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ANDRII B. ILIENKO AND JOSEF G. STEINEBACH

STOCHASTICALLY LIPSCHITZIAN FUNCTIONS AND LIMIT THEOREMS FOR FUNCTIONALS OF SHOT NOISE PROCESSES

Let θ be a short memory shot noise process. For wide classes of "stochastically Lipschitzian" (SL) and "stochastically locally Lipschitzian" (SLL) non-linear functions $K\colon \mathbb{R} \to \mathbb{R}$, we prove asymptotic normality of the normalized integrals $\Theta_K(T) = \int_0^T K(\theta(t))\,dt$ as $T \to \infty$. We also consider various examples of SL and SLL functions.

1. Introduction

Let $(\zeta(s),\, s \in \mathbb{R})$ be a Lévy process. It is well-known that ζ can be represented in the following form: $\zeta(s) = \xi(s) + \sigma w(s)$, $s \in \mathbb{R}$, where ξ is a Lévy process without Gaussian component, $\sigma \geq 0$, and w is a Wiener process suitably extended to $\mathbb{R}$ (see, e.g., Kallenberg (2002), p. 290, Corollary 15.7). Let Π be the spectral measure of ξ, and $\Pi_k = \int_{\mathbb{R}} x^k\,\Pi(dx)$, $\overline{\Pi}_k = \int_{\mathbb{R}} |x|^k\,\Pi(dx)$, $k \geq 1$, be the corresponding spectral and absolute spectral moments. In what follows, we will suppose that $\overline{\Pi}_k < \infty$, $k = 1, 2$, and that the process ξ (and so ζ) is centered. Further, let $(g(t),\, t \in \mathbb{R})$ be a non-random real function with $\int_{\mathbb{R}} g^2(t)\,dt < \infty$. Then the following Wiener-type integral is well-defined:

$$\theta(t) = \int_{\mathbb{R}} g(t-s)\,d\zeta(s), \qquad t \in \mathbb{R}. \tag{1}$$

This defines the *shot noise process* with the *response function* g and the *driving Lévy process* ζ. The process θ is strictly stationary and mean-square continuous. More details on the shot noise processes of this form can be found in Chapter 5 of Buldygin and Kozachenko (2000).

This paper focuses on limit theorems for non-linear functionals of shot noise processes. More precisely, we consider the integrated process of the form

$$\Theta_K(T) = \int_0^T K(\theta(t))\,dt, \qquad T > 0, \tag{2}$$

where $K\colon \mathbb{R} \to \mathbb{R}$ is a non-random continuous function, and the integral is interpreted as a mean-square Riemann one. We establish two $\sqrt{T}$-CLT's for *stochastically Lipschitzian* and *stochastically locally Lipschitzian* K's, respectively. It should be noted that the integrals defined in (2) play a role in parameter estimation of shot noise processes, being connected with many useful statistics (as, e.g., moment estimators).

In Bulinskii and Molchanov (1991) as well as Giraitis et al. (1993), similar problems were studied for related shot noise processes and fields with random frequencies and compound Poisson driving measures. In these papers, the authors considered two classes of K's: polynomials and exponential functions; the latter was necessary for applications to Burgers' equation with shot noise initial conditions. The first paper deals with short

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memory shot noise processes and fields, whereas the second one focuses on the case of long range dependence.

In our setting, the process θ exhibits short memory if $g \in \mathbb{L}_1(\mathbb{R})$, and long memory otherwise. In some sense, the discrete analogues of short memory shot noise processes are two-sided (non-causal) linear sequences with summable weights, i.e.

$$X_n = \sum_{i\in\mathbb{Z}} a_{n-i}\epsilon_i, \qquad n \in \mathbb{Z}, \tag{3}$$

where $(\epsilon_i,\ i \in \mathbb{Z})$ is an i.i.d. sequence of innovations with $\mathbf{E}\epsilon_0 = 0$, $\mathbf{E}\epsilon_0^2 < \infty$, and $\sum_{i\in\mathbb{Z}} |a_i| < \infty$. There exists an extensive literature on limit theorems for functionals of linear sequences (one- or two-sided, with summable or non-summable weights, with Gaussian or generally distributed innovations; see, e.g., Giraitis (1985), Giraitis and Surgailis (1989), Ho and Hsing (1997), and references therein). In their recent paper, Cheng and Ho (2005) considered a CLT for a wide class of non-linear continuous functionals in the above setting (3). Their theorem was then used to derive an analogous result for some discontinuous K's (e.g., for indicator functions). The authors used the ℓ-approximation method proposed by Ho and Hsing (1997) for one-sided linear sequences. We use a more direct approach, which allows (at the expense of some additional requirements on the response function g) to extend the class of admissible continuous functionals further and to weaken the moment assumptions.

2. Main results

First we recall the definition of Lipschitzian function. We give it in a slightly unusual form to make the further definitions more natural.

Definition 0. *A continuous function $K\colon \mathbb{R} \to \mathbb{R}$ is called (globally) Lipschitzian if there is $L > 0$ such that*

$$\big(K(x_1 + x_2) - K(x_1)\big)^2 \le L \cdot x_2^2 \tag{4}$$

for each $(x_1, x_2) \in \mathbb{R}^2$.

Now, in a similar manner, we introduce the classes of *stochastically Lipschitzian* and *stochastically locally Lipschitzian* functions, which play an important role in what follows. Let $\mathbb{L}_2(\Omega, \mathbb{R}^2)$ as usual denote the space of all two-dimensional random vectors $\vec{\xi} = (\xi_1, \xi_2)$ on the given probability space with $\mathbf{E}\xi_i^2 < \infty$, $i = 1, 2$.

Definition 1. *Let $\mathcal{M} \subset \mathbb{L}_2(\Omega, \mathbb{R}^2)$. We shall say that a continuous function $K\colon \mathbb{R} \to \mathbb{R}$ is stochastically Lipschitzian w.r.t. $\mathcal{M}$ if there is $L > 0$ such that*

$$\mathbf{E}\big(K(\xi_1 + \xi_2) - K(\xi_1)\big)^2 \le L \cdot \mathbf{E}\xi_2^2 \tag{5}$$

for each $\vec{\xi} \in \mathcal{M}$.

Definition 2. *Let $\mathcal{M} \subset \mathbb{L}_2(\Omega, \mathbb{R}^2)$. We shall say that a continuous function $K\colon \mathbb{R} \to \mathbb{R}$ is stochastically locally Lipschitzian w.r.t. $\mathcal{M}$ if there are $\epsilon > 0$, $L > 0$ such that*

$$\mathbf{E}\big((K(\xi_1 + \xi_2) - K(\xi_1)) \cdot \mathbb{1}_{\{|\xi_2| \le \epsilon\}}\big)^2 \le L \cdot \mathbf{E}\xi_2^2 \tag{6}$$

for each $\vec{\xi} \in \mathcal{M}$. Here $\mathbb{1}_A$ denotes the indicator function of the event A.

We shall write $K \in SL(\mathcal{M})$ or $K \in SLL(\mathcal{M})$ if the function K is stochastically Lipschitzian or stochastically locally Lipschitzian w.r.t. $\mathcal{M}$, respectively.

Let $\mathcal{B}(\mathbb{R})$ as usual denote the Borel σ-algebra on $\mathbb{R}$.

Definition 3. *Let θ be a shot noise process as in (1), and*

$$\mathcal{M}_\theta = \Big\{\vec{\xi} \in \mathbb{L}_2(\Omega, \mathbb{R}^2)\colon \xi_i = \int_{A_i} g(-s)\, d\zeta(s),\ i = 1, 2; A_1, A_2 \in \mathcal{B}(\mathbb{R}),\ A_1 \cap A_2 = \emptyset\Big\}.$$

We shall say that a function $K\colon \mathbb{R}\to\mathbb{R}$ is stochastically Lipschitzian or stochastically locally Lipschitzian w.r.t. the shot noise process θ (and write $K\in SL_\theta$ or $K\in SLL_\theta$) if $K\in SL(\mathcal{M}_\theta)$ or $K\in SLL(\mathcal{M}_\theta)$, respectively.

Remark 1. Note that $\vec{\xi}\in\mathcal{M}_\theta$ implies that ξ_1 and ξ_2 are independent.

It should be noted that the condition $K\in SL_\theta$ may be regarded as a particular case of the more general condition (7) in Cheng and Ho (2005). Theorem 3 below shows that the condition $K\in SL_\theta$ *alone* ensures the central limit theorem for a wide range of response functions g.

It is clear from (5) that the class SL_θ is closed under linear combinations and includes all Lipschitzian (in the usual sense) functions on $\mathbb{R}$. The next statements show that SL_θ may actually include a lot of non-Lipschitzian functions.

Theorem 1. *Suppose that $g\in\mathbb{L}_2(\mathbb{R})\cap\mathbb{L}_\infty(\mathbb{R})$. Let m be a positive integer, K_i, $i=1,\dots,m$, be Lipschitzian (in the usual sense) functions on $\mathbb{R}$, $K=\prod_{i=1}^m K_i$, and $\overline{\Pi}_{2m}<\infty$. Then $K\in SL_\theta$.*

Theorem 2. *Let the following conditions hold:*

i) $g\in\mathbb{L}_2(\mathbb{R})\cap\mathbb{L}_\infty(\mathbb{R})$;
ii) *the m-th derivative $K^{(m)}$ exists on $\mathbb{R}$ and is Lipschitzian (in the usual sense) for some non-negative integer m;*
iii) $\sup_{A\in\mathcal{B}(\mathbb{R})}\mathbf{E}\Big(K^{(i)}\big(\int_A g(-s)\,d\zeta(s)\big)\Big)^2<\infty$ *for all $i=1,\dots,m$;*
iv) $\overline{\Pi}_{2m+2}<\infty$.

Then $K\in SL_\theta$.

Remark 2. Condition ii) is clearly satisfied if $K^{(m+1)}$ exists and is bounded on $\mathbb{R}$.

Example 1. Let $g\in\mathbb{L}_2(\mathbb{R})\cap\mathbb{L}_\infty(\mathbb{R})$. Consider *a generalized polynomial* $P=(P(x),x\in\mathbb{R})$ of the following form:

$$P(x)=b_0+\sum_{i=1}^n b_i^+x^{\alpha_i}\mathbb{1}_{[0,\infty)}(x)+\sum_{i=1}^n b_i^-(-x)^{\alpha_i}\mathbb{1}_{(-\infty,0)}(x), \tag{7}$$

with $n\in\mathbb{N}$, $b_0,b_i^+,b_i^-\in\mathbb{R}$, $\alpha_i\in[1,m]$, $i=1,\dots,n$, and $m\in\mathbb{N}$. Suppose also that $\overline{\Pi}_{2m}<\infty$. Then $P\in SL_\theta$.

Indeed, the term $f_\alpha^+(x)=x^\alpha\mathbb{1}_{[0,\infty)}(x)$, $x\in\mathbb{R}$, $\alpha\in[1,m]$, may be written as follows:

$$\begin{aligned}
f_\alpha^+(x)&=\big(x^\alpha\mathbb{1}_{[1,\infty)}(x)+\mathbb{1}_{(-\infty,1)}(x)\big)+\big(x^\alpha\mathbb{1}_{[0,1)}(x)-\mathbb{1}_{(-\infty,1)}(x)\big)\\
&=\big(x\mathbb{1}_{[1,\infty)}(x)+\mathbb{1}_{(-\infty,1)}(x)\big)^{\lfloor\alpha\rfloor}\big(x^{\{\alpha\}}\mathbb{1}_{[1,\infty)}(x)+\mathbb{1}_{(-\infty,1)}(x)\big)\\
&\quad+\big(x^\alpha\mathbb{1}_{[0,1)}(x)-\mathbb{1}_{(-\infty,1)}(x)\big)=p_1^{\lfloor\alpha\rfloor}(x)p_{\{\alpha\}}(x)+q_\alpha(x),\qquad x\in\mathbb{R},
\end{aligned}$$

where $\lfloor\cdot\rfloor$ and $\{\cdot\}$ stand for the floor function and the fractional part of α, respectively, $p_\beta(x)=x^\beta\mathbb{1}_{[1,\infty)}(x)+\mathbb{1}_{(-\infty,1)}(x)$, $x\in\mathbb{R}$, $\beta\in[0,1]$, and $q_\beta(x)=x^\beta\mathbb{1}_{[0,1)}(x)-\mathbb{1}_{(-\infty,1)}(x)$, $x\in\mathbb{R}$, $\beta\in[1,\infty)$. It is straightforward to check that the functions p_β and q_β are Lipschitzian on $\mathbb{R}$ in the usual sense for the above values of β. Hence, by Theorem 1, $f_\alpha^+\in SL_\theta$. The functions $f_\alpha^-(x)=(-x)^\alpha\mathbb{1}_{(-\infty,0)}(x)$, $x\in\mathbb{R}$, $\alpha\in[1,m]$, may be considered in a similar way. Since SL_θ is closed under linear combinations, the result follows.

The notion of *stochastic local Lipschitzianity* can be made clear by the following general example.

Example 2. Let K be a differentiable function on $\mathbb{R}$, and

$$Q_\epsilon(x) = \sup_{|\delta|\le\epsilon} |K'(x+\delta)|, \qquad x \in \mathbb{R}, \tag{8}$$

$$L = \sup_{A\in\mathcal{B}(\mathbb{R})} \mathbf{E} Q_\epsilon^2\Big(\int_A g(-s)\, d\zeta(s)\Big) < \infty \qquad \text{for some } \epsilon > 0. \tag{9}$$

In the notations of Definition 3, by the mean value theorem,

$$\mathbf{E}\big((K(\xi_1+\xi_2) - K(\xi_1)) \cdot \mathbb{1}_{\{|\xi_2|\le\epsilon\}}\big)^2 \le \mathbf{E}(Q_\epsilon^2(\xi_1)\cdot\xi_2^2) \le L \cdot \mathbf{E}\xi_2^2,$$

i.e. $K \in SLL_\theta$. Here we used the independence of ξ_1 and ξ_2.

We may now weaken the assumptions on the response function g in Example 1.

Example 3. Consider again the generalized polynomial P given by (7). Let all assumptions of Example 1 hold except $g \in \mathbb{L}_\infty(\mathbb{R})$. We suppose instead that $g \in \mathbb{L}_2(\mathbb{R}) \cap \mathbb{L}_{2m}(\mathbb{R})$. Then $P \in SLL_\theta$.

Indeed, as in Example 1, it is sufficient to consider only the function $f_\alpha^+(x) = x^\alpha \mathbb{1}_{[0,\infty)}(x)$, $x \in \mathbb{R}$, $\alpha \in [1, m]$. The function f_1^+ is Lipschitzian on $\mathbb{R}$, so $f_1^+ \in SLL_\theta$. If $\alpha > 1$, then f_α^+ is differentiable on $\mathbb{R}$. Moreover, $Q_1^2(x) = \alpha^2(x+1)^{2\alpha-2}\mathbb{1}_{[-1,\infty)}(x) \le m^2(x+1)^{2m-2}$, $x \in \mathbb{R}$, where Q_ϵ is defined by (8) with f_α^+ in place of K. Hence, (9) follows from Lemma 3 below with $K(x) = x+1$, $x \in \mathbb{R}$. Thus $P \in SLL_\theta$ in view of Example 2.

Next we state two central limit theorems for $K \in SL_\theta$ and $K \in SLL_\theta$.

Theorem 3. *Let θ be a shot noise process as in (1), and $K \in SL_\theta$. Suppose that the following conditions hold:*

i) $\int_0^\infty \big(\int_s^\infty g^2(t)\, dt\big)^{1/2} ds < \infty$;
ii) $\int_{-\infty}^0 \Big(\int_{-\infty}^s g^2(t)\, dt\Big)^{1/2} ds < \infty$.

Then

$$T^{-1/2}\big(\Theta_K(T) - T \cdot \mathbf{E}K(\theta(0))\big) \xrightarrow[T\to\infty]{d} \mathcal{N}(0, \sigma^2), \tag{10}$$

with

$$\sigma^2 = \lim_{T\to\infty} T^{-1}\operatorname{Var}(\Theta_K(T)) < \infty, \tag{11}$$

where $\xrightarrow{d}$ denotes convergence in distribution.

Remark 3. Note that conditions i) and ii) hold if

$$g(s) = o(|s|^{-\frac{3}{2}-\delta}), \quad s \to \pm\infty, \quad \text{for some } \delta > 0. \tag{12}$$

Theorem 4. *Let θ be a shot noise process as in (1), and $K \in SLL_\theta$. Suppose that condition* (12) *holds, and*

i) $\overline{\Pi}_4 < \infty$;
ii) $\sup_{A\in\mathcal{B}(\mathbb{R})} \mathbf{E}K^{8/3}\big(\int_A g(-s)\, d\zeta(s)\big) < \infty$.

Then (10) *and* (11) *follow.*

3. Notations and preliminary lemmas

In this section, we introduce some notations and prove a few auxiliary results. Together with the process θ, let us consider, for $A \in \mathcal{B}(\mathbb{R})$, the processes $\theta_A = \big(\int_{\mathbb{R}} g(t-s)\mathbb{1}_{(-A)}(t-s)\, d\zeta(s),\ t \in \mathbb{R}\big)$. In particular, for $\Delta(a) = [-a, a]$, $a > 0$, we have

$$\theta_{\Delta(a)}(t) = \int_{t-a}^{t+a} g(t-s)\, d\zeta(s), \qquad t \in \mathbb{R}. \tag{13}$$

Note that, in the notations of Definition 3,

$$\theta_{A_i}(0) = \int_{A_i} g(-s)\, d\zeta(s) = \xi_i, \qquad i = 1, 2. \tag{14}$$

Further, let $\kappa_l = \kappa_l(\theta(\cdot))$ denote the l-th semi-invariant of $\theta(\cdot)$, which does not depend on the time parameter by virtue of the strict stationarity. Analogously, put $\kappa_{A,l} = \kappa_l(\theta_A(\cdot))$. Finally, we introduce the standard norms in $\mathbb{L}_p(\mathbb{R})$, $p \in [1, \infty)$, and in $\mathbb{L}_\infty(\mathbb{R})$ by setting $\|g\|_p = (\int_{\mathbb{R}} |g(s)|^p\, ds)^{1/p}$, $\|g\|_\infty = \operatorname{ess\,sup}_{s\in\mathbb{R}} |g(s)|$.

Lemma 1. *Assume that $\overline{\Pi}_1, \overline{\Pi}_{2k} < \infty$ and $g \in \mathbb{L}_2(\mathbb{R}) \cap \mathbb{L}_{2k}(\mathbb{R})$ for some $k \geq 1$. Then the semi-invariants κ_l exist for all $l = 2, \dots, 2k$, and $\kappa_l = (\Pi_l + \delta_{l,2}\sigma^2)\|g\|_l^l$ for these l, where δ_{ij} denotes Kronecker's delta.*

Proof. It is an immediate consequence of Lemma 2.2 in Buldygin and Kozachenko (2000), p.152. □

Lemma 2. *Assume that $\overline{\Pi}_1, \overline{\Pi}_{2k} < \infty$ for some $k \geq 1$, and $g \in \mathbb{L}_2(\mathbb{R}) \cap \mathbb{L}_\infty(\mathbb{R})$. Then there is a positive constant D_k such that $\mathbf{E}\theta_A^{2k}(\cdot) \leq D_k \mathbf{E}\theta_A^2(\cdot)$ for each $A \in \mathcal{B}(\mathbb{R})$.*

Proof. It is straightforward from Lemma 1, applied to the process θ_A, that

$$|\kappa_{l,A}| \leq (\overline{\Pi}_l + \delta_{l,2}\sigma^2)\|g\|_l^l \leq \frac{\overline{\Pi}_l + \delta_{l,2}\sigma^2}{\Pi_2 + \sigma^2} \cdot \|g\|_\infty^{l-2}\kappa_{2,A}, \qquad l = 2, \dots, 2k.$$

Hence, by the Leonov-Shiryaev formula (see, e.g., Shiryaev (1995), p. 290, Theorem 6),

$$\mathbf{E}\theta_A^{2k}(\cdot) = \sum_{l_1+\dots+l_q=2k} \frac{1}{q!} \frac{(2k)!}{l_1! \cdot \ldots \cdot l_q!} \prod_{p=1}^{q} \kappa_{l_p,A} \leq P(\kappa_{2,A}), \tag{15}$$

where P is a polynomial of degree k with zero constant term and non-negative coefficients, which do not depend on A. (Here we used that $\kappa_{1,A} = \mathbf{E}\theta_A(\cdot) = 0$.) By Lemma 1, we have for $m = 1, \dots, k$, $A \in \mathcal{B}(\mathbb{R})$:

$$\kappa_{2,A}^m \leq (\Pi_2 + \sigma^2)^{m-1}\|g\|_2^{2m-2}\kappa_{2,A} = (\Pi_2 + \sigma^2)^{m-1}\|g\|_2^{2m-2}\mathbf{E}\theta_A^2(\cdot). \tag{16}$$

A combination of (15) and (16) completes the proof. □

Lemma 3. *Assume $\overline{\Pi}_1, \overline{\Pi}_{2k} < \infty$ for some $k \geq 1$, together with $g \in \mathbb{L}_2(\mathbb{R}) \cap \mathbb{L}_{2k}(\mathbb{R})$. Let K be Lipschitzian (in the usual sense). Then $S_k = \sup_{A\in\mathcal{B}(\mathbb{R})} \mathbf{E}K^{2k}(\theta_A(\cdot)) < \infty$.*

Proof. Let L be the Lipschitz constant of K. Then

$$\mathbf{E}K^{2k}(\theta_A(0)) \leq 2^{2k-1}\mathbf{E}\big(K(\theta_A(0)) - K(0)\big)^{2k} + 2^{2k-1}K^{2k}(0) \leq 2^{2k-1}L^k\, \mathbf{E}\theta_A^{2k}(0) + 2^{2k-1}K^{2k}(0). \tag{17}$$

Lemma 1 (applied to θ_A) and the Leonov-Shiryaev formula show that the semi-invariants $\kappa_{A,l}$, $l = 2, \dots, 2k$, and the moments $\mathbf{E}\theta_A^{2k}(\cdot)$ are uniformly bounded in A. Thus, in view of (17), the proof is complete. □

Remark 4. Clearly, $(0, \theta_A(0))^T = \big(0, \int_A g(-s)\, d\zeta(s)\big)^T \in \mathcal{M}_\theta$. Therefore, the proof (and the Lemma) remains true if K is only *stochastically* Lipschitzian w.r.t. θ and $k = 1$.

Lemma 4. *Let θ be a shot noise process as in (1), and $K\colon \mathbb{R} \to \mathbb{R}$ be a Borel function. Then the process $K(\theta)$ is strictly stationary. If, in addition, K is continuous and $\mathbf{E}K^2(\theta(\cdot)) < \infty$, then $K(\theta)$ is mean-square continuous.*

Proof. The strict stationarity of $K(\theta)$ follows from that of θ. The mean-square continuity is a consequence of the following implications:

$$\theta(t) \xrightarrow[t\to t_0]{\mathbb{L}_2(\Omega)} \theta(t_0) \implies \theta(t) \xrightarrow[t\to t_0]{Pr.} \theta(t_0) \implies K(\theta(t)) \xrightarrow[t\to t_0]{Pr.} K(\theta(t_0)) \implies K(\theta(t)) \xrightarrow[t\to t_0]{\mathbb{L}_2(\Omega)} K(\theta(t_0)), \tag{18}$$

where $\xrightarrow{\mathbb{L}_2(\Omega)}$ and $\xrightarrow{Pr.}$ denote convergence in mean square and in probability, respectively. The first implication in (18) is clear, the second one follows from the continuity of K (see, e.g., Kallenberg (2002), p. 64, Lemma 4.3), and the third one is a consequence of the uniform integrability of $K^2(\theta)$, which in turn follows from the strict stationarity of $K(\theta)$. □

The second assertion of Lemma 4 shows that the integrals in (2) are well-defined as mean-square Riemann ones.

Lemma 5. *Let the conditions of Theorem 3 or Theorem 4 hold. Then*

i) $P_1(a) = a \cdot \mathbf{E}\big(K(\theta(0)) - K(\theta_{\Delta(a)}(0))\big)^2 \to 0$ *as* $a \to \infty$;
ii) $P_2(a) = \int_{2a}^{\infty} \mathbf{E}\big|K(\theta(0)) - K(\theta_{\Delta(\tau/2)}(0))\big|\, d\tau < \infty$ *for* $a > 0$ *(and thus* $\lim_{a\to\infty} P_2(a) = 0$*);*
iii) $P_3(a) = \int_{2a}^{\infty} \mathbf{E}\big|K(\theta_{\Delta(\tau/2)}(0)) - K(\theta_{\Delta(a)}(0))\big|\big|K(\theta(\tau)) - K(\theta_{\Delta(\tau/2)}(\tau))\big|\, d\tau < \infty$ *for* $a > 0$, *and* $\lim_{a\to\infty} P_3(a) = 0$.

Proof. We only prove the first assertion, the other ones can be proved in a similar way. Under the conditions of Theorem 3, by (5) and Lemma 1 applied to the process $\theta - \theta_{\Delta(a)}$,

$$P_1(a) \le aL\mathbf{E}\big(\theta(0) - \theta_{\Delta(a)}(0)\big)^2 = aL(\Pi_2 + \sigma^2) \int_{\Delta^C(a)} g^2(s)\, ds,$$

where $\Delta^C(a)$ denotes the complement to $\Delta(a)$. Hence, the assertion follows from conditions i) and ii) of Theorem 3. At the same time, under the conditions of Theorem 4, we have by (6)

$$\begin{aligned}
P_1(a) &= a\mathbf{E}\big(K(\theta(0)) - K(\theta_{\Delta(a)}(0)) \cdot \mathbb{1}_{\{|\theta(0)-\theta_{\Delta(a)}(0)| \le \epsilon\}}\big)^2 \\
&\quad + a\mathbf{E}\big(K(\theta(0)) - K(\theta_{\Delta(a)}(0)) \cdot \mathbb{1}_{\{|\theta(0)-\theta_{\Delta(a)}(0)| > \epsilon\}}\big)^2 \\
&\le aL\mathbf{E}(\theta(0) - \theta_{\Delta(a)}(0))^2 \\
&\quad + a\mathbf{E}\big(\big(K(\theta(0)) - K(\theta_{\Delta(a)}(0))\big)^2 \mathbb{1}_{\{|\theta(0)-\theta_{\Delta(a)}(0)| > \epsilon\}}\big) = U_1(a) + U_2(a).
\end{aligned}$$

As above, $\lim_{a\to\infty} U_1(a) = 0$. By Hölder's inequality applied to $U_2(a)$, we get

$$U_2(a) \le \big(\mathbf{E}\big(K(\theta(0)) - K(\theta_{\Delta(a)}(0))\big)^{8/3}\big)^{3/4} \cdot a\big(\mathbb{P}\{|\theta(0) - \theta_{\Delta(a)}(0)| > \epsilon\}\big)^{1/4}.$$

By condition ii) of Theorem 4, the first factor is bounded in a. The second one can be estimated by the Chebyshev inequality and the Leonov-Shiryaev formula:

$$\begin{aligned}
a\big(\mathbb{P}\{|\theta(0) - \theta_{\Delta(a)}(0)| > \epsilon\}\big)^{1/4} &\le a\epsilon^{-1}\big(\mathbf{E}(\theta(0) - \theta_{\Delta(a)}(0))^4\big)^{1/4} \\
&\le a\epsilon^{-1}\big(\kappa_4(\theta(0) - \theta_{\Delta(a)}(0)) + 3\kappa_2^2(\theta(0) - \theta_{\Delta(a)}(0))\big)^{1/4}.
\end{aligned}$$

Now, by Lemma 1 applied to the process $\theta - \theta_{\Delta(a)}$, we have

$$a\Big(\mathbb{P}\{|\theta(0) - \theta_{\Delta(a)}(0)| > \epsilon\}\Big)^{1/4}$$
$$\le a\epsilon^{-1}\Big(\Pi_4 \int_{\Delta^C(a)} g^4(s)\,ds + 3(\Pi_2+\sigma^2)^2\Big(\int_{\Delta^C(a)} g^2(s)\,ds\Big)^2\Big)^{1/4}.$$

Thus, condition (12) of Theorem 4 yields the first assertion. □

4. Proofs of the main results

In this section, we give the proofs of Theorems 1, 2, 3 and 4.

Proof of Theorem 1. Let D_k, L_i and S_{ik}, $i, k = 1, \dots, n$, denote the constants from Lemma 2, the Lipschitz constants of the functions K_i, and the constants from Lemma 3 for the functions K_i, respectively. Set $\bar{D} = \max_{k=1,\dots,n} D_k$, $\bar{L} = \max\{1, L_1, \dots, L_n\}$ and $\bar{S} = \max_{i,k=1,\dots,n} S_{ik}$.

Let $\mathcal{N}_n$ denote the set of all *proper* subsets of $\{1, \dots, n\}$, i.e. $\mathcal{N}_n = \big\{N \subset \{1,\dots,n\} : N \neq \{1,\dots,n\}\big\}$, and $\vec{\xi} = (\xi_1, \xi_2)^T \in \mathcal{M}_\theta$. It can easily be seen that

$$\prod_{i=1}^n K_i(\xi_1+\xi_2) - \prod_{i=1}^n K_i(\xi_1) = \sum_{N\in\mathcal{N}_n} \prod_{i\in N} K_i(\xi_1) \prod_{i\notin N}(K_i(\xi_1+\xi_2) - K_i(\xi_1)).$$

Hence, by the Lipschitzianity of K_i,

$$\mathbf{E}\Big(\prod_{i=1}^n K_i(\xi_1+\xi_2) - \prod_{i=1}^n K_i(\xi_1)\Big)^2$$
$$\le (2^n-1)\sum_{N\in\mathcal{N}_n} \mathbf{E}\Big(\prod_{i\in N} K_i^2(\xi_1)\prod_{i\notin N}(K_i(\xi_1+\xi_2)-K_i(\xi_1))^2\Big)$$
$$\le (2^n-1)\sum_{N\in\mathcal{N}_n}\Big(\prod_{i\notin N} L_i^2\Big)\mathbf{E}\Big(\prod_{i\in N} K_i^2(\xi_1)\Big)\mathbf{E}\Big(\prod_{i\notin N}\xi_2^2\Big).$$

Let $\operatorname{card} N$ denote the cardinality of the set N. By the inequality

$$\prod_{i\in N} K_i^2(\xi_1) \le (\operatorname{card} N)^{-1}\sum_{i\in N} K_i^{2\operatorname{card} N}(\xi_1),$$

we have

$$\mathbf{E}\Big(\prod_{i=1}^n K_i(\xi_1+\xi_2) - \prod_{i=1}^n K_i(\xi_1)\Big)^2$$
$$\le (2^n-1)\sum_{N\in\mathcal{N}_n}(\operatorname{card} N)^{-1}\bar{L}^{2n-2\operatorname{card} N}\Big(\sum_{i\in N}\mathbf{E}K_i^{2\operatorname{card} N}(\xi_1)\Big)\mathbf{E}\xi_2^{2n-2\operatorname{card} N}.$$

Thus, by (14), Lemma 2 and Lemma 3, we finally obtain

$$\mathbf{E}\Big(\prod_{i=1}^n K_i(\xi_1+\xi_2) - \prod_{i=1}^n K_i(\xi_1)\Big)^2 \le (2^n-1)\sum_{N\in\mathcal{N}_n}\bar{L}^{2n}\bar{S}\bar{D}\,\mathbf{E}\xi_2^2 = (2^n-1)^2\bar{L}^{2n}\bar{S}\bar{D}\,\mathbf{E}\xi_2^2,$$

which completes the proof. □

Proof of Theorem 2. By Taylor's formula,

$$K(\xi_1+\xi_2) - K(\xi_1) = \sum_{i=1}^m \frac{K^{(i)}(\xi_1)}{i!}\xi_2^i + \Big(\frac{K^{(m)}(\eta) - K^{(m)}(\xi_1)}{m!}\xi_2^m\Big),$$

where $|\eta - \xi_1| \le |\xi_2|$. Hence, denoting by L the Lipschitz constant of K, we have

$$\mathbf{E}\big(K(\xi_1+\xi_2) - K(\xi_1)\big)^2 \le (m+1)\sum_{i=1}^m \frac{\mathbf{E}(K^{(i)}(\xi_1))^2}{(i!)^2}\mathbf{E}\xi_2^{2i} + \frac{L(m+1)}{(m!)^2}\mathbf{E}\xi_2^{2m+2}.$$

Thus, by (14) and Lemma 2,

$$\mathbf{E}\big(K(\xi_1+\xi_2)-K(\xi_1)\big)^2 \le (m+1)\Big(\sum_{i=1}^{m}\frac{D_i\mathbf{E}(K^{(i)}(\xi_1))^2}{(i!)^2}+\frac{LD_{m+1}}{(m!)^2}\Big)\mathbf{E}\xi_2^2.$$

Condition iii) of the Theorem shows that the coefficient of $\mathbf{E}\xi_2^2$ on the right-hand side is bounded in $\vec{\xi}\in\mathcal{M}$ (or — equivalently — in $A\in\mathcal{B}(\mathbb{R})$). So, $K\in SL_\theta$. $\square$

Proof of Theorems 3, 4. Let $m=\mathbf{E}K(\theta(\cdot))$, $m_a=\mathbf{E}K(\theta_{\Delta(a)}(\cdot))$. The random variable on the left-hand side of (10) is equal to

$$\begin{aligned}(19)\qquad & T^{-1/2}(\Theta_K(T)-Tm)=T^{-1/2}\int_0^T(K(\theta_{\Delta(a)}(t))-m_a)\,dt\\ &+T^{-1/2}\int_0^T(K(\theta(t))-K(\theta_{\Delta(a)}(t))-m+m_a)\,dt=I_1(a,T)+I_2(a,T).\end{aligned}$$

Let $\lfloor\cdot\rfloor$ and $\lceil\cdot\rceil$ denote the floor and ceiling function, respectively. The first summand may be further rewritten as follows:

$$\begin{aligned}I_1(a,T)=&T^{-1/2}\sum_{k=1}^{\lfloor T\rfloor}\int_{k-1}^{k}(K(\theta_{\Delta(a)}(t))-m_a)\,dt+T^{-1/2}\int_{\lfloor T\rfloor}^{T}(K(\theta_{\Delta(a)}(t))-m_a)\,dt\\ =&T^{-1/2}\sum_{k=1}^{\lfloor T\rfloor}J_k(a)+T^{-1/2}J(a,T).\end{aligned}$$

Denote by $\mathcal{F}_{(t_1,t_2]}$ the σ-algebra generated by $\{\zeta(t),t_1<t\le t_2\}$. By (13), it is easily seen that $\theta_{\Delta_a}(t)$ are $\mathcal{F}_{(t-a,t+a]}$-measurable, and so $J_k(a)$ are $\mathcal{F}_{(k-1-a,k+a]}$-measurable. Hence, $(J_k(a),k\in\mathbb{N})$ is a $\lceil 2a+1\rceil$-dependent strictly stationary random sequence. Moreover, $\mathbf{E}J^2(a,T)$ is bounded in $T\in(0,\infty)$. Thus, by Diananda (1953), $I_1(a,T)\xrightarrow{d}\mathcal{N}(0,\sigma_a^2)$ as $T\to\infty$ with $\sigma_a^2=\lim_{T\to\infty}T^{-1}\operatorname{Var}\int_0^T K(\theta_{\Delta(a)}(t))\,dt$.

So, in order to prove Theorems 3 and 4 it is enough to show that

$$(20)\qquad \lim_{a\to\infty}\limsup_{T\to\infty}\mathbf{E}I_2^2(a,T)=0.$$

Let $C_a=(C_a(\tau),\tau\in\mathbb{R})$ be the correlation function of the centered stationary process $K(\theta)-K(\theta_{\Delta(a)})-m+m_a$, i.e.

$$C_a(\tau)=\mathbf{E}\big(K(\theta(0))-K(\theta_{\Delta(a)}(0))-m+m_a\big)\big(K(\theta(\tau))-K(\theta_{\Delta(a)}(\tau))-m+m_a\big).$$

For $T>2a$, the term $\mathbf{E}I_2^2(a,T)$ may be written in the following form:

$$\begin{aligned}\mathbf{E}I_2^2(a,T)&=T^{-1}\int_0^T\int_0^T C_a(t-s)\,ds\,dt=2T^{-1}\int_0^T\int_s^T C_a(t-s)\,ds\,dt\\ &=2T^{-1}\int_0^{T-2a}\int_0^{2a}C_a(\tau)\,ds\,d\tau+2T^{-1}\int_0^{T-2a}\int_{2a}^{T-s}C_a(\tau)\,ds\,d\tau\\ &+2T^{-1}\int_{T-2a}^{T}\int_0^{T-s}C_a(\tau)\,ds\,d\tau=R_1(a,T)+R_2(a,T)+R_3(a,T).\end{aligned}$$

Clearly, $|R_1(a,T)|\le 4aC_a(0)\le 4a\mathbf{E}\big(K(\theta(0))-K(\theta_{\Delta(a)}(0))\big)^2$.

Hence, $\lim_{a\to\infty}\limsup_{T\to\infty}R_1(a,T)=0$ by the first assertion of Lemma 5. It is also easy to see that $|R_3(a,T)|\le 8a^2T^{-1}C_a(0)\xrightarrow[T\to\infty]{}0$. Moreover, $|R_2(a,T)|=2T^{-1}\big|\int_{2a}^T(T-\tau)C_a(\tau)\,d\tau\big|\le 2\int_{2a}^T|C_a(\tau)|\,d\tau$. Hence, in order to prove (20) we only need to show that $C_a\in\mathbb{L}_1(\mathbb{R}^+)$, and

$$(21)\qquad \lim_{a\to\infty}\int_{2a}^{\infty}|C_a(\tau)|\,d\tau=0.$$

For $\tau \geq 2a$, we may write $C_a(\tau)$ in the following form:

$$\begin{aligned}C_a(\tau) = &\mathbf{E}\big(K(\theta_{\Delta(\tau/2)}(0)) - K(\theta_{\Delta(a)}(0)) - m_{\tau/2} + m_a\big)\\ &\big(K(\theta_{\Delta(\tau/2)}(\tau)) - K(\theta_{\Delta(a)}(\tau)) - m_{\tau/2} + m_a\big)\\ &+\mathbf{E}\big(K(\theta_{\Delta(\tau/2)}(0)) - K(\theta_{\Delta(a)}(0)) - m_{\tau/2} + m_a\big)\\ &\big(K(\theta(\tau)) - K(\theta_{\Delta(\tau/2)}(\tau)) - m + m_{\tau/2}\big)\\ &+\mathbf{E}\big(K(\theta(0)) - K(\theta_{\Delta(\tau/2)}(0)) - m + m_{\tau/2}\big)\\ &\big(K(\theta_{\Delta(\tau/2)}(\tau)) - K(\theta_{\Delta(a)}(\tau)) - m_{\tau/2} + m_a\big)\\ &+\mathbf{E}\big(K(\theta(0)) - K(\theta_{\Delta(\tau/2)}(0)) - m + m_{\tau/2}\big)\\ &\big(K(\theta(\tau)) - K(\theta_{\Delta(\tau/2)}(\tau)) - m + m_{\tau/2}\big)\\ &= V_1(a,\tau) + V_2(a,\tau) + V_3(a,\tau) + V_4(\tau).\end{aligned}$$

Consider V_i, $i = 1,\dots,4$, in more detail. The two factors in V_1 are easily seen to be centered and measurable w.r.t. the σ-algebras $\mathcal{F}_{(-\tau/2,\,\tau/2]}$ and $\mathcal{F}_{(\tau/2,\,3\tau/2]}$, respectively. So, they are independent, and $V_1(a,\tau) = 0$ for all $a > 0$ and $\tau \geq 2a$.

Now we prove that

$$\lim_{a\to\infty} \int_{2a}^{\infty} |V_2(a,\tau)|\, d\tau = 0. \tag{22}$$

Clearly,

$$\begin{aligned}|V_2(a,\tau)| \leq &\mathbf{E}\big|K(\theta_{\Delta(\tau/2)}(0)) - K(\theta_{\Delta(a)}(0))\big| \cdot \mathbf{E}\big|K(\theta(\tau)) - K(\theta_{\Delta(\tau/2)}(\tau))\big|\\ +&\mathbf{E}\big|K(\theta_{\Delta(\tau/2)}(0)) - K(\theta_{\Delta(a)}(0))\big|\big|K(\theta(\tau)) - K(\theta_{\Delta(\tau/2)}(\tau))\big| = V_2'(a,\tau) + V_2''(a,\tau).\end{aligned}$$

Under the conditions of Theorem 3 (Theorem 4), the first factor in $V_2'(a,\tau)$ is uniformly bounded in $\tau \geq 2a$ by the Lyapunov inequality and Remark 4 (respectively condition ii) of Theorem 4). Hence, (22) holds with $V_2'(a,\tau)$ and $V_2''(a,\tau)$ in place of $V_2(a,\tau)$ in view of assertions ii) and iii) of Lemma 5, respectively.

The terms $V_3(a,\tau)$ and $V_4(\tau)$ can be treated in a similar way. This proves (21), which completes the proof of Theorems 3 and 4. □

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Department of Mathematical Analysis and Probability Theory, National Technical University of Ukraine (KPI), Prospekt Peremogy 37, 03056 Kiev, Ukraine
E-mail address: a_ilienko@ukr.net

Mathematical Institute of the University of Cologne, Weyertal 86-90, 50931 Cologne, Germany
E-mail address: jost@math.uni-koeln.de