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DISCRETE TIME APPROXIMATION OF COALESCING STOCHASTIC FLOWS ON THE REAL LINE

We have constructed an approximation for the Harris and Arratia flows using a sequence of independent stationary Gaussian processes as a perturbation. We have established which relationship should be between the step of approximation and the smoothness of the covariance of perturbing processes in order that the approximating functions converge to the Arratia flow.

1. Introduction

It is well-known [1] that the solution to the Cauchy problem for the SDE

$$\begin{cases} dx(t) = a(x(t))dt + b(x(t))dw(t) \\ x(0) = u_0 \end{cases} \tag{1}$$

with continuously differentiable functions a and b having bounded derivatives can be obtained via the discrete time approximation. Namely, if we define a sequence $\{x_n^m\}$ by the rule

$$x_0^m = x_0 \in \mathbb{R},\ x_{n+1}^m = x_n^m + \frac{1}{m}a(x_n^m) + \frac{1}{\sqrt{m}}b(x_n^m)\xi_n, \tag{2}$$

where $\{\xi_n, n \geq 1\}$ is a sequence of independent standard Gaussian random variables, then the random functions

$$x_m(t) = m\left(\frac{k+1}{m} - t\right)x_k^m + m\left(t - \frac{k}{m}\right)x_{k+1}^m,\ t \in \left[\frac{k}{m}; \frac{k+1}{m}\right], k = 0, \ldots, m-1$$

weakly converge in $C([0,1])$ to the solution of (1).

In this paper, we study a difference approximation similar to (2) for coalescing stochastic flows. As is known [2], such flows are not generated by a Gaussian white noise in the space of vector fields. In order to understand how the flow with coalescence is arranged, we can consider its difference approximation. As a perturbation, we select a sequence of Gaussian stationary processes. In order to allow the coalescence of the trajectories of individual particles in the limit, the covariance functions of these processes are chosen to be less and less smooth at the origin. On the other hand, for the limit flow to preserve the order, the step of approximation must be sufficiently small. The relationship between the step of approximation and the smoothness of the covariance of perturbing processes explains, to some extent, the structure of singular stochastic flows.

2. SDE and stochastic flows on the real line

The main object of the article is the Harris flow of Brownian motions on $\mathbb{R}$. Let φ be a continuous real positive definite function on $\mathbb{R}$ such that $\varphi(0) = 1$ and φ is Lipschitz outside any neighborhood of zero.

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Definition 1. The Harris flow with φ being its local characteristic is a family $\{x(u,\cdot); u \in \mathbb{R}\}$ of Brownian martingales with respect to the joint filtration such that

1) for every $u_1 \le u_2$ and $t \ge 0$,

$$x(u_1,t) \le x(u_2,t),$$

2) the joint characteristics are

$$d\langle x(u_1,\cdot), x(u_2,\cdot)\rangle(t) = \varphi(x(u_1,t) - x(u_2,t))dt.$$

It is known that the Harris flow exists [3]. If the function φ is smooth enough, the Harris flow can be obtained as a flow of solutions to SDE. Namely, for a sequence of standard Wiener processes $\{w_k; k \ge 1\}$, consider the SDE

$$dx(u,t) = \sum_{k=1}^{\infty} a_k(x(u,t))dw_k(t), \tag{3}$$

where $a = (a_k)_{k\ge 1}$ is a Lipschitz mapping from $\mathbb{R}$ to l_2 such that

$$\sum_{k=1}^{\infty} a_k^2 \equiv 1$$

and

$$\sum_{k=1}^{\infty} a_k(u)a_k(v) = \varphi(u-v).$$

Then the flow corresponding to (3) is the Harris flow with the local characteristic φ, and furthermore it is a flow of homeomorphisms. Note, that the Harris flow could be coalescent [3] and, in this case, may not be generated by SDE. For this reason, it is interesting to consider discrete approximations for the flow built in a similar way, as approximations to SDE. Consider a sequence of independent stationary Gaussian processes $\{\xi_n(u); u \in \mathbb{R}, n \ge 1\}$ with zero mean and a covariation function Γ. Suppose that Γ is continuous. Define a sequence of random mappings $\{x_n; n \ge 0\}$ by the rule

$$x_0(u) = u,\ x_{n+1}(u) = x_n(u) + \xi_{n+1}(x_n(u)),\ u \in \mathbb{R}. \tag{4}$$

Note that the continuity of Γ implies that the processes $\{\xi_n; n \ge 1\}$ have measurable modifications. This allows us to substitute x_n into ξ_{n+1}. The independence of $\{\xi_n; n \ge 1\}$ guarantees that $\xi_{n+1}(x_n(u))$ does not depend on the choice of these modifications. We need the following description of one and two-point motions of $\{x_n; n \ge 0\}$.

Lemma 1. *The sequences $\{x_n(u); n \ge 0\}$ and $\{x_n(u_2) - x_n(u_1); n \ge 0\}$ have the same distributions as the sequences $\{y_n(u); n \ge 0\}$, $\{z_n(u); n \ge 0\}$, which are defined by the rules*

$$y_0 = u,\ y_{n+1} = y_n + \eta_n,$$

$$z_0 = u_2 - u_1,\ z_{n+1} = z_n + \sqrt{2\Gamma(0) - 2\Gamma(z_n)}\eta_n,$$

where $\{\eta_n; n \ge 1\}$ is a sequence of independent standard normal variables.

The proof of the lemma can be obtained easily by calculating the conditional distributions of x_{n+1} under given $x_0, \ldots, x_n$, and is omitted.

It follows from Lemma 1 that the sequence of random mappings $\{x_n; n \ge 0\}$ is similar to the Harris flow. All its one-point motions are Gaussian symmetric random walks. But the mappings x_n for $n \ge 1$ are not monotone. In the next section, we will prove that any m-point motion of $\{x_n; n \ge 0\}$ approximates the m-point motion of the Harris flow.

3. m-POINT MOTIONS

In this section, we consider the limit behavior of x_n under a suitable normalization. Let us define the random functions

$$\widetilde{x}_n(u,t) = n\left(\frac{k+1}{n} - t\right)x_k(u) + n\left(t - \frac{k}{n}\right)x_{k+1}(u),$$

$$u \in \mathbb{R},\ t \in \left[\frac{k}{n}; \frac{k+1}{n}\right], k = 0, \ldots, n-1.$$

Our first result is related to the n-point motions of $\widetilde{x}_n$.

Theorem 1. *Let Γ be a continuous positive definite function on $\mathbb{R}$ such that $\Gamma(0) = 1$, and Γ has two continuous bounded derivatives. Suppose that $\widetilde{x}_n$ is built upon a sequence $\{\xi_k; k \geq 1\}$ with covariance $\frac{1}{\sqrt{n}}\Gamma$.*

Then, for every $u_1, \ldots, u_l \in \mathbb{R}$, the random processes $\{\widetilde{x}_n(u_j, \cdot), j = 1, \ldots, l\}$ weakly converge in $C([0;1], \mathbb{R}^l)$ to the l-point motion of the Harris flow with the local characteristic Γ.

Proof. It follows from Lemma 1 and the invariance principle that, for every $j = 1, \ldots, l$, $\widetilde{x}_n(u_j, \cdot)$ weakly converges in $C([0;1])$ to the Brownian motion which starts from u_j. Then it remains to prove that any limit point of $\{\widetilde{x}_n(u_j, \cdot), j = 1, \ldots, l\}$ coincides with the l-point motion of the Harris flow. Without loss of generality, we suppose that the whole sequence $\{\widetilde{x}_n(u_j, \cdot), j = 1, \ldots, l\}$ weakly converges. For a function $f \in C^3(\mathbb{R})$ with bounded derivatives, consider the random processes

$$y_n(t) = \widetilde{x}_n(u_{j+1}, t) - \widetilde{x}_n(u_j, t),$$

$$z_n(t) = f(y_n(t)) - f(u_{j+1} - u_j) - \int_0^t (1 - \Gamma(y_n(s)))f''(s)ds.$$

Following the known procedure (see, e.g., [4]), it is easy to verify that $\{z_n; n \geq 1\}$ weakly converges to a certain martingale. Consequently, the weak limit of y_n satisfies the martingale problem for the operator

$$Af(x) = (1 - \Gamma(x))\frac{d^2}{dx^2}f(x).$$

Since the martingale problem now has a unique solution [4], the weak limit of y_n is the solution to the Cauchy problem

$$\begin{cases} dy(t) = \sqrt{2 - 2\Gamma(y(t))}dw(t), \\ y(0) = u_{j+1} - u_j. \end{cases}$$

The solution to this SDE has the strong Markov property. Consequently, y is nonnegative for $u_{j+1} - u_j > 0$. Hence, the weak limit of $\{\widetilde{x}_n(u_j, \cdot); j = 1, \ldots, l\}$ preserves the order. It remains to check the form of the joint characteristic, which can be done in a standard way. The theorem is proved. $\square$

The previous result is based on the uniqueness of a solution to the SDE related to a stochastic flow. We now consider the convergence of difference approximations to the n-point motions of the Arratia flow. We recall that Arratia's flow [5] is the Harris flow with the local characteristic $\Gamma = 1\!\!\mathrm{I}_{\{0\}}$. In this flow, any two trajectories coalesce into a single one in a finite time.

Theorem 2. *Suppose that, for every $m \geq 1$, $\widetilde{x}_m$ is built upon a sequence $\{\xi^m_n; n \geq 1\}$, where the independent identically distributed processes ξ^m_n have the covariance function Γ_m which satisfies the Lipschitz condition. For $m \geq 1$, we define*

$$C_m = \sup_{\mathbb{R}} \frac{2 - 2\Gamma_m(x)}{x^2}.$$

If

1) $\lim_{m\to\infty} \frac{C_m e^{C_m}}{m} = 0$,

2) for every $\delta > 0 \ \ \sup_{\mathbb{R}\setminus[-\delta;\delta]} |\Gamma_m(x)| \to 0, m \to \infty$,

then the random processes $\{\widetilde{x}_m(u_1, \cdot), \dots, \widetilde{x}_m(u_l, \cdot); m \geq 1\}$ weakly converge to the l-point motion of Arratia's flow starting from $u_1, \dots, u_l$.

Proof. Like the proof of Theorem 1, we have the weak compactness of $\{(\widetilde{x}_m(u_1, \cdot), \dots, \widetilde{x}_m(u_l, \cdot); m \geq 1)\}$ in $C([0; 1], \mathbb{R}^l)$ and the weak convergence of $x_m(u_i, \cdot)$ to a Wiener process. Consequently, for any limit point of $\{(\widetilde{x}_m(u_1, \cdot), \dots, \widetilde{x}_m(u_l, \cdot); m \geq 1)\}$, it is enough to check the mutual characteristics and the order preserving property. For $u_i < u_{i+1}$, the difference process $y_m(t) = \widetilde{x}_m(u_{i+1}, t) - \widetilde{x}_m(u_{i+1}, t)$ are equidistributed with the difference approximation v_m to the solution of the SDE

$$\begin{cases} d\widetilde{y}_m(t) = \sqrt{2 - 2\Gamma_m(\widetilde{y}_m(t))}dw(t), \\ \widetilde{y}_m(0) = u_{i+1} - u_i. \end{cases}$$

It is known [1] that

$$E \sup_{[0;1]} (v_m(t) - \widetilde{y}_m(t))^2 \leq C \frac{C_m e^{C_m}}{m}.$$

Note that $\widetilde{y}_m$ is nonnegative. Consequently, for every $r > 0$,

$$P\{\inf_{[0;1]} y_m < -r\} = P\{\inf_{[0;1]} v_m < -r\} \to 0, m \to \infty.$$

Hence, the weak limit of any subsequence of $\{y_m; m \geq 1\}$ is nonnegative. The completion of the proof can be done exactly as in the previous theorem, by using the martingale approximation and the fact that any nonnegative martingale remains at zero after hitting zero. The theorem is proved. □

4. Convergence of random maps

In this section, we will consider the convergence of $\{\widetilde{x}_n; n \geq 1\}$ as random maps to the corresponding maps from a stochastic flow. Let us begin with the case of smooth Γ. Define the sequence

$$x^m_{n+1}(u) = x^m_n(u) + \frac{1}{\sqrt{m}} \xi_{n+1}(x^m_n(u)), \tag{5}$$

where $\{\xi_n; n \geq 1\}$ is a sequence of independent stationary centered Gaussian processes with the covariance function Γ satisfying the inequality

$$\forall\, u \in \mathbb{R}: \ 1 - \Gamma(u) \leq Cu^2$$

with some constant C. Define the Harris flow x corresponding to Γ. Note that x has a modification $x(u, t), u \in \mathbb{R}, t \in [0; 1]$ continuous with respect to both variables. Really, using the martingale inequality, we can obtain

$$E \sup_{s\in[0;t]} (x(u, s) - x(v, s))^2 \leq$$

$$\leq 2(u - v)^2 + 2E \int_0^t (2 - 2\Gamma(x(u, s) - x(v, s)))ds \leq$$

$$\leq 2(u-v)^2 + 4C\int_0^t \sup_{r\in[0;s]} (x(u,r)-x(v,r))^2 ds.$$

Consequently, for some $\widetilde{c}$,

$$E \sup_{t\in[0;1]} (x(u,t)-x(v,t))^2 \leq \widetilde{c}(u-v)^2.$$

This inequality together with the Kolmogorov condition gives us the desired property.

The next statement asserts the convergence of our approximations to a stochastic flow in the case of smooth Γ.

Theorem 3. *The random functions $\{\widetilde{x}_m = x_m^m; m \geq 1\}$ converge in distribution in the space $C([a;b])$ to the random function x for an arbitrary interval $[a;b]$.*

Proof. The convergence of finite-dimensional distributions was proved in Theorem 1. It remains to check the weak compactness of $\{\widetilde{x}_m; m \geq 1\}$. For arbitrary $u, v \in \mathbb{R}$, we have

$$E(x_{n+1}^m(u) - x_{n+1}^m(v))^2 = E(x_n^m(u) - x_n^m(v))^2 + \frac{1}{m}E(2 - 2\Gamma(x_n^m(u) - x_n^m(v))) \leq$$

$$\leq E(x_n^m(u) - x_n^m(v))^2 + 2Cm^{-1}E(x_n^m(u) - x_n^m(v))^2.$$

Consequently,

$$E(\widetilde{x}_m(u) - \widetilde{x}_m(v))^2 \leq (u-v)^2(1+\frac{2C}{m})^m \leq e^{2C}(u-v)^2.$$

The obtained estimation gives the desired weak compactness. The theorem is proved. $\square$

To obtain an approximation of Arratia's flow, we need some additional results about the convergence of smooth stochastic flows to Arratia's flow. Let us consider the following SDE with the space-time white noise (Wiener sheet) W:

$$(6)\qquad \begin{aligned} &dz(u,t) = \int_{\mathbb{R}} \varphi(z(u,t)-p)W(dp,dt), \\ &z(u,0) = u, \ u \in \mathbb{R}. \end{aligned}$$

Here, $\varphi \in C_0^\infty(\mathbb{R})$ and $\int_{\mathbb{R}} \varphi^2(u)du = 1$ (see [6, 7] about equations of type (6)). All what we need here is a statement that, under our condition on φ, the unique strong solution to (6) exists and is the Harris flow corresponding to the local characteristic

$$\Gamma(u) = \int_{\mathbb{R}} \varphi(-p)\varphi(u-p)dp.$$

It was proved in [8] that the n-point motions of solutions z_ε to (6), which corresponds to φ_ε with the property $\operatorname{supp}\varphi_\varepsilon \subset [-\varepsilon;\varepsilon]$, converge in distribution to the n-point motions of the Arratia flow when $\varepsilon \to 0$.

Consider discrete approximations of z. For every $n \geq 1$, we define

$$(7)\qquad \begin{aligned} &z_0^n(u) = u, \\ &z_{k+1}^n(u) = z_k^n(u) + \int_{\frac{k}{n}}^{\frac{k+1}{n}} \int_{\mathbb{R}} \varphi(z_k^n(u)-p)W(dp,dt), \\ &k = 0, \ldots, n-1. \end{aligned}$$

It can be easily checked that every z_k^n has a continuous modification. The next theorem gives a speed of convergence of z_n^n to $z(\cdot,1)$ in the space $C([0;1])$. Define

$$L^2 = \int_{\mathbb{R}} \varphi'(p)^2 dp.$$

Theorem 4. *There exist such positive constants C', C'', and C''' that, for every $n \geq 1$,*

$$E\|z_n^n - z(\cdot, 1)\| \leq \frac{C'}{\sqrt{n}} \exp\{(C''L^2 + C'''L^4)e^{4L^2} + L^2\}(L^2+1), \tag{8}$$

where $\|\cdot\|$ is the uniform norm in $C([0;1])$.

Proof. For $k = 1, \ldots, n$, let us consider

$$E\left(z_k^n(0) - z\left(0, \frac{k}{n}\right)\right)^2 = E\left(z_{k-1}^n(0) - z\left(0, \frac{k-1}{n}\right)\right)^2 +$$

$$+E\int_{\frac{k-1}{n}}^{\frac{k}{n}} \int_{\mathbb{R}} (\varphi(z_{k-1}^n(0) - p) - \varphi(z(0,s) - p))^2 dpds \leq$$

$$\leq E\left(z_{k-1}^n(0) - z\left(0, \frac{k-1}{n}\right)\right)^2 + L^2 E\int_{\frac{k-1}{n}}^{\frac{k}{n}} (z_{k-1}^n(0) - z(0,s))^2 ds =$$

$$= E\left(z_{k-1}^n(0) - z\left(0, \frac{k-1}{n}\right)\right)^2 \left(1 + L^2\frac{1}{n}\right) +$$

$$+L^2 E\int_{\frac{k-1}{n}}^{\frac{k}{n}} \left(z(0) - z\left(0, \frac{k-1}{n}\right)\right)^2 ds =$$

$$= E\left(z_{k-1}^n(0) - z\left(0, \frac{k-1}{n}\right)\right)^2 \left(1 + \frac{L^2}{n}\right) + \frac{L^2}{2n^2}.$$

Consequently,

$$E(z_n^n(0) - z(0,1))^2 \leq \frac{L^2}{n^2} e^{L^2}.$$

Note that, under our conditions on φ, the random functions $\{z_k^n\}$ and z have continuous derivatives with respect to the spatial variable.

We denote these derivatives by y_k^n and y. Then, for $k = 1, \ldots, n$,

$$y_k^n(u) = y_{k-1}^n(u)\left(1 + \int_{\frac{k-1}{n}}^{\frac{k}{n}} \int_{\mathbb{R}} \varphi'(z_{k-1}^n(u) - p) W(dp, dt)\right),$$

and

$$dy(u,t) = y(u,t)\int_{\mathbb{R}} \varphi'(z(u,t) - p) W(dp, dt).$$

Hence,

$$y_k^n(u) - y\left(u, \frac{k}{n}\right) = y_{k-1}^n(u) - y\left(u, \frac{k}{n}\right) +$$

$$+\int_{\frac{k-1}{n}}^{\frac{k}{n}} \int_{\mathbb{R}} [y_{k-1}^n(u)\varphi'(z_{k-1}^n(u) - p) - y(u,s)\varphi'(z(u,s) - p)] W(dp, dt) =$$

$$= y_{k-1}^n(u) - y\left(u, \frac{k-1}{n}\right) +$$

$$+\int_{\frac{k-1}{n}}^{\frac{k}{n}} \int_{\mathbb{R}} \left[\left(y_{k-1}^n(u) - y\left(u, \frac{k-1}{n}\right)\right)\varphi'(z_{k-1}^n(u) - p) +\right.$$

$$+y\left(u, \frac{k-1}{n}\right)\left(\varphi'(z_{k-1}^n(u) - p) - \varphi'\left(z\left(u, \frac{k-1}{n} - p\right) +\right.\right.$$

$$+y\left(u, \frac{k-1}{n}\right)\left(\varphi'\left(z\left(u, \frac{k-1}{n}\right) - p\right) - \varphi'(z(u,s) - p)\right) +$$

$$\left.+\varphi'(z(u,s) - p)\left(y\left(u, \frac{k-1}{n}\right) - y(u,s)\right)\right] W(dp, ds).$$

Then

$$E\left(y_k^n(u)-y\left(u,\frac{k}{n}\right)\right)^2=E\left(y_{k-1}^n(u)-y\left(u,\frac{k-1}{n}\right)\right)^2+$$
$$+4E\int_{\frac{k-1}{n}}^{\frac{k}{n}}\int_{\mathbb{R}}\left(y_{k-1}^n(u)-y\left(u,\frac{k-1}{n}\right)\right)^2\varphi'(z_{k-1}^n(u)-p)^2dpds+$$
$$+4E\int_{\frac{k-1}{n}}^{\frac{k}{n}}\int_{\mathbb{R}}y\left(u,\frac{k-1}{n}\right)^2(\varphi'(z_{k-1}^n(u)-p)-\varphi'\left(z\left(u,\frac{k-1}{n}\right)-p\right)^2dpds+$$
$$+4E\int_{\frac{k-1}{n}}^{\frac{k}{n}}\int_{\mathbb{R}}y\left(u,\frac{k-1}{n}\right)^2\left(\varphi'(z\left(u,\frac{k-1}{n}\right)-p\right)-\varphi'(z(u,s)-p))^2dpds+$$
$$+4E\int_{\frac{k-1}{n}}^{\frac{k}{n}}\int_{\mathbb{R}}\varphi'(z(u,s)-p)^2\left(y\left(u,\frac{k-1}{n}\right)-y(u,s)\right)^2dpds\le$$
$$\le E\left(y_{k-1}^n(u)-y\left(u,\frac{k-1}{n}\right)\right)^2\cdot(1+\frac{4}{n}L^2)+$$
$$+\frac{4}{n}Ey\left(u,\frac{k-1}{n}\right)^2\cdot L^2\left(z_{k-1}^n(u)-z\left(u,\frac{k-1}{n}\right)\right)^2+$$
$$+4Ey\left(u,\frac{k-1}{n}\right)^2L^2\int_{\frac{k-1}{n}}^{\frac{k}{n}}\left(z\left(u,\frac{k-1}{n}\right)-z(u,s)\right)^2ds+$$
$$+4L^2E\int_{\frac{k-1}{n}}^{\frac{k}{n}}\left(y\left(u,\frac{k-1}{n}\right)-y(u,s)\right)^2ds.$$

Note that the processes $z(u,t), t\in[0;1]$ and

$$\eta(t)=\int_0^t\int_{\mathbb{R}}\varphi'(z(u,s)-p)W(dp,ds),\ t\in[0;1]$$

are continuous martingales with the characteristics

$$\langle z(u,\cdot)\rangle(t)=t,\ \langle\eta\rangle(t)=L^2t.$$

Consequently, $z(u,\cdot)$ and η are Wiener processes. It follows from this that

$$y(u,t)=\exp\{\eta(t)-\frac{t}{2}L^2\}.$$

Hence,

$$Ey\left(u,\frac{k-1}{n}\right)^2\int_{\frac{k-1}{n}}^{\frac{k}{n}}\left(z\left(u,\frac{k-1}{n}\right)-z(u,s)\right)^2ds=$$
$$=Ey\left(u,\frac{k-1}{n}\right)^2E\int_{\frac{k-1}{n}}^{\frac{k}{n}}\left(z\left(u,\frac{k-1}{n}\right)-z(u,s)\right)^2ds\le\frac{1}{2n^2}e^{L^2},$$
$$E\int_{\frac{k-1}{n}}^{\frac{k}{n}}\left(y\left(u,\frac{k-1}{n}\right)-y(u,s)\right)^2ds=$$
$$=E\int_{\frac{k-1}{n}}^{\frac{k}{n}}\left(\int_{\frac{k-1}{n}}^{s}\int_{\mathbb{R}}y(u,r)\varphi'(z(u,r)-p)W(dp,dr)\right)^2ds=$$
$$=L^2E\int_{\frac{k-1}{n}}^{\frac{k}{n}}\int_{\frac{k-1}{n}}^{s}y(u,r)^2drds\le\frac{1}{2n^2}L^2\cdot e^{L^2}.$$

Furthermore,

$$Ey\left(u,\frac{k-1}{n}\right)^2\left(z_{k-1}^n(u)-z\left(u,\frac{k-1}{n}\right)\right)^2\le$$

$$\leq \sqrt{Ey\left(u, \frac{k-1}{n}\right)^4}\sqrt{E\left(z^n_{k-1}(u) - z\left(u, \frac{k-1}{n}\right)\right)^4} \leq e^{3L^2} C_2 \frac{L^2}{n} e^{L^2}.$$

In the last inequality, the martingale property of x^n and x was used.

Finally, we obtain

$$E\left(y^n_k(u) - y\left(u, \frac{k}{n}\right)\right)^2 \leq$$

$$\leq E\left(y^n_{k-1}(u) - y\left(u, \frac{k}{n}\right)\right)^2 \left(1 + \frac{4L^2}{n} + \frac{4}{n} L^4 C_2 e^{4L^2}\right) + \frac{c_3}{n^2}(L^2+1)e^{L^2}.$$

Consequently,

$$E(y^n_n(u) - y(u,1))^2 \leq \frac{c_4}{n} \exp\{(c_5 L^2 + c_6 L^4)e^{4L^2} + L^2\}(L^2+1).$$

To obtain an estimation for the uniform norm $\|z^n_n - z(\cdot, 1)\|$, we proceed as follows:

$$E\|z^n_n - z(\cdot,1)\| \leq E|z^n_n(0) - z(0,1)| + E\int_0^1 |y^n_n(u) - y(u,1)|du \leq$$

$$\leq \frac{c_7}{\sqrt{n}} \exp\{(c_8 L^2 + c_9 L^4)e^{4L^2} + L^2\}(L^2+1).$$

The theorem is proved. □

The obtained estimation can be used to prove the convergence of the difference approximation to Arratia's flow. We will establish this convergence using the Lévy–Prokhorov distance. Let us recall its definition.

Definition 2. [9]. For two nondecreasing càdlàg functions f, g on $[0;1]$, the Lévy–Prokhorov distance is

$$\begin{aligned} \rho(f,g) = \inf\{\varepsilon > 0 :\ \forall\, u \in [0;1] :& \\ f(u-\varepsilon) - \varepsilon \leq g(u) \leq f(u+\varepsilon) + \varepsilon& \\ g(u-\varepsilon) - \varepsilon \leq f(u) \leq g(u+\varepsilon) + \varepsilon\}.& \end{aligned}$$

It is well known [9] that the convergence in this distance is equivalent to the convergence at every point of continuity of the limit function. Also note that

$$\rho(f,g) \geq d(f,g),$$

where $d(f,g)$ is the Skorokhod distance between f and g [9].

Take a function $\psi \in C_0^\infty$ with $\operatorname{supp}\psi \subset [-1;1]$ such that

$$\int_{\mathbb{R}} \psi^2(u)du = 1.$$

For arbitrary $\varepsilon > 0$, we define

$$\psi_\varepsilon(u) = \frac{1}{\varepsilon^{1/2}} \psi\left(\frac{u}{\varepsilon}\right),$$

$$\Gamma_\varepsilon(u) = \frac{1}{\varepsilon}\int_{\mathbb{R}} \psi_\varepsilon(p)\psi_\varepsilon(u+p)dp.$$

Here, the parameter ε is associated with the smoothness of Γ_ε. In order to approximate the Arratia flow, we have to take $\varepsilon \to 0$. For independent Gaussian processes $\{\xi_n; n \geq 1\}$ with the covariance $\{\Gamma_{\varepsilon_n}\}$, let us construct the sequences

$$x^n_{k+1}(u) = x^n_k(u) + \frac{1}{\sqrt{n}} \xi_n(x^n_k(u)).$$

The next theorem shows that x^n_n can be used to approximate the Arratia flow.

Theorem 5. *Suppose that* $\varepsilon_n \to 0, n \to \infty$,

$$\frac{1}{\varepsilon_n^2} = o(\mathrm{lln}\, n), \ n \to \infty.$$

Then the random functions x_n^n *converge weakly in* $D([0;1])$ *to the value of the Arratia flow* $x(\cdot, 1)$.

Proof. Consider the sequence of SDE

$$dz_{\varepsilon_n}(u,t) = \int_{\mathbb{R}} \psi_\varepsilon(z_{\varepsilon_n}(u,t) - p) W(dp, dt).$$

As was mentioned at the beginning of this section, for every $u_1, \ldots, u_m \in [0;1]$,

$$(z_{\varepsilon_n}(u_1, 1), \ldots, z_{\varepsilon_n}(u_m, 1)) \text{ weakly converge to } (x(u_1, 1), \ldots, x(u_m, 1)).$$

Hence [10], $z_{\varepsilon_n}(\cdot, 1)$ weakly converge to $x(\cdot, 1)$ in the Lévy–Prokhorov distance. For every $n \geq 1$, the sequence $x_1^n, \ldots, x_n^n$ is equidistributed with the discrete approximations to z_{ε_n} from Theorem 4. Consequently, x_n^n is equidistributed with $\widetilde{x}_n$ such that

$$E\|\widetilde{x}_n - z_{\varepsilon_n}(\cdot, 1)\| \leq \frac{C'}{\sqrt{n}} \exp\{(C'' L_{\varepsilon_n}^2 + C''' L_{\varepsilon_n}^4) e^{4L_{\varepsilon_n}^2} + L_{\varepsilon_n}^2\}(L_{\varepsilon_n}^2 + 1),$$

where $L_{\varepsilon_n}^2 = \frac{1}{\varepsilon_n} \int_{\mathbb{R}} \psi'(p)^2 dp$. Hence,

$$E\|\widetilde{x}_n - z_{\varepsilon_n}(\cdot, 1)\| \to 0, \ n \to \infty.$$

Since, for continuous functions f and g, the Skorokhod distance

$$d(f, g) \leq \|f - g\|,$$

x_n^n weakly converges to $x(\cdot, 1)$ in $D([0;1])$. The theorem is proved. □

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