true even for the values of $c = 0.95$ and $c = 2$.

5. SUMMARY

The Wishart distribution and normal distribution are widely spread in both statistics and probability theory with numerous and useful applications in finance, economics, environmental sciences, biology, etc. Different functions involving a Wishart matrix and a normal vector have been studied in statistical literature recently. However, to the best of our knowledge, combinations of a singular Wishart matrix and a singular normal vector have not been investigated up to now.

In this paper we analyse the product of a singular Wishart matrix and a singular Gaussian vector. A very useful stochastic representation of this product is obtained, which is later used to derive its characteristic function as well as to provide an efficient way how to simulate the elements of the product in practice. With the use of the derived stochastic representation, there is no need in generating a large dimensional Wishart matrix. Its application speeds up simulation studies where the product of a singular Wishart matrix and a singular normal vector is present. Furthermore, we prove the asymptotic normality of the product under the double asymptotic regime. In a numerical study, a good performance of the obtained asymptotic distribution is documented. It is also noted that for the values $c = 0.95$ and $c = 2$, it produces a very good approximation of the corresponding finite sample distribution obtained by applying the derived stochastic representation.

Acknowledgements

Taras Bodnar and Stanislas Muhinyuza appreciate the financial support of SIDA via the project 1683030302. Stepan Mazur acknowledges the financial support from the project "Models for macro and financial economics after the financial crisis" (Dnr: P18-0201) funded by Jan Wallander and Tom Hedelius Foundation.

The authors are grateful to the referee and the editor for their suggestions, which have improved the presentation of the paper.

References

1. J. M. Bernardo, A. F. M. Smith, *Bayesian theory*, Chichester: Wiley, 1994.
2. O. Bodnar, *Sequential surveillance of the tangency portfolio weights*, International Journal of Theoretical and Applied Finance, **12** (2009), 797–810.
3. T. Bodnar, S. Mazur, Y. Okhrin, *On exact and approximate distributions of the product of the Wishart matrix and normal vector*, Journal of Multivariate Analysis, **122** (2013), 70–81.
4. T. Bodnar, S. Mazur, K. Podgórski, *Singular inverse Wishart distribution and its application to portfolio theory*, Journal of Multivariate Analysis, **143** (2016), 314–326.
5. T. Bodnar, Y. Okhrin, *Properties of the singular, inverse and generalized inverse partitioned Wishart distributions*, Journal of Multivariate Analysis, **99** (2008), 2389– 2405.
6. T. Bodnar, Y. Okhrin, *On the product of inverse Wishart and normal distributions with applications to discriminant analysis and portfolio theory*, Scandinavian Journal of Statistics, **38** (2011), 311–331.
7. M. Britten-Jones, *The sampling error in estimates of mean-variance efficient portfolio weights*, The Journal of Finance, **54** (1999), 655–671.
8. J. A. Díaz-García, R. G. Jáimez, K. V. Mardia, *Wishart and pseudo-Wishart distributions and some applications to shape theory*, Journal of Multivariate Analysis, **63** (1997), 73–87.
9. G. H. Givens, J. A. Hoeting, *Computational statistics*, John Wiley & Sons, 2012.
10. A. Gupta, D. Nagar, *Matrix Variate Distributions*, Chapman and Hall/CRC, Boca Raton, 2000.
11. A. Gupta, T. Varga, T. Bodnar, *Elliptically contoured models in statistics and portfolio theory*, Springer, 2013.
12. J. D. Jobson, B. Korkie, *Estimation for Markowitz efficient portfolios*, Journal of the American Statistical Association, **75** (1980), 544–554.
13. R. Kan, G. Zhou, *Optimal portfolio choice with parameter uncertainty*, Journal of Financial and Quantitative Analysis, **42** (2007), 621–656.
14. C. Khatri, *A note on Mitra's paper "A density free approach to the matrix variate beta distribution"*, Sankhyā: The Indian Journal of Statistics, Series A, **32** (1970), 311–318.
15. I. Kotsiuba, S. Mazur, *On the asymptotic and approximate distributions of the product of an inverse Wishart matrix and a gaussian random vector*, Theory of Probability and Mathematical Statistics, **93** (2015), 95–104.
16. R. J. Muirhead, *Aspects of Multivariate Statistical Theory*, Wiley, New York, 1982.
17. D. Pappas, K. Kiriakopoulos, G. Kaimakamis, *Optimal portfolio selection with singular covariance matrix*, International Mathematical Forum, **5** (2010), 2305–2318.
18. S. B. Provost, E. M. Rudiuk, *The exact distribution of indefinite quadratic forms in noncentral normal vectors*, Annals of the Institute of Statistical Mathematics, **48** (1996), 381–394.
19. A. C. Rencher, W. F. Christensen, *Methods of multivariate analysis, third edition*, Wiley Online Library, 2012.
20. M. Srivastava, C. Khatri, *An introduction to multivariate statistics*, North-Holland, New York, 1979.
21. M. S. Srivastava, *Singular Wishart and multivariate beta distributions*, The Annals of Statistics, **31** (2003), 1537–1560.
22. H. Uhlig, *On singular Wishart and singular multivariate beta distributions*, The Annals of Statistics, **22** (1994), 395–405.

Department of Mathematics, Stockholm University, Roslagsvägen 101, SE-10691 Stockholm, Sweden
E-mail address: taras.bodnar@math.su.se

Unit of Statistics, School of Business, Örebro University, SE-70182 Örebro, Sweden
E-mail address: stepan.mazur@oru.se

Department of Mathematics, Stockholm University, Roslagsvägen 101, SE-10691 Stockholm, Sweden
Department of Mathematics, College of Science and technology, University of Rwanda, P.O. Box 3900, Kigali-Rwanda
E-mail address: stanislas.muhinyuza@math.su.se

Institute of Statistics, Leibniz University of Hannover, D-30167 Hannover, Germany
E-mail address: parolya@statistik.uni-hannover.de

Received 28.04.2018

ДОБУТОК СИНГУЛЯРНОЇ ВИПАДКОВОЇ МАТРИЦІ ВІШАРТА ТА СИНГУЛЯРНОГО НОРМАЛЬНОГО ВИПАДКОВОГО ВЕКТОРА У ВЕЛИКИХ РОЗМІРНОСТЯХ

Т. БОДНАР, С. МАЗУР, С. МУХІНЮЗА, Н. ПАРОЛЯ

Анотація. У статті ми розглядаємо добуток сингулярної випадкової матриці Вішарта та сингулярного нормального випадкового вектора. Отримано дуже корисне стохастичне представлення цього добутку, за допомогою якого виводиться його характеристична функція та асимптотичний розподіл при подвійному асимптотичному режимі. Також, із використанням методу Монте-Карло, показано хороші результати апроксимації, отримані за допомогою виведеного багатовимірного асимптотичного розподілу в умовах скінченної вибірки.

ПРОИЗВЕДЕНИЕ СИНГУЛЯРНОЙ СЛУЧАЙНОЙ МАТРИЦЫ ВИШАРТА И СИНГУЛЯРНОГО НОРМАЛЬНОГО СЛУЧАЙНОГО ВЕКТОРА В БОЛЬШОЙ РАЗМЕРНОСТИ

Т. БОДНАР, С. МАЗУР, С. МУХИНЮЗА, Н. ПАРОЛЯ

Аннотация. В статье мы рассматриваем произведение сингулярной случайной матрицы Вишарта и сингулярного нормального случайного вектора. Получено очень полезное стохастическое представление этого произведения, с помощью которого выводится его характеристическая функция и асимптотическое распределение при двойном асимптотическом режиме. Также, с использованием метода Монте-Карло, показаны хорошие результаты аппроксимации, полученные с помощью выведенного многомерного асимптотического распределения в условиях конечной выборки.